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A Study on Equitable Restrained Domination in Graphs

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Abstract: We study equitable restrained domination in graphs, a concept combining restrained and equitable domination. A set $D \subseteq V(G)$ is an equitable restrained dominating set if it is both restrained and equitable and the minimum cardinality of such a set is the equitable restrained domination number $\gamma_r^e(G)$. In this paper, we establish bounds for $\gamma_r^e(G)$ and explore its relationships with other domination parameters. Additionally, we examine the effect of graph operation, the join of two graphs on $\gamma_r^e(G)$.

Keywords: Equitable restrained domination, restrained domination, equitable domination, domination number, graph operations

1. Introduction

Graph theory has emerged as one of the most dynamic branches of discrete mathematics, with wide-ranging applications in computer science, communication networks, biological systems, and social sciences. Among its central notions, the concept of domination has been studied extensively due to both its theoretical significance and practical relevance. A dominating set in a graph ensures that every vertex is either in the set or adjacent to a vertex in the set, making it a fundamental tool for solving problems related to monitoring, resource allocation, and network security.

We consider a simple, finite, connected and undirected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The order of G is the cardinality of $V(G)$ and the size of G is the cardinality of $E(G)$. For a vertex $v \in V(G)$, the open neighbourhood is defined as $N_G(v) = \{u \in V(G) : uv \in E(G)\}$, and the degree of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex with no neighbours is called an isolated vertex, a vertex of degree one is called a leaf, and a support vertex is a vertex adjacent to a leaf. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two disjoint graphs. The join of G and H, denoted by $G + H$, is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. That is, the join is obtained from the disjoint union of G and H by adding all possible edges between $V(G)$ and $V(H)$. For further graph-theoretic terminology, we refer to Harary [3], while for standard terminology related to domination theory, we refer to Haynes *et al.* [4].

A subset $D \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex of D. If no proper subset $D' \subset D$ is a dominating set, then D is called a minimal dominating set. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma(G)$, and a dominating set of cardinalities $\gamma(G)$ is referred to as a γ −set of G. Domination has been a central topic of research in graph theory, leading to the development of several important variants.

One such variant is restrained domination, introduced by Telle [6] , further explore in Domke *et al.* [2]. A subset $D \subseteq V(G)$ is called a restrained dominating set if every vertex in $V(G) \setminus D$ has a neighbour in D as well as a neighbour in $V(G) \setminus D$. The minimum cardinality of a restrained dominating set is called the restrained domination number of G, denoted by $\gamma_r(G)$ and a restrained dominating set of cardinality $\gamma_r(G)$ is called a γ_r −set of G.

Another important extension is equitable domination, which has been studied in detail by Swaminathan and Dharmalingam [5], Anitha *et al.* [1], and Polinar-Sandueta [10], with applications to balancing degree conditions in networks. A subset $D \subseteq V(G)$ is called an equitable dominating set if for every $v \in V(G) \setminus D$, there exists a neighbour $u \in D$ such that $|d_G(u) - d_G(v)| \leq 1$. The minimum cardinality of such a set is called the equitable domination number of G, denoted by $\gamma^e(G)$. Further developments in this direction have also been discussed in the works of Dharmalingam [11], Thasneem and Menon [9], and Maglanque [13], highlighting various generalizations and connections to structural graph properties.

The concept of equitable restrained domination arises naturally by combining restrained domination and equitable domination. It was first introduced by Kulli [8], and further studied by Vaidya and Ajani [7]. A dominating set D of G is called an equitable restrained dominating set if D is both restrained and equitable. The minimum cardinality of such a set is called the equitable restrained domination number of G, denoted by $\gamma_r^e(G)$ and a set of cardinality $\gamma_r^e(G)$ is called a γ_r^e −set of G. This parameter has gained recent attention, with related investigations on degree equitable restrained double domination by Hosamani *et al.* [12].

In this paper, we investigate equitable restrained domination, highlighting its relationship with restrained and equitable domination. we examine the effect of the join operation on $\gamma_r^e(G)$ and propose a conjecture characterizing trees for which $\gamma_r^e(T) = n - 2$. We establish bounds for $\gamma_r^e(G)$ in general graphs and examine its behavior under the join operation.

