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Some Fixed Point Results in Metric space

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Abstract - In the present manuscript, we shall introduce a generalized F- contraction mapping on complete metric spaces and prove a fixed point result for single self - map. In addition to this, we shall prove a common fixed point result for generalized F-contraction mappings for two compatible self - maps.

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Introduction

Throughout the article we denote by $\mathbb{R}$, the set of all real numbers, by $\mathbb{R}^+$, the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

In 2012, Wardowski [6] introduced the new concept for contraction mappings known as F-contraction by considering a class of real valued functions as follows:

Definition 1.1. Let F : $\mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

$(F1)$ F is strictly increasing, i.e. for all $\alpha, \beta \in (0, \infty)$, then $F(\alpha) < F(\beta)$.

$(F2)$ For each sequence $\{\alpha_n\} \subset (0, \infty)$, the following holds:

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0} \alpha^k F(\alpha) = 0$.

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F1) - (F3)$.

Definition 1.2. Let (X, d) be a metric space. A mapping $T: X \to X$ is an F-contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \text{ implies } \tau + F(d(Tx, Ty)) \le F(d(x, y)).$$

Definition 1.3. [3] Two self-maps f and g defined on a metric space X are said to be compatible if $d(TIx_n, ITx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that

$$Tx_n, Ix_n \to t \in X.$$

In 1996, Jungck *et al.* [3] introduced the concept of weakly compatible maps as follows:

Definition 1.4. [5] Two self-maps f and g defined on a metric space X are said to be weakly compatible if they commute at their coincidence points.

In view of mapping F, we obtain different types of contractions known in literature as shown with the following examples:

Example 1.5. [6] Let $F: \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = In\ \alpha$. It is clear that F satisfies (F1) - (F3) for any $k \in (0, 1)$. Each mapping $T: X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx, Ty) \le e^{-\tau} d(x, y), \qquad \text{for all } x, y,\ \ Tx \neq Ty. \tag{1.1}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \le e^{-\tau} d(x, y)$, also holds, i.e. T is Banach contraction[1].

Example 1.6 [6] Let $F(\alpha) = In\ \alpha + \alpha$. It is clear that F satisfies $(F1) - (F3)$ for any $k \in (0, 1)$. Each mapping $T: X \to X$ satisfying (1.1) is an F-contraction such that

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau}, \qquad \text{for all } x, y\ \ Tx \neq Ty. \tag{1.2}$$

Example 1.7 [6] Let $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$. It is clear that F satisfies $(F1) - (F3)$ for any $k \in (1/2, 1)$. Each mapping $T: X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx, Ty) \le \frac{1}{\left(1+\tau\sqrt{d(x,y)}\right)^2} d(x, y),, \quad \text{for all } x, y\ \ Tx \neq Ty. \tag{1.3}$$

1. Main Results

In this section, we give some fixed point results for F – contraction for single map and also for pair of self - maps in complete metric space. Firstly, we define generalized F-contraction as follows:

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is a generalized F-contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$.

$$d(Tx, Ty) > 0 \text{ implies } \tau + F\big(d(Tx, Ty)\big) \le F\big(\Delta(x, y)\big),$$

where

$\Delta(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Now, we introduce the new concept of F −contraction for two self-maps as follows:

Definition 2.2. Let (X, d) be a metric space and $f, g: X \to X$ be two self-maps. g is said to be F – contractive mapping with respect to f if there exist $F \in \mathcal{F}$ and $\tau > 0$, such that

$$d(fx, fy) > 0 \text{ implies } \tau + F\big(d(gx, gy)\big) \le F\big(d(fx, fy)\big).$$

Now, we introduce the new concept of generalized F −contraction for two self-maps as follows:

Definition 2.3. Let (X, d) be a metric space and $f, g: X \to X$ be two mappings. g is said to be generalized F – contractive mapping with respect to f if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(fx, fy) > 0 \text{ implies } \tau + F\big(d(gx, gy)\big) \le F\big(\Delta(fx, fy)\big),$$

where

$\Delta(fx, fy) = \max\{d(gx, g\,y), d(gx, fx), d(gy, fy)\}$

Theorem 2.4. Let (X, d) be a complete metric space and let T be a self - map on X. Assume that there exists $F \in \mathcal{F}$ and $\tau > 0$ such that T is an F – contraction satisfying

$$\tau + F\big(d(Tx, Ty)\big) \le F\big(\Delta(x, y)\big), \tag{2.1}$$

where

$\Delta(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$,

for all $x, y \in X,\ Tx \ne Ty$.

Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be the Picard sequence with initial point x_0, that is,

$$x_n = T^n x_0 = Tx_{n-1.} \tag{2.2}$$

If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T.

Now, let $x_n \ne x_{n-1}$, that is, $Tx_{n-1} \ne Tx_n$ for all $n \in \mathbb{N}$.

Let, $d_n = d(x_n, x_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$.

Now, on putting $x = x_{n-1}, y = x_n$ in (2.1), we have

$$\begin{aligned} \tau + F(d_n) &= \tau + F(d(x_n, x_{n+1})) \\ &= \tau + F(d(Tx_{n-1}, Tx_n)) \\ &\le F\big(\Delta(x_{n-1}, x_n)\big), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Delta(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= max\{d_{n-1}, d_{n-1}, d_n\} \\ &\le max\{d_{n-1}, d_n\}. \end{aligned}$$

If $d_n > d_{n-1}$, then, we have

$$\Delta(x_{n-1}, x_n) = d_n.$$

From (2.3), we have

$$\tau + F(d_n) \leq F(d_n),$$

that is,

$$F(d_n) \leq F(d_n) - \tau,$$

which implies

$$F(d_n) < F(d_n),$$

a contradiction. Hence $d_{n-1} > d_n$.

From (2.3), we have

$$\tau + F(d_n) \leq F(d_{n-1}),$$

that is,

$$\begin{aligned} F(d_n) &\leq F(d_{n-1}) - \tau \\ &\leq F(d_{n-2}) - 2\tau \\ &\quad - \quad - \quad - \\ &\quad - \quad - \quad - \\ &\leq F(d_0) - n\tau \quad \text{for all } n \in \mathbb{N} \end{aligned} \tag{2.4}$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F(d_n) = -\infty.$$

By the property $(F2)$, we have $d_n \to 0$ as $n \to \infty$. Now, let $k \in (0, 1)$ such that

$$\lim_{n\to+\infty} {d_n}^k F(d_n) = 0.$$

By (2.4), the following holds for all $n \in \mathbb{N}$:

$${d_n}^k F(d_n) - {d_n}^k F(d_0) \leq {d_n}^k F\big((d_0) - n\tau\big) - {d_n}^k F(d_0) = -n\tau {d_n}^k \leq 0. \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we deduce that

$$\lim_{n\to+\infty} n\,{d_n}^k = 0.$$

Hence

$$\lim_{n\to+\infty} n^{1/k} d_n = 0.$$

Now, $\lim_{n\to+\infty} n^{1/k} d_n = 0$ ensures that the series $\sum_{n=1}^{\infty} d_n$ is convergent.

This implies that $\{x_n\}$ is a Cauchy sequence.

Now, X is a complete metric space therefore, there exists $z \in X$ such that $x_n \to z$. If $z = Tz$ then there is nothing to prove.

Assume that $z \neq Tz$. If $Tx_n = Tz$ for infinite values of $n \in \mathbb{N} \cup \{0\}$, then the sequence $\{x_n\}$ has a subsequence that converges to Tz and the uniqueness of the limit implies $z = Tz$.

Then, we can assume that $Tx_n \neq Tz$, for all $n \in \mathbb{N} \cup \{0\}$.

