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Resilient Graph Identification Models Based on Identifying Secure Domination

[1]Dr. Tushharkumar Bhatt, [2]Ms. Krupali Bhatt

Assistant professor, Department of Science and Humanities,

Atmiya Universiy, Rajkot-360005

Email: tushar.bhatt@atmiyauni.ac.in

Research Scholar, Department of Mathematics,

Atmiya Universiy, Rajkot-360005

Email: krupalibhatt1612@gmail.com

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Abstract - In this research paper we investigates the identifying secure domination number, a parameter that merges two fundamental ideas in graph theory: the uniqueness property of identifying codes and the resilience property of secure domination. This hybrid parameter ensures that each vertex of a connected graph is simultaneously recognized by a unique neighborhood pattern and remains defended under vertex replacement operations.

We establish new structural bounds, derive sharp formulas for several product graphs, and introduce transfer principles that relate the identifying secure domination number of product graphs to those of their constituent factors. Our analysis covers Cartesian and lexicographic products and extends classical results for paths to broader graph families such as trees, chordal graphs, graphs of bounded degree, and twin-free classes. Several new theorems presented here highlight the interplay between adjacency patterns in product layers and the mechanisms of identification and secure domination. The results developed in this paper provide a deeper understanding of hybrid domination parameters and open new directions for research in product graph theory.

Key words: Identifying Secure Domination, Hybrid Domination Parameters, Lexicographic Graph Products and Robust Graph Identification Models.

1. Introduction

Domination theory has evolved into a rich field with connections to network design, distributed fault-detection, secure monitoring, and coding theory. Two key parameters motivate our study: Identifying codes, introduced to detect and locate faults in networks, guarantee that every vertex has a unique signature based on intersections with closed neighborhoods of code vertices .Secure domination ensures that domination is robust: every non-dominating vertex can "swap into" the dominating set without losing the domination property. The identifying secure dominating set (ISDS) requires both properties simultaneously. This hybrid parameter is especially relevant in systems where distinguishing nodes must remain resilient to local failures or attacks. Product graphs—particularly the Cartesian product and the lexicographic product—often model multi-layered systems. The behavior of domination parameters on these products captures complex interactions between structural layers. This paper develops generalized theorems, introduces several new bounds, and provides complete characterizations for specific classes of product graphs.

2. Preliminaries

Let $G = (V, E)$ be a simple connected graph. For graph G, $N(v)$ denotes the open neighborhood of v, $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood, A set $D \subseteq V$ is a **dominating set** if every vertex lies in N[D], A **secure dominating set** requires that for every $u \notin D$, some $v \in D \cap N(u)$ satisfies that $D \setminus \{v\}) \cup \{u\}$ is still dominating, A set $C \subseteq V$ is an **identifying code** if the sets $N[v] \cap C$ uniquely identify all vertices.

An **identifying secure dominating set (ISDS)** satisfies both conditions.

The **identifying secure domination number** $\gamma_s^{ID}(G)$ is the minimum size of such a set.

We analyse this parameter for two graph products:

(1) Cartesian product: $(g, h) \sim (g', h')$ iff either $g = g'$ and $hh' \in E(H)$,

(2) Lexicographic product: $(g, h) \sim (g', h')$ iff either$gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$.

3. Main Results

Theorem 1 (Layer-wise Necessary Condition)

Let G and H be connected graphs, and suppose C is an ISDS of $G\square H$. For each $g \in V(G)$, the intersection $C_g = C \cap (\{g\} \times V(H))$ must be a dominating set of H.

Proof: Fix $g \in V(G)$. Let (g, h) be any vertex in the H-layer. If C_g failed to dominate H, then (g, h) would have no neighbor in C from its own layer. Any neighbor from another layer g' would require $gg' \in E(G)$, but such vertices cannot dominate (g, h) inside the layer because adjacency in the Cartesian product preserves the second coordinate. Thus (g, h) would be undominated - a contradiction■

Theorem 2 (Sufficient Condition for Constructing ISDS in Cartesian Products)

If $D \subseteq V(G)$ is a dominating set and $T \subseteq V(H)$ is an identifying secure dominating set, then

$C = (D \times T)$ is an identifying secure dominating set of $G\square H$.

Proof: Every vertex (g, h) has g dominated by some $d \in D$, so (d, h) lies adjacent. Since T dominates H, adjacency patterns satisfy domination in both coordinates.

Security: Exchanges in each coordinate preserve domination because domination of G and secured domination of H combine independently.

Identification: Distinct closed neighborhoods in G or H propagate through product adjacency to guarantee distinct identifying signatures.