2. Main Results

Theorem 2.1. For every graph G of order n, we have $1 \leq \gamma_r^e(G) \leq n$. Further, the lower bound is attained if and only if $G = K_1 \vee H$ where H is a complete graph or a $(n-1, n-2)$ −biregular graph. The upper bound is attained if and only if G does not contain an edge $uv \in E(G)$, other than a pendent edge, which satisfies the following conditions:

1. There exist vertices $x \in N_G(u)$ and $y \in N_G(v)$ such that $|N_G(u)| \geq 2, |N_G(v)| \geq 2$.
2. There exist vertices $x \in N_G(u)$ and $y \in N_G(v)$ such that $|d_G(u) - d_G(x)| \leq 1$ and $|d_G(v) - d_G(y)| \leq 1$.

Proof. The lower bound follows from the definition of equitable restrained domination. Now, consider the case when the lower bound is attained. Suppose $\gamma_r^e(G) = 1$, then G has a central vertex v such that for any $u \in V(G) \setminus \{v\}$, we have $|d_G(u) - d_G(v)| \leq 1$, which implies $d_G(u) = n - 1$ or $d_G(u) = n - 2$. Clearly, the possible degrees of $u \in V(G) \setminus \{v\}$ are as above. Therefore, G is either a complete graph or an $(n-1, n-2)$ −biregular graph.

Conversely, it is clear for the complete graph. Now, assume G is an $(n-1, n-2)$ −biregular graph. Then G contains a central vertex, so $\gamma_r^e(G) = 1$, which implies $\gamma_r^e(G) = 1$.

Now consider the upper bound. Suppose $\gamma_r^e(G) = n$ and G contains an edge which satisfies the conditions in the hypothesis. Then $V(G) \setminus \{u, v\}$ will form an equitable restrained dominating set in G, hence $\gamma_r^e(G) \leq n - 2$, a contradiction. Therefore, G must not contain an edge as stated in the hypothesis of the theorem. □

Theorem 2.2. Let T be a tree of order *n*. Then $\gamma_r^e(T) = n$ if and only if T does not contain an edge $uv \in E(T)$ such that u has a neighbor $x \neq v$ and v has a neighbor $y \neq u$ satisfying $|d_G(u) - d_G(x)| \leq 1$ and $|d_G(v) - d_G(y)| \leq 1$.

Proof. The proof follows similar to earlier theorem. □

Theorem 2.3. For any tree T, $\gamma_r^e(T) \geq |\ell(T)|$. Moreover, equality holds if and only if $T \cong P_4$.

Proof. First, every leaf must belong to any equitable restrained dominating set. If a leaf $l \notin D$, its neighbours must lie in D, but then l is isolated in $T \setminus D$, contradicting the restrained condition. Hence $\ell(T) \subseteq D$, so $\gamma_r^e(T) \geq |\ell(T)|$.

Now, if $\gamma_r^e(T) = |\ell(T)|$, then $\ell(T)$ itself is a $\gamma_r^e(T)$ −set. Thus $\ell(T)$ dominates T, and $T \setminus \ell(T)$ has no isolated vertices. Therefore, every internal vertex must have degree 2. The only tree satisfying this is the path P_4, where

$\gamma_r^e(P_4) = |\ell(P4)| = 2$. Hence equality holds iff $T \cong P_4$. □

Observation: For any tree T of order n, $\gamma_r^e(T) \neq n - 1$.

Justification. If $\gamma_r^e(T) < n$, then there exists an edge $uv \in E(T)$ satisfying the equitable and restrained conditions, which allows at least two vertices to be simultaneously excluded from the dominating set, forcing $\gamma_r^e(T) \leq n - 2$.

We now consider trees for which $\gamma_r^e(T) = n - 2$ and construct a family of such trees.