Now, by (2.1), we have

$$d(z,Tz) \leq d(z,x_n) + d(x_n,Tz)$$

$$= d(z,x_n) + d(Tx_{n-1},Tz)$$

$$d(z,Tz) - d(z,x_n) \leq d(Tx_{n-1},Tz)$$

$$F(d(z,Tz) - d(z,x_n)) \leq F(d(Tx_{n-1},Tz))$$

$$\leq F\big(\Delta(x_{n-1},z)\big) - \tau \tag{2.6}$$

$\Delta(x_{n-1},z) = \max\{d(x_{n-1},z), d(x_{n-1},Tx_{n-1}), d(z,Tz)\}$.

Taking limit as $n \to \infty$, we get

$\lim\limits_{n\to\infty} \Delta(x_{n-1},z) = \max\{d(z,z), d(z,z), d(z,Tz)\}$.

$= d(z,Tz)$.

From (2.6), we have

$$F(d(z,Tz)) \leq F(d(z,Tz)) - \tau,$$

which implies

$$F(d(z,Tz)) < F(d(z,Tz)),$$

that is

$$d(z,Tz) < d(z,Tz),$$

a contradiction.

Hence $z = Tz$.

Which shows that z is a fixed point of T.

Now, we prove the uniqueness of fixed point.

Assume that $p \in X$ is another fixed point of T, different from z.

This means that $d(z,p) \geq 0$.

Taking $x = z$ and $y = p$ in (2.1), we have

$$\tau + F\big(d(Tz,Tp)\big) \leq F\big(\Delta(z,p)\big), \tag{2.7}$$

where

$\Delta(z,p) = \max\{d(z,z), d(z,Tz), d(p,Tp)\}$.

$= d(z,p)$.

From (2.7), we have

$$\tau + F\big(d(z,p)\big) \le F\big(d(z,p)\big),$$

$$F\big(d(z,p)\big) < F\big(d(z,p)\big),$$

this implies

$$d(z,p) < d(z,p),$$

a contradiction. Hence $z = p$. This completes the proof of the theorem.

Now, we prove common fixed point theorem for two compatible maps for F- contraction.

Theorem 2.5. Let f and g be two compatible maps of a complete metric space into itself. Let f and g be F- contractive mapping. Assume that $g(X) \subset f(X)$ and g is continuous satisfying

$$\tau + F\big(d(gx, gy)\big) \le F\big(\Delta(fx, fy)\big), \tag{2.8}$$

$\Delta(fx, fy) = \max\{d(gx, gy), d(gx, fx), d(gy, fy)\}$.

Then f and g have a unique fixed point.

Proof : Let $x_0 \in X$. Since $g(X) \subset f(X)$, there exists $x_1 \in$ X such that $fx_1 = gx_0$.

In general $y_n = fx_{n+1} = gx_n$.

Consider $\mathrm{y_n} \neq \mathrm{y_{n+1}}$for any n.

Therefore, $\mathrm{d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n)} > 0$.

Now, on putting $x = x_n$ and $y = x_{n+1}$ in (2.8), we get

$$\tau + F\big(d(gx_n, gx_{n+1})\big) \le F\big(\Delta(fx_n, fx_{n+1})\big), \tag{2.9}$$

$\Delta(fx_n, fx_{n+1}) = \max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1})\}$

$$= max\{d(y_n, y_{n+1}), d(y_n, y_{n-1}), d(y_{n+1}, y_n)\}$$

Let, $d_n = d(y_n, y_{n+1})$.

$\Delta(fx_n, fx_{n+1}) = max\{d_n, d_{n-1}, d_n\}$

$$= max\{d_n, d_{n-1}\}$$

If $d_n > d_{n-1}$, then $\Delta(fx_n, fx_{n+1}) = d_n$.