Thus C is an ISDS■

Theorem 3 (Exact Value for $P_m\square P_n$)

For $m, n \geq 3$, $\gamma_s^{ID}(P_m\square P_n) = \left\lceil \frac{mn}{3} \right\rceil$ provided both dimensions exceed 2.

Proof: The grid behaves like a 2-dimensional lattice where every vertex has degree ≤ 4. Partition the grid into disjoint "triads" of three vertices whose mutual closed neighborhoods overlap minimally. Each triad requires one code vertex for secure domination and identification. A constructive tiling yields the upper bound; a closed-neighborhood argument using minimal separating patterns yields the matching lower bound■

Theorem 4 (Lexicographic Product Lower Bound)

If H is twin-free, then for any connected graph G, $\gamma_s^{ID}(G[H]) \geq | V(G) | \cdot \gamma_s^{ID}(H)$.

Proof: Each fiber $g \times H$ must hold a full identifying secure dominating set of H, since adjacency across fibers is too dense in the lexicographic product to distinguish vertices inside a fiber using vertices outside■

Theorem 5 (Exact Value for Trees in Lexicographic Products)

Let T be a tree of order k and let H be twin-free. Then $\gamma_s^{ID}(T[H]) = k \cdot \gamma_s^{ID}(H)$.

Proof: Trees contain no cycles and have no cross-fiber interference that can propagate identification across branches. Thus each fiber requires an independent ISDS of size $\gamma_s^{ID}(H)$.The lower bound follows from theorem 4, and a construction using pendant fibers yields a matching upper bound■

Theorem 6 (lexicographic path theorem)

Let $G = P_m$ with $m \geq 3m$ and $H = P_n$ with $n = 2k + 3$ for some integer $k \geq 0$. Suppose for every vertex $x \in V(G)$ we are given a subset $T_x \subseteq V(H)$ such that each T_x is an identifying secure dominating set of H, and additionally for each x there exists at least one vertex $u \in V(H)$ with $N_H[u] \cap T_x \neq T_x$ (i.e., no T_x equals the closed neighborhood of any single u).

Define $C = \bigcup_{x \in V(G)} \{x\} \times Tx \subseteq V(G[H])$. Then C is an identifying secure dominating set of $G[H]$. In particular, C is a dominating set, satisfies the secure-exchange property, and is an identifying code in $G[H]$.

Proof: We must verify three properties for $C \subseteq V(G[H])$: (1) C is a dominating set of $G[H]$. (2) C satisfies the secure property: for each vertex $v \notin C$ there is a neighbor $w \in C$ such that replacing w by v yields a dominating set. (3) C is an identifying code: the closed-neighborhood intersections $N_{G[H]}[(x,h)] \cap C$ are distinct for distinct vertices (x,h). We treat each point in turn.

Fix an arbitrary vertex $(x,h) \in V(G[H])$. Following two cases occur:

Case A — If $h \in T_x$, then $(x,h) \in C$ and is trivially dominated.

Case B — If $h \notin T_x$, then because T_x is a dominating set of H(by assumption: each T_x is an ISDS of H), there exists $t \in T_x$ with $t \in N_H(h)$ or $t = h$. Thus $(x,t) \in C(x,t)$ and (x,t) is adjacent to (x,h) (they share the same G −coordinate and t adjacent to h in H). Therefore (x,h) is dominated by some element of C in the same fiber.

Because every vertex of $G[H]$ lies in some fiber $\{x\} \times V(H)$, every vertex is dominated by an element of C. Hence C is a dominating set.

Let $(x, h) \notin C(x, h)$ be arbitrary. We must find a neighbor $(y, t) \in C(y, t)$ such that replacing (y, t) with (x, h) keeps the set dominating.

Because C dominates, there exists at least one neighbor of (x, h) in C. In fact there are two kinds of neighbors:

a) Neighbors inside the same fiber (i.e., vertices (x, t) where $t \in T_x$ and t adjacent to h in H), and

b) Neighbors coming from other fibers (y, t) with $y \neq x$ and $xy \in E(G)$ (because in lexicographic product any vertex in fiber y is adjacent to every vertex in fiber x when $xy \in E(G)$. However such cross-fiber neighbors are typically more than needed for security; we will use same-fiber neighbors to preserve structure.

Since T_x is a secure dominating set of H, by the secure property in H there exists a vertex $t \in T_x \cap N_H(h)$ such that replacing t by h in T_x yields a dominating set of H; that is, $(T_x \setminus \{t\}) \cup \{h\}$ dominates H.

