Panamerican Mathematical Journal
ISSN: 1064-9735
Vol 31 No. 1 (2021)

Dark Energy Models in Relativistic Cosmology Framewor

Dr. Amit Prakash

Katra Baradri,Bhagwan Bazar, Chapra

E-mail - a.amitprakash@gmail.com

Article History:
Received- *25-05-2021*
Revised- *04 -07- 2021*
Accepted- *16-07- 2021*

Abstract

Late-time accelerated universe expansion is one of the most significant problems of the contemporary theoretical physics. In this paper, we have presented a detailed mathematical review of the dark energy models in the relativistic cosmology. Starting with the Einstein Field Equations and FriedmannLemaitreRobertsonWalker (FLRW) metric, we have obtained the basic kinematic conditions of cosmic acceleration. Although the conventional model of the standard percent Λ CDM has great phenomenological consistency with observational data, its physical interpretation suffers seriously due to theoretical obstacles namely the cosmological constant problem and the cosmic coincidence problem. This review therefore critically assesses other mathematical structures. Models of dynamical dark energy that are time-varying scalar field models such as quintessence and phantom energy, and the generalized Chaplygin Gas fluid mechanics are all analyzed. Moreover, we discuss geometric adjustments to General Relativity, namely, f(R) gravity, which mathematically resemble the dark energy without the need to introduce exotic macroscopic fluids. Lastly, we address the importance of observational constraints (Type Ia Supernovae, Baryon Acoustic Oscillations, and the Cosmic Microwave Background) to the above theoretical models, with a focus on the implication of the current tension in Hubble. As a result of this synthesis lies the need to have accuracy kinematic measurements of the next generation cosmological surveys in order to conclusively determine that a static vacuum energy nor a dynamically evolving gravitational sector.

Keywords Relativistic Cosmology; Dark Energy; Λ CDM Model; Modified Gravity; Scalar Fields; Cosmic Acceleration; Hubble Tension; General Relativity.

1. Introduction

In the late twentieth century, the modern cosmology changed and there was a paradigm shift after the finding that the universe is not only expanding but also accelerating. This discovery, which had been determined independently by using high-redshift Type Ia supernovae by the Supernova Search Team (Riess et al., 1998) and the Supernova Cosmology Project (Perlmutter et al., 1999) essentially invalidated the previously agreed assumption that the universe was matter-dominated and decelerating. In order to explain this rapid growth in the context of the General Relativity, there must be a repulsive, gravitationally anti-attractive element dominating the existing cosmological energy balance. This mysterious element is universally known as the Dark Energy.

At present, the conventional cosmological model is the Lambda Cold Dark Matter (Λ CDM) model, which has been representing this acceleration as a cosmological constant (Λ) an addition that was initially put forward by Albert Einstein. This mathematical understanding of dark energy regards the dark energy as the natural vacuum energy of space itself. It is marked by an equation of state parameter in which the pressure (p) is dependent on energy density (ρ),

$$w = \frac{p}{\rho c^2} = -1$$

Although the Λ CDM model of the paramount success in the fitting of the modern observational data, even with the precision measurements of the Cosmic Microwave Background and baryon acoustic oscillations (Planck Collaboration, 2020), the model is fraught with serious theoretical difficulties. The most salient one is the problem of cosmological constant. Quantum field theory anticipates the existence of a significantly 120 orders of magnitude higher vacuum energy density than the observed amount of Λ necessary to cause the current acceleration of the universe (Carroll, 2001; Padmanabhan, 2003). Moreover, the model is plagued by the problem of coincidence where it is doubted that both the dark energy density as well as the dark matter density coincidentally coincide with the magnitude of each other present at this particular epoch in cosmic history (Peebles and Ratra, 2003).

Out of these theoretical discrepancies, endless space of other mathematical models has been developed (Bamba et al., 2012). These alternatives may largely be classified into two broad cosmological groups:

1. Dynamical Dark Energy: These theories incorporate a time-changing scalar field, like quintessence, phantom energy, in which the equation of state parameter w is no longer strictly equal to -1 (Zlatev, Wang and Steinhardt, 1999; Caldwell, 2002). The other outstanding fluid model is the Chaplygin gas that tries to make the dark matter and dark energy one identical fluid model (Bento, Bertolami and Sen, 2002).
2. Modified Gravity: This type does not involve the introduction of an exotic new energetic fluid. Rather it modifies the geometrical sector of the Einstein Field Equations on the large cosmological scales. The best example is f (R) gravity where the Ricci scalar R is substituted with a function f (R) of arbitrary value in the gravitational action (Sotiriou and Faraoni, 2010; De Felice and Tsujikawa, 2010).