Let $S(n, k)$ denote the tree obtained from the path P_n on vertices $v_1, v_2, \dots, v_n$ by attaching exactly $k \geq 1$ leaves to each of the vertices $v_2, v_3, \dots, v_n$. The vertex v_1 has no additional leaf neighbors. Then:

$$|V(S(n,k))| = n + (n-1)k, \quad |E(S(n,k))| = (n-1)(k+1).$$

Now define the operation Ω on a tree T as follows. Let $s \in V(T)$ be a support vertex. The operation Ω is:

• Choose a pendant neighbor ℓ of s, and attach either $|d_T(s) - 1|$ or $|d_T(s) - 2|$ new leaves to ℓ.

• To each remaining leaf neighbor of s, attach exactly one new pendant vertex.

Note that $\gamma_r^e(S(2,1)) = 3 = n$ for $n = 3$, so $S(2,1)$ itself does not satisfy $\gamma_r^e(T) = n - 2$. However, every tree obtained by applying Ω at least once to $S(2,1)$ satisfies $\gamma_r^e(T) = n - 2$. We therefore define:

$$\mathcal{F} = \{T \mid T\ is\ obtained\ from\ S(2,1)\ by\ applying\ \Omega\ at\ least\ once\}$$

Theorem 2.4 (**Conjecture**). Let T be a tree of order $n \geq 4$. Then $\gamma_r^e(T) = n - 2$ if and only if $T \in \mathcal{F}$.

Theorem 2.5. Let G be a graph. Then $\gamma(G) = \gamma_r(G) = \gamma_r^e(G)$ if and only if there exists a γ −set D of G such that for every vertex $u \in V \setminus D, N_G^e(u) \cap D \neq \emptyset$ and $N_G(u) \cap (V \setminus D) \neq \emptyset$.

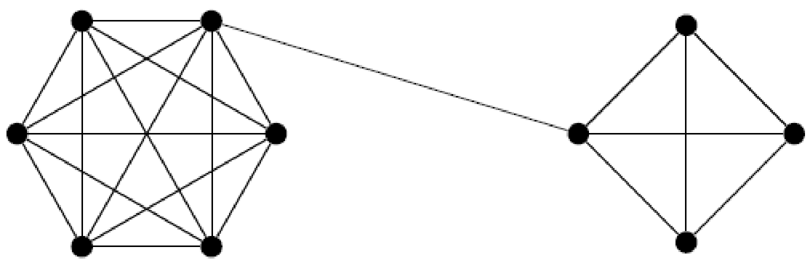
Proof. Assume that $\gamma(G) = \gamma_r(G) = \gamma_r^e(G)$. Suppose, for contradiction, that for every γ −set D of G, there exists a vertex $u \in V \setminus D$ such that either $N_G^e(u) \cap D = \emptyset$ or $N_G(u) \cap (V \setminus D) = \emptyset$.

Let D be a γ_r^e −set of G. Then D is an equitable restrained dominating set with cardinality $\gamma_r^e(G) = \gamma(G)$. Therefore, D is also a γ −set. By our assumption, there exists $u \in V \setminus D$ such that either $N_G^e(u) \cap D = \emptyset$ or $N_G(u) \cap (V \setminus D) = \emptyset$, which contradicts the definition of an equitable restrained dominating set.

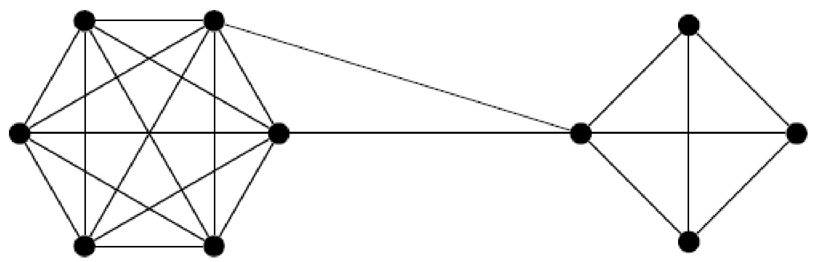
Conversely, suppose there exists a γ −set D of G such that for every $u \in V \setminus D$, $N_G^e(u) \cap D \neq \emptyset$ and $N_G(u) \cap (V \setminus D) \neq \emptyset$. Then D satisfies the conditions for an equitable restrained dominating set, so $\gamma_r^e(G) \leq |D|$. Since D is a γ −set, we have $\gamma(G) = |D|$. Also, by standard domination inequalities, $\gamma(G) \leq \gamma_r(G) \leq \gamma_r^e(G)$. Therefore, $\gamma(G) = \gamma_r(G) = \gamma_r^e(G)$. □