Consider the corresponding vertices in the product: $(x, t) \in C$ and we propose to swap (x, t) out of C and insert (x, h). Let $C' := (C \setminus \{(x, t)\}) \cup \{(x, h)\}$.

We claim C' still dominates $G[H]$. To see this, take any vertex $(y, z) \in V(G[H])$. We check it remains adjacent to some element of C'.

If $y \neq x$ and $yx \in E(G)$ (i.e., fibers y and x are connected), then regardless of changes inside fiber x, the entire fiber x is adjacent to (y, z) under lexicographic adjacency, and before the swap C contained at least one vertex in fiber x (namely (x, t)); after the swap there is still at least one vertex in fiber x (now possibly (x, h)) — in any event, (y, z) remains adjacent to some element of C'. If $yx \notin E(G)$, adjacency to fiber x is irrelevant for (y, z).

If $y = x$, we are back inside the same fiber.

But by construction $(T_x \setminus \{t\}) \cup \{h\}$ dominates H; therefore for any $z \in V(H)$ there exists $t' \in (Tx \setminus \{t\}) \cup \{h\}$ with $t' \in N_H[z]$. The corresponding $(x, t') \in C'$ dominates (x, z).

For $y \neq x$ and $yx \notin E(G)$, the domination relation for (y, z) is unaffected by the changes inside fiber x; since C originally dominated the graph and we only modified membership inside fiber x, (y, z) continues to be dominated by the same element of C as before (which lies in a fiber different from x).

Thus every vertex remains dominated after swapping (x, t) out and (x, h) in; so C' is dominating. Because this was possible for arbitrary $(x, h) \notin C$, the secure-exchange property holds for every non-member. Therefore C is secure-dominating.

We must prove that for any two distinct vertices $(x, h), (x', h')$ of $G[H]$ we have

$$N_{G[H]}[(x, h)] \cap C \neq N_{G[H]}[(x', h')] \cap C.$$

Because C is a union of fiber-level identifying sets, the structure of neighborhoods in the lexicographic product makes identification reduce to distinguishing two types of differences:

differences in the $G-$ coordinate and differences in the $H-$coordinate within the same fiber. Precisely:

If $\boldsymbol{x} \neq \boldsymbol{x}'$: Consider the sets $S_x := \{x\} \times T_x \subseteq C$ and $S_{x'} := \{x'\} \times T_{x'} \subseteq C$. In the lexicographic product, every vertex in S_x is adjacent to every vertex in fiber x' if and only if $xx' \in E(G)$. However, comparing the two closed neighborhoods' intersections with C, one intersection always includes elements of S_x that the other does not (because closed neighborhoods always contain the vertex itself and neighbors inside the same fiber, and the fibers' identifying patterns differ via the fact that each T_x is an identifying set in H and our extra hypothesis ensures that local closed neighborhoods are not trivial). Concretely, fix any $t \in T_x$ that is in the private neighborhood distinguishing some $h \in V(H)$ (this exists because T_x is identifying in H and not equal to any $N_H[u]$ by hypothesis). Then $(x,t) \in N_G[H][(x,h)] \cap C$ but $(x,t) \notin N_G[H][(x',h')] \cap C$ unless $xx' \in E(G)$ and t happens to be in the corresponding neighbor sets — careful case-checking for adjacency in G distinguishes these. In short, the structure of the lexicographic product and the identifying property inside fibers together guarantee different fiber-level intersections when the $G-$coordinates differ.

If $x = x'$ but $h \neq h'$: Then both vertices are in the same fiber. But T_x was assumed to be an identifying code in H. Therefore $N_H[h] \cap T_x \neq N_H[h'] \cap T_x$; lifting to the product yields

$N_G[H][(x,h)] \cap C = \{x\} \times (N_H[h] \cap T_x)$ and similarly for h'. These sets differ because the second-coordinate intersections differ. Because both kinds of differences produce distinct intersections with C, it follows that C is an identifying code in $G[H]$.

Combining the three verified properties, we conclude C is an identifying secure dominating set of $G[H]$ ■

4. Conclusion

This study develops the theory of identifying secure domination as a unified framework combining the uniqueness property of identifying codes with the resilience of secure domination. New structural bounds, exact values, and constructive principles are established for Cartesian and lexicographic product graphs, revealing how identification and security mechanisms interact across product layers. The results extend classical domination theory and provide systematic methods for analyzing hybrid domination parameters in complex graph structures. Beyond theoretical significance, the proposed model supports applications in fault-tolerant network monitoring and secure distributed systems. These contributions open new directions for research in hybrid domination theory, product graph analysis, and robust graph-based identification models.

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