The paper presents a stringent review of these dark energy models in the relativistic cosmology framework mathematically. This study isolates and resolves the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric and Friedmann equations that led to the discovery of the kinematic conditions of cosmic acceleration in a systematic manner. The standard percent Λ CDM model as well as dynamical scalar field models (Copeland, Sami and Tsujikawa, 2006) and other gravitational models will be mathematically examined and the viability of each model contrasted in terms of mathematical versatility and observational limitations.

2. The Relativistic Framework: General Relativity and Cosmology

In order to mathematically explain the phenomenon of the dark energy and accelerated universe expansion, the frameworks of relativistic cosmology have to be laid down. This structure is regulated by General Theory of Relativity by Albert Einstein that implies the gravitational dynamics of the universe is dictated by the interaction between the geometry of spacetime and the matter-energy content that lives in it (Clifton et al., 2012).

2.1 The Einstein Field Equations

The modern cosmology theory starts with the Einstein Field Equations (EFE). These non-linear partial differential equations provide the relationship between the local energy and the momentum of whatever is present in space and time and the curvature. The EFE when the cosmological constant term is added can be written in the form of a tensors as:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In this expression:

- $R_{\mu\nu}$ is the Ricci curvature tensor, which describes how the volume of a geodesic ball changes in curved spacetime.
- R is the Ricci scalar (the trace of the Ricci tensor).
- $g_{\mu\nu}$ is the metric tensor, encoding the geometry of spacetime.
- Λ represents the cosmological constant.
- G is Newton's gravitational constant, and c is the speed of light in a vacuum.
- $T_{\mu\nu}$is the energy-momentum tensor describing the physical cosmic fluid.

2.2 The Cosmological Principle and the FLRW Metric

Cosmologists use the Cosmological Principle in order to resolve the EFE on a cosmic scale. According to this principle, the universe is homogeneous (appears the same place to place) and isotropic (appears the same direction to direction) on sufficiently large spatial scales (which is generally greater than 100 Megaparsecs).

The application of these symmetries has a drastic effect of limiting the possible geometries of space-time. A solution to the EFE that is homogeneous and isotropic and unique, the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is unique (Peebles and Ratra, 2003). In spherical comoving coordinates (t, r, θ, ϕ), the FLRW line element is given by:

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + sin^2\theta d\phi^2) \right]$$

In this case, t represents the cosmic time of a comoving observer and a(t) represents the cosmic scale factor which is a very important dimensionless time-dependent function that quantifies the relative expansion (or contraction) of the universe. The parameter k is the global spatial curvature of the universe where k is discrete: k = +1 (closed, spherical geometry), k = 0 (flat, Euclidean geometry) and k = -1 (open, hyperbolic geometry).

2.3 The Energy-Momentum Tensor

In order to assess the right-hand side of the EFE, the matter and energy content of the universe is described as a perfect fluid, which is completely characterised by the proper rest-mass energy density characterized by the percent rh(t) and its isotropic pressure characterised by p(t). The energy-momentum of a perfect fluid is:

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + p g_{\mu\nu}$$

where u_μ is the four-velocity of the cosmic fluid. To comoving observers, the spatial components of the fluid velocity are also zero, that is, the fluid is in rest relative to the expanding coordinate grid.

2.4 The Friedmann Equations and the Acceleration Condition

Replacing the FLRW parameter and the perfect fluid energy-momentum tensors by their original expressions into the Einstein Field Equations we now get the basic equations of cosmological dynamics: the Friedmann Equations.

The 00 (time-time) equation of the EFE gives the First Friedmann Equation, also known as the energy constraint equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}$$

in which $H = \dot{a}$ is the Hubble parameter, the fractional rate of expansion of the universe, and dot a is the derivative of scale factor and t.