Remark 2.1. The following are infinite classes of graphs for which $\gamma_r(G) = \gamma_r^e(G)$:

1. All regular graphs and all biregular graphs with bi-regularity $(k, k + 1)$.
2. Consider the graph formed by joining a vertex of K_n to a vertex of K_{n-2}.



3. The graph obtained by joining a vertex of K_{n-2} to any two vertices of K_n.

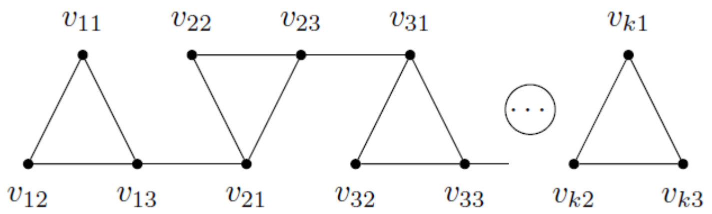


4. In these constructions, we have $\gamma_r(G) = \gamma_r^e(G)$. Continuing this construction iteratively, we can construct graphs for which $\gamma_r(G) = \gamma_r^e(G) = k$, for any positive integer k.

Proposition 2.1. For every $k \in \mathbb{N}$, there exists a connected graph G such that $\gamma(G) = \gamma_r(G) = \gamma^e(G) = \gamma_r^e(G) = k$.

Proof. For $k = 1$, take $G = K_3$; then the result clearly holds.

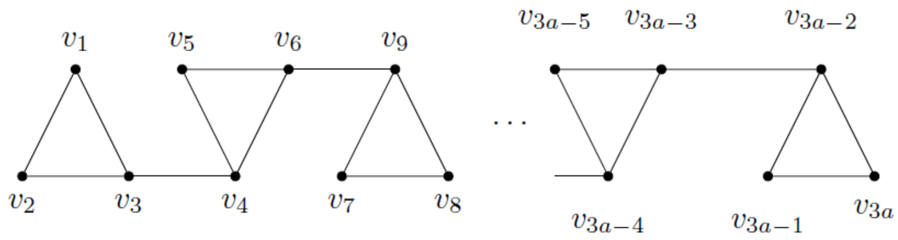
Let $k \geq 2$. Take k copies of K_3, where the vertices of the i^{th} copy are labeled as v_{i1}, v_{i2}, v_{i3}. Connect v_{i3} to $v_{(i+1)1}$ for all $i = 1, 2, \dots, k - 1$ to obtain the resulting graph.



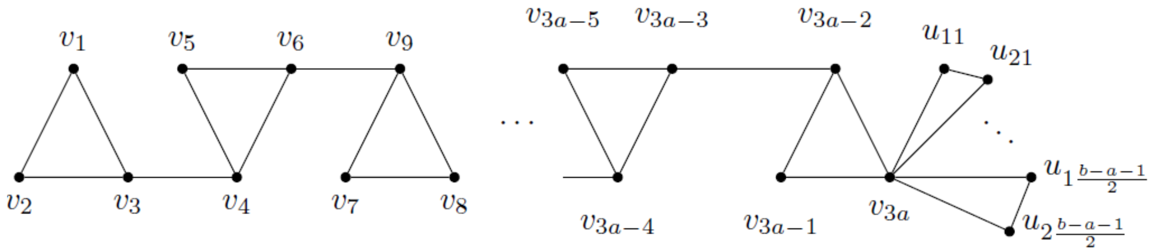
Then, $\gamma(G) = \gamma_r(G) = \gamma^e(G) = \gamma_r^e(G) = k$, by choosing the set of k vertices $\{v_{13}, v_{23}, v_{33}, \dots, v_{k3}\}$. □

Proposition 2.2. For any integers a, b with $a < b$, $b - a \geq 5$ and $b - a$ is odd, one can construct a connected graph G such that $\gamma_r(G) = a$ and $\gamma_r^e(G) = b$.