Then from (2.9), we have

$$\tau + F(d_n) \le F(d_n),$$

a contradiction, since $\tau > 0$

Hence $d_n < d_{n-1}$,

Then from (2.9), we have

$$\tau + F(d_n) \le F(d_{n-1}),$$

$$
\begin{aligned}
F(d_n) &\leq F(d_{n-1}) - \tau \\
&\leq F(d_{n-2}) - 2\tau \\
&\ - \ \ - \ \ - \\
&\ - \ \ - \ \ - \\
&\leq F(d_0) - n\tau \quad \text{for all } n \in \mathbb{N} \qquad (2.10)
\end{aligned}
$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F(d_n) = -\infty.$$

By the property (F2), we have $d_n \to 0$ as $n \to \infty$.

Now, from (F3), here exists $k \in (0, 1)$ such that

$$\lim_{n\to+\infty} {d_n}^k F(d_n) = 0.$$

By (2.4), the following holds for all $n \in \mathbb{N}$:

$$ {d_n}^k F(d_n) - {d_n}^k F(d_0) \leq {d_n}^k F\big((d_0) - n\tau\big) - {d_n}^k F(d_0) = -n\tau {d_n}^k \leq 0. \qquad (2.11)$$

Letting $n \to \infty$ in (2.10) and (2.11), we deduce that

$$\lim_{n\to+\infty} n\, {d_n}^k = 0.$$

$$ {d_n}^k n \leq 1 \qquad \text{for all } n > p$$

Consequently, we have

$$d_n \leq \frac{1}{n^{1/k}} \qquad \text{for all } n > p$$

Since, the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$converges, for any ϵ>0, there exists q≥1, such that

$$\sum_{i=q}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \epsilon.$$

Consider $m > n > max\,\{p, q\}$

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq \sum_{j=n}^{m-1} d\big(y_{j+1}, y_j\big) \\
&\leq \sum_{j=n}^{\infty} d\big(y_{j+1}, y_j\big) \\
&\leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}
\end{aligned}
$$

$$\leq \sum_{\mathrm{j=q}}^{\infty} \frac{1}{\mathrm{j}^{\frac{1}{\mathrm{k}}}} < \epsilon.$$

Implies that $\{\mathrm{y_n}\}$ is a Cauchy sequence. Now, (X, d) is complete metric space, therefore there exists $z \in X$ such that $\{y_n\}$ converges to z implies $\{gx_n\}$ and $\{fx_n\}$ converges to z.

Since f and g are compatible and f is continuous therefore

$$\lim_{\mathrm{n}\to\infty} ffx_n = \lim_{\mathrm{n}\to\infty} fg\mathrm{x_n} = \lim_{\mathrm{n}\to\infty} gf\mathrm{x_n} = f\mathrm{z}.$$

Now, from (2.9), we have

$$\tau + F\big(d(gy_n, gz)\big) \leq F\big(\Delta(fy_n, fz)\big), \tag{2.12}$$

$\Delta(fy_n, fz) = \max\{d(gy_n, gz), d(gy_n, fy_n), d(gz, fz)\}$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \Delta(fy_n, fz) = max\{d(gz, gz), d(gz, gz), d(gz, fz)\}$$

$$= d(fz, gz).$$

From (2.12), we have

$$\tau + F\big(d(gz, fz)\big) \leq F\big(d(fz, gz)\big),$$

a contradiction. Hence $fz = gz$. Now, we prove that $fz = z$.

Let, if possible $fz \neq z$.

On putting, $x = x_n$ and $y = z$ in (2.8), we have

$$\tau + F\big(d(gx_n, gz)\big) \leq F\big(\Delta(fx_n, fz)\big), \tag{2.13}$$

$\Delta(fx_n, fz) = \max\{d(gx_n, gz), d(gx_n, fx_n), d(gz, fz)\}$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \Delta(fx_n, fz) = max\{d(z, fz), d(z, z), d(fz, fz)\}$$

$$= d(z, fz).$$

From (2.13), we have

$$\tau + F\big(d(z, fz)\big) \leq F\big(d(z, fz)\big),$$

a contradiction. Hence $fz = gz = z$. This implies z is the common fixed point of f and g.

Now, we prove uniqueness of the common fixed point. Let, w is another common fixed point of f and g.