The components of the space, iii, give the Second Friedmann Equation, or Raychaudhuri equation or acceleration equation:

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

This second equation is critical for understanding dark energy. Let us temporarily assume a universe without a cosmological constant ($\Lambda = 0$). In this scenario, the acceleration of the scale factor, ddot a, depends entirely on the term -$\left(\rho + \frac{3p}{c^2}\right)$. Because standard ordinary matter (baryons) and dark matter are effectively pressureless ($p \approx 0$), and radiation has a positive pressure ($p = \rho \frac{c^2}{3}$), the term ($\rho + \frac{3p}{c^2}$) is always strictly positive for conventional cosmic components. Therefore, gravity acts as a purely attractive force, resulting in a decelerating universe ($\ddot{a} < 0$).

To mathematically explain the current accelerating universe ($\ddot{a} > 0$) we need to add a large, negative pressure fluid component to the universe. In particular, the kinematic requirement on the dark energy is:

$$p < \frac{-\rho c^2}{3}$$

Or, in terms of the equation of state parameter $w = \frac{p}{\rho c^2}$:

$$w < \frac{-1}{3}$$

This is a fundamental inequality which must be either satisfied or avoided by any cosmological model that tries to explain the dark energy by any of the following methods: either by a constant of the energy of a vacuum, or by a dynamical scalar field, or by geometric remedies to gravity (Copeland, Sami and Tsujikawa, 2006).

3. The Standard Model: The Cosmological Constant (Λ)

The most common and easiest mathematical explanation of the accelerated expansion of the universe is the addition of a cosmological constant, representing percent Λ, to the Einstein Field Equations. This is known as the standard model of cosmology when it is combined with Cold Dark Matter (CDM), and is also better known as the Λ CDM model. The physical interpretation of the phenomenology of the percentage of Λ is successful but the explanation poses immense theoretical problems (Carroll, 2001).

3.1 Mathematical Nature of the Cosmological Constant

The cosmological constant can be shifted to the right-hand side of the Einstein Field Equations, originally to allow Einstein to mathematically allow a static universe (which Einstein himself abandoned). In this expression, it is not a geometrical curvature term, but as an effective source of energy and momentum that is evenly spread over the vacuum of spacetime (Padmanabhan, 2003).

We can define the effective energy density ρ_Λ and effective pressure p_Λ associated with the cosmological constant as:

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

$$p_\Lambda = \frac{-\Lambda c^4}{8\pi G}$$

By substituting the first equation into the second, we uncover the crucial relationship between the pressure and energy density of the cosmological constant:

$$p_\Lambda = -\rho_\Lambda c^2$$

Applying the general equation of state parameter $w = \frac{p}{\rho c^2}$, we find that for the cosmological constant:

$$w_\Lambda = -1$$

This is a strictly constant equation of state that satisfies perfectly the acceleration equation $w < \frac{-1}{3}$in the preceding section. Since ρ_Λ is entirely unaffected by the expansion of the universe (as opposed to matter which is diluted by a^{-3} or radiation diluted by a^{-4})), the vacuum energy is not diluted. The amount of constant matter density in the universe decreases as the cosmic scale a(t) grows, causing the diluting matter density in the universe to eventually be overtaken by the constant energy density of the vacuum, marking the epoch of accelerated expansion (Copeland, Sami and Tsujikawa, 2006).

3.2 The Λ CDM Model and Cosmic Composition

In order to measure the present universe composition, the cosmologists make use of the dimensionless density parameters, which are referred to as the percentage of Ω. The density of any cosmic component i i.e. the density ratio of the cosmic component i and the critical density i.e. the density needed to satisfy a spatially flat universe defined as the ratio of energy density of a cosmic component ρ_i to the critical density i.e. density ρ_c :

$$\Omega_i = \frac{\rho_i}{\rho_c}$$

where the critical density is mathematically defined from the First Friedmann equation as:

$$\rho_c = \frac{3H^2}{8\pi G}$$

In the Λ CDM framework, the total energy density of the universe is the sum of radiation (r), non-relativistic matter (m, including both baryonic and cold dark matter), and dark energy (Λ). The Friedmann equation can be rewritten in terms of these density parameters at the present epoch (denoted by the subscript 0):

$$\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0} = 1$$

where $\Omega_{k,0} = \frac{-kc^2}{a_0^2 H_0^2}$represents the spatial curvature density parameter. Precision cosmological observations, particularly those from the Cosmic Microwave Background radiation measured by the Planck satellite, indicate that the universe is remarkably flat, meaning $\Omega_{k,0} \approx 0$ (Planck Collaboration, 2020). Consequently, the present-day universe is dominated by dark energy and matter: $\Omega_{\Lambda,0} \approx 0.68 \wedge \Omega_{m,0} \approx 0.32$.