Proof. Take a copies of K_3 and join them one after another as shown in the construction.



Now attach $(b - a - 1)/2$ copies of K_2 to the vertex v_{3a} by joining each to a distinct new vertex. Let the new K_2 vertices be labeled u_{1i} and u_{2i} for $1 \leq i \leq (b - a - 1)/2$.



Clearly, $\gamma_r(G) = a$, as choosing one vertex from each triangle suffices, and all non-dominated vertices have neighbors both in and outside the dominating set.

Now we prove that $\gamma_r^e(G) = b$. Define

$$D = \{v_1, v_4, v_7, \ldots, v_{3a-1}, v_{3a}\} \cup \{u_{1i}, u_{2i} \mid 1 \le i \le (b - a - 1)/2\}$$

To ensure domination, we include v_{3a} as well, making $|D| = b$.

Every vertex $u \in V(G) \setminus D$ is adjacent to some $v \in D$ such that $|d_G(u) - d_G(v)| \le 1$, and u is adjacent to another vertex in $V(G) \setminus D$. Thus, D is an equitable restrained dominating set, so $\gamma_r^e(G) \le b$.

Let D' be any equitable restrained dominating set of G with minimum cardinality. The vertices $\{v_{3a}, u_{1i}, u_{2i}\}$ are equitable isolates — they must be included in every equitable restrained dominating set. So,

$$|D'| \ge b - a.$$

Let $I = \{v_{3a}, u_{1i}, u_{2i} | 1 \le i \le (b - a - 1)/2\}$. Then $D' \setminus I$ is an equitable restrained dominating set of $G \setminus I$. Hence,

$$\begin{aligned} \gamma_r^e(G\ I) &\le |D' \setminus I| \\ a &\le |D'| - (b - a) \\ a &\le |D'| - b + a \\ b &\le |D'| \\ b &\le \gamma_r^e(G) \end{aligned}$$

Thus, $\gamma_r^e(G) = b$. □

Corollary 2.1. The difference $\gamma_r^e - \gamma_r$ can be made arbitrarily large.

Theorem 2.6. Let G and H be graphs. If a subset D of $V(G + H)$ is an equitable restrained dominating set then one of the following holds:

1. $D \subseteq V(G)$ and D is an equitable restrained dominating set in G.
2. $D \subseteq V(H)$ and D is an equitable restrained dominating set in H.
3. $D \cap V(G) \ne \emptyset$ and $D \cap V(H) \ne \emptyset$.

Proof. Suppose D is an equitable restrained dominating set in $G + H$. Consider the following cases:

Case 1: $D \cap V(H) = \emptyset$. Then $D \subseteq V(G)$. Let $x \in V(G) \setminus D$. Since D is an equitable restrained dominating set in $G + H$, there exists $u \in D$ such that $ux \in E(G + H)$ and $|d_{G+H}(u) - d_{G+H}(x)| \le 1$, and $V(G + H) \setminus D$ contains no isolated vertices. As $D \cap V(H) = \emptyset$, it follows that $u \in V(G)$ and $ux \in E(G)$. Also, $V(G + H) \setminus D = (V(G) \setminus D) \cup V(H)$ has no isolated vertices. Hence, $D \subseteq V(G)$ and D is an equitable restrained dominating set in G.

Case 2: $D \cap V(G) = \emptyset$. Then $D \subseteq V(H)$. Let $y \in V(H) \setminus D$. Since D is an equitable restrained dominating set in $G + H$, there exists $v \in D$ such that $vy \in E(G + H)$ and $|d_{G+H}(v) - d_{G+H}(y)| \le 1$, and $V(G + H) \setminus D$ contains no isolated vertices. As $D \cap$

$V(G) = \emptyset$, it follows that $v \in V(H)$ and $vy \in E(H)$. Also, $V(G + H) \setminus D = (V(H) \setminus D) \cup V(G)$ has no isolated vertices. Hence, $D \subseteq V(H)$ and D is an equitable restrained dominating set in H.