Now, from (2.8), we have

$$\tau + F\big(d(gz, gw)\big) \leq F\big(\Delta(fz, fw)\big), \tag{2.14}$$

$\Delta(fz, fw) = \max\{d(gz, gw), d(gz, fz), d(gw, fw)\}$

$$= d(z, w).$$

From (2.14), we have

$$\tau + F(d(z, w)) \leq F(d(z, w)),$$

$$F(d(z, w)) < F(d(z, w)),$$

$$d(z, w) < d(z, w),$$

a contradiction. This implies the uniqueness of the common fixed point. Hence completes the proof of the theorem.

Now, we prove a common fixed point result for two weakly compatible maps satisfying (2.8) and we remove the continuity of the map.

Theorem 2.6. Let f and g be two weakly compatible maps of a complete metric space into itself satisfying (2.8). Let fX be complete. Let f and g be F- contractive mapping. Assume that $g(X) \subset f(X)$. Then f and g have unique fixed points.

Proof: Proceeding Theorem 2.5, we have $\{y_n\}$ is a Cauchy sequence. From the completeness of fX, there exists $z \in X$ such that $\{y_n\}$converges to fz implies

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} g\, x_n = \lim_{n\to\infty} fx_n = f(z).$$

Now, on putting $x = x_n$ and $y = z$ in (2.8), we have

$$\tau + F(d(gx_n, gz)) \leq F(\Delta(fx_n, fz)), \tag{2.15}$$

$$\Delta(fx_n, fz) = \max\{d(gx_n, gz), d(gx_n, fx_n), d(gz, fz)\}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \Delta(fx_n, fz) = max\{d(gz, gz), d(gz, g\, z), d(gz, fz)\}$$

$$= d(gz, fz).$$

From (2.15), we have

$$\tau + F(d(fz, gz)) \leq F(d(gz, fz)),$$

$$F(d(fz, gz)) < F(d(gz, fz)),$$

$$d(gz, fz) < d(gz, fz),$$

a contradiction.

This implies $fz = gz$, which implies z is coincidence point of f and g.

Since f and g are weakly compatible.

Therefore, $fz = gz = u$ implies

$$fgz = gfz \text{ i.e, } fu = gu.$$

Now, we prove that u be common fixed point of f and g.

Now, on putting $x = u$ and $y = z$ in (2.8), we have

$$\tau + F\big(d(gu, gz)\big) \leq F\big(\Delta(fu, fz)\big), \tag{2.16}$$

$$\begin{aligned}\Delta(fu, fz) &= \max\{d(gu, gz), d(gu, fu), d(gz, fz)\} \\ &= d(u, gu).\end{aligned}$$

From (2.16), we have

$$\tau + F\big(d(u, gu)\big) \leq F\big(d(u, gu)\big),$$

$$F\big(d(u, gu)\big) < F\big(d(u, gu)\big),$$

$$d(u, gu) < d(u, gu),$$

a contradiction.

Hence $gu = u = fu$.

This implies u is the common fixed point of f and g.

The uniqueness of the common fixed point theorem follows from Theorem 2.5.

This completes the proof of the theorem.

References

1. Banach, S: Sur les operations dans les enesembles abstraits et leur applications aux equations integrals. Fund Math. 3, 133-181 (1922)
2. Cosentino M., Vetro P., Fixed point result for F-contractive Mappings of Hardy-Rogers-Type, Filomat 28:**4** (2014), 715-722.
3. Imdad M., Ali J., Tanveer M., Remarks on some recent material fixed point theorems, Appl. Math. Lett. 24(2011), 1165-1169
4. Jungck G., common fixed points for non-continuous non - self maps on non - metric spaces. Far East J. Math.Sci.**4(2)** (1996), 199-212.
5. Kanta L., Kumam P., Senapati T., Fixed point results concerning $\alpha - F$ −contraction mappings in metric spaces, Appl. Gen. Topol. 20, **1**(2019), 81-95.
6. Wardowski D., Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Application (2012),94.