The dynamical evolution of the Hubble parameter H(a) in a flat Λ CDM universe is governed by the following equation:

$$H^2(a) = H_0^2\left[\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}\right]$$

This equation is the best illustration of the thermal history of the universe, which started as a radiation-dominated phase, then moved on to a matter-dominated phase and finally to the dark-energy-dominated phase.

3.3 Theoretical Shortcomings of Λ

Although the percent Λ CDM model has all the spectacular consensus with observational data of Type Ia supernovae, baryon acoustic oscillations and CMB, it is deeply flawed in theory.

1. The Cosmological Constant Problem (The Fine-Tuning Problem): Provided that the physical interpretation of the zero-point energy of the quantum field theory is the interpretation of a percentage of the Λ, there is a massive inconsistency. The theoretical value of the expected energy density of the vacuum, obtained by taking the sum of the zero-point energies of all known quantum fields in the cutoff scale of the Planck mass, is astronomically greater than the observed value, when quantum field theorists perform the calculation of this quantity. Particularly, the phenomenological prediction is about 10^{120} times greater than the theoretical prediction that astrophysics demands Λ (Carroll, 2001). This is the biggest difference between theory and observation in the history of physics that seems to demand an unexplainable and extreme fine-tuning of parameters to balance out the large quantum-vacuum contributions (Padmanabhan, 2003).

2. The Cosmic Coincidence Problem: The second significant problem is a timing problem. The evolution equation of $H^2(a)$ indicates that the matter density decreases with a as a^{-3}, and the density of the vacuum energy is a constant. During most of cosmic history, the dark energy was significantly less dominant than matter. Dark energy will be significantly stronger than matter in the remote future infinity. But, we are fortuitously in the exceedingly short, transitional cosmic epoch in which the two percentages ρ_m and ρ_Λ are equal of the same order. The percent Λ CDM model does not provide any mathematical or physical process which could result in this coincidence happening at this particular time; it simply has to be a given starting condition of the universe (Peebles and Ratra, 2003, Zlatev, Wang and Steinhardt, 1999).

Since these two monumental theoretical challenges cannot be solved using the standard model, other mathematical frames have been created by the cosmological community.

Table 1: The Cosmic Energy Inventory (Standard Λ CDM Model)

Cosmic Component	**Density Parameter Symbol**	**Approximate Present Value ($\Omega_{i,0}$)**	**Evolution with Scale Factor (a)**	**Kinematic Role**
Radiation (Photons & Neutrinos)	Ω_r	approx 10^{-5}	$\rho_r \propto a^{-4}$	Dominated the very early universe; drives deceleration.
Baryonic (Ordinary) Matter	Ω_b	approx 0.05	$\rho_b \propto a^{-3}$	Forms stars, gas, and dust; drives deceleration.
Cold Dark Matter (CDM)	Ω_c	approx 0.27	$\rho_c \propto a^{-3}$	Provides the gravitational scaffolding for galaxies.
Dark Energy (Vacuum Energy)	Ω_Λ	approx 0.68	$\rho_\Lambda =$ Constant	Drives the current epoch of accelerated expansion.
Spatial Curvature	Ω_k	approx 0	$\rho_k \propto a^{-2}$	Indicates the universe is geometrically flat.

4. Dynamical Dark Energy Models

Since the cosmological constant (Λ) has such severe issues of theoretical fine-tuning and coincidence, a natural extension of the standard model is to assume that dark energy is not a fixed vacuum energy, but a dynamical, time-varying part. In such models, the parameter w of the equation of state is not defined to be equal to -1 but is a function of the scale factor a(t) or the cosmic time. This dynamical evolution provides a way out of the coincidence problem in that the present epoch of acceleration may be described as a natural extension of the dynamical history of the universe and not a finely-tuned initial condition (Copeland, Sami and Tsujikawa, 2006).