Case 3: If neither Case 1 nor Case 2 holds, then $D \cap V(G) \neq \emptyset$ and $D \cap V(H) \neq \emptyset$. □

Theorem 2.7. Let G and H be graphs of order n and m, respectively. Then $\gamma_r^e(G + H) = 1$ if and only if either $D_1 = \{u\}$ is a γ_r^e −set of G and $d_H(v) \geq m - 2$ for all $v \in V(H)$, or $D_2 = \{x\}$ is a γ_r^e −set of H and $d_G(y) \geq n - 2$ for all $y \in V(G)$.

Proof. Assume that $\gamma_r^e(G + H) = 1$. Then clearly $\Delta(G + H) = m + n - 1$. Suppose $D_1 = \{u\} \subseteq V(G)$ is a γ_r^e −set of $G + H$. This means that for every $v \in V(G + H)$ with $v \neq u$, we have:

$$|d_{G+H}(u) - d_{G+H}(v)| \leq 1$$

$$|(n + m - 1) - d_{G+H}(v)| \leq 1$$

This implies $d_{G+H}(v) \geq n + m - 2$. Therefore, for all $v \in V(H)$, we obtain $d_H(v) \geq m - 2$. Similarly, if $D_2 = \{x\} \subseteq V(H)$ is a γ_r^e −set of $G + H$, then it follows that $d_G(y) \geq n - 2$ for all $y \in V(G)$.

Conversely, suppose $D_1 = \{u\}$ is a γ_r^e −set of G and $d_H(v) \geq m - 2$ for all $v \in V(H)$. Since $D_1 = \{u\}$ is a γ_r^e −set, by definition we have $\Delta(G) = n - 1$. This means that in $G + H$, $d_{G+H}(u) = n - 1 + m$. Also, $d_H(v) \geq m - 2$ for all $v \in V(H)$ implies $d_{G+H}(v) \geq m - 2 + n$ in $G + H$. Hence,

$|d_{G+H}(u) - d_{G+H}(v)| \leq |(n - 1 + m) - (m - 2 + n)| = 1.$

Therefore, $D_1 = \{u\}$ is a γ_r^e −set of $G + H$, implying $\gamma_r^e(G + H) = 1$. The same argument applies symmetrically if $D_2 = \{x\}$ is a γ_r^e −set of H. □

Corollary 2.2. Let G and H be graphs of order n and m, respectively. If $\Delta(G) = n - 1$, $\delta(G) \geq n - 2$, and $d_H(u) \geq m - 2$ for all $u \in V(H)$, then $\gamma_r^e(G + H) = 1$.

Theorem 2.8. Let D_1 and D_2 be the minimal nontrivial γ_r^e −sets of G and H, respectively, with $|D_1| \neq 1$ and $|D_2| \neq 1$. Then $D_1 \cup D_2$ is not always a γ_r^e −set of $G + H$, but it is a γ_r −set of $G + H$.

Proof. Since D_1 and D_2 are γ_r^e −sets of G and H, each outside vertex is dominated and has a neighbor outside. In $G + H$, any $w \in V(G + H) \setminus (D_1 \cup D_2)$ inherits this property: if $w \in V(G) \setminus D_1$ it connects to D_1 and to $V(G) \setminus D_1$, and similarly for $w \in V(H) \setminus D_2$. Thus $D_1 \cup D_2$ is a restrained dominating set of $G + H$.

However, vertices from D_1 and D_2 acquire additional adjacency across the join, so their degrees may differ by more than one. Therefore, $D_1 \cup D_2$ need not be a γ_r^e −set of $G + H$, but it is always a γ_r −set. □

3. Conclusion

We established bounds and characterizations for $\gamma_r^e(G)$, analyzed its gap with related parameters, and studied its behavior under joins. We further observed that no tree T satisfies $\gamma_r^e(T) = n - 1$, and proposed a conjecture characterizing all trees for which $\gamma_r^e(T) = n - 2$ in terms of the operation Ω applied to $S(2,1)$. These findings extend domination theory by linking equitable and restrained conditions, offering scope for further study in structured graphs and algorithms.

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