4.1 Quintessence

Quintessence is the most notable category of dynamical models of the dark energy. Quintessence adds a canonical scalar field, denoted by ϕ, which has gravity interacting in the weakest possible way and rolling slowly down a potential well $V(\phi)$ (Caldwell, Dave and Steinhardt, 1998; Zlatev, Wang and Steinhardt, 1999).

The dynamics of a universe with a quintessence field, in addition to the usual matter S_m is of the form:

$$S = \int d^4 x\sqrt{-g}\left[\frac{Rc^4}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_m$$

By varying this action with respect to the metric tensor in a spatially flat FLRW background, we can derive the energy density ρ_ϕ and pressure p_ϕ associated with the homogeneous scalar field (where spatial derivatives $\partial_i\phi = 0$):

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

In this case $\frac{1}{2}\dot{\phi}^2$ is the kinetic energy of the field, and $V(\phi)$ is its potential energy. The quintessence equation of state parameter is consequently dynamically defined as:

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$

To have quintessence cause acceleration in the cosmos, we remember the kinematic condition $w < -1$ over 3. This, looking at the equation of w_ϕ, can only be mathematically fulfilled when there is dominance of the potential energy over kinetic energy that is, $\dot{\phi}^2 < V(\phi)$. This slow-roll state implies that the field has to change very slowly with cosmic time.

The quintessence field itself is dynamically evolved by the Klein-Gordon equation obtained by varying the action by ϕ, or by wandering the equation of conservation of energy-momentum. $\nabla_\mu T^{\mu\nu} = 0$:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

The 3 H dot ϕ is referred to as Hubble friction. Expansion rate H gradually dilutes the dynamics of the scalar field, and automatically the slow-roll conditions required at late times to dominate the dark energy are satisfied (Zlatev, Wang and Steinhardt, 1999).

4.2 Phantom Dark Energy

Although quintessence confines the equation of state to the range $-1 < w_\phi < 1$, there is also no clear observational evidence that $w < -1$ is excluded (Planck Collaboration, 2020; Abbott et al.,). $w < -1$ defines a scalar field that is called a phantom energy (Caldwell, 2002).

In order to obtain $w < -1$ mathematically, the kinetic energy of the scalar field needs to be negative, which is against the null energy condition of General Relativity. Energy density and pressure of a phantom field is:

$$\rho_\phi = \frac{-1}{2}\dot{\phi}^2 + V(\phi)$$

$$p_\phi = \frac{-1}{2}\dot{\phi}^2 - V(\phi)$$

The equation of state becomes:

$$w_\phi = \frac{\frac{-1}{2}\dot{\phi}^2 - V(\phi)}{\frac{-1}{2}\dot{\phi}^2 + V(\phi)} < -1$$

The phantom dark energy has disastrous cosmological implications. Since $w < -1$ the energy density of the phantom field increases in fact with the expansion of the universe. The rate of expansion grows violently rapid as $a(t) \to \infty$, ultimately coming to rest all the fundamental forces. The mathematical extrapolation of this model indicates that the scale factor will have an infinite value in some finite proper time, ripping up galaxies, stellar systems, atoms and even the spacetime in a phenomenon termed the Big Rip (Caldwell, 2002).

4.3 The Chaplygin Gas Model

Another method of dynamical scalar fields is to alter the basic perfect fluid as used in the cosmological model. The Chaplygin gas model postulates an exotic fluid that is meant to come up with a unified dark matter and dark energy into one dark sector (Kamenshchik, Moschella and Pasquier, 2001). The equations of state The original Chaplygin gas is determined by the mathematically exotic equation of state:

$$p = \frac{-A}{\rho}$$

Assuming that A is a positive constant. This was further generalized into the Generalized Chaplygin Gas (GCG) model (Bento, Bertolami and Sen, 2002), introducing a free parameter α:

$$p = \frac{-A}{\rho^\alpha}$$

where $0 \leq \alpha \leq 1$.

We replace the equation of state of GCG in the fluid continuity equation (which is obtained by the First Law of Thermodynamics in an expanding universe):

$$\dot{\rho} + 3H(\rho + p) = 0$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho - \frac{A}{\rho^{\alpha}}\right) = 0$$

This differential equation is integrable precisely to determine the energy density of the form of the scale factor a:

$$\rho(a) = \left[A + Ba^{-3(1+\alpha)}\right]^{\frac{1}{1+\alpha}}$$

B being a constant of integration. The mathematical genius of the GCG model is the fact that it has asymptotic limits:

1. **At early times** (when a is very small), the term B $a^{-3(1+\alpha)}$ heavily dominates. The density approximates to $\rho \approx B^{\frac{1}{1+\alpha}}a^{-3}$. This mathematically perfectly reproduces the a^{-3} law of dilution of pressureless dust, that is, the Chaplygin gas behaves in the same way as the Cold Dark Matter in the early universe and can form proper structures (Bento, Bertolami and Sen, 2002).
2. **At late times** (when a is very large), the $a^{-3(1+\alpha)}$ term approaches zero. The density asymptotically approaches a constant value: $\rho \approx A^{\frac{1}{1+\alpha}}$. This regime causes the fluid to behave as a cosmological constant with $w = -1$ which accelerates in the late time.

Therefore, the Generalized Chaplygin gas is a beautiful mathematical way to unify, to have only

one dark ingredient to describe not only the aggregation of galaxies, but also the accelerating cosmology.

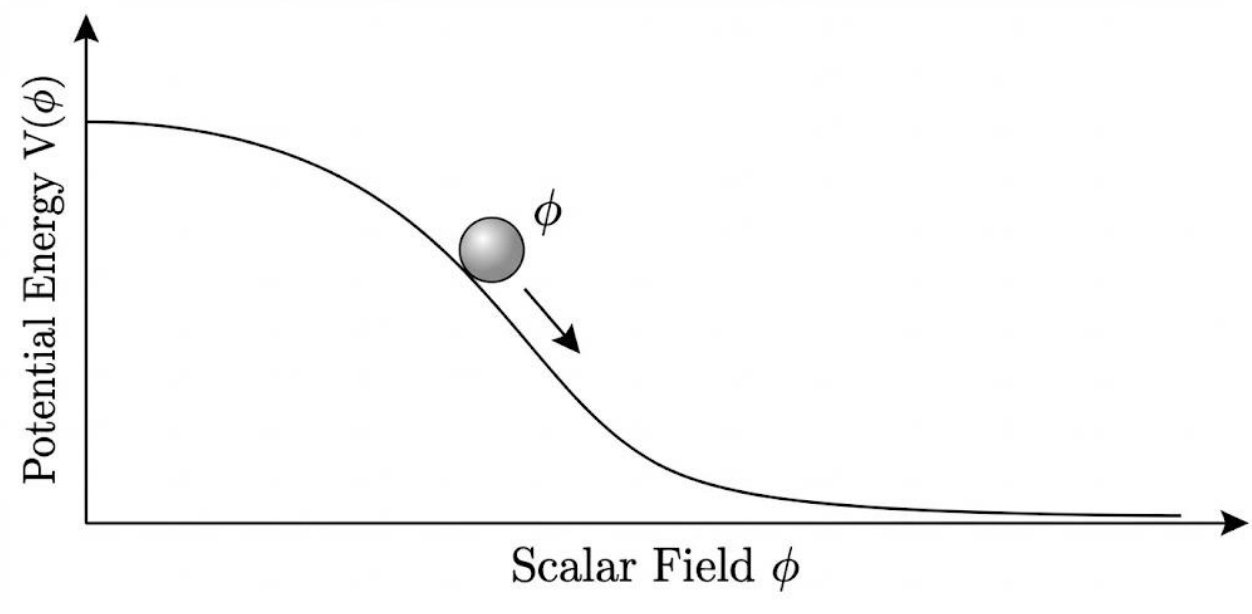

Figure 2: Quintessence Scalar Field Potential

5. Alternative Approach: Modified Gravity

All the models described above, the cosmological constant, quintessence, phantom energy, and the Chaplygin gas, have in common one similarity: the assumption that the General Relativity theory of gravity is correct at all scales, and that cosmic acceleration must be due to some mysterious, unobserved negative pressure fluid. This solution alters the energy momentum surface $T_{\mu\nu}$ on the right hand side of the Einstein Field equity.

Nonetheless, an even more radically different mathematical course of action would be to suggest that nothing like a dark energy exists at all. Rather, the late-time acceleration has been observed, which could be the initial indicator that General Relativity does not hold in the immense, cosmological scales (Clifton et al., 2012). We can obtain the accelerating universe mathematically by simply changing the geometry of the sector (the left-hand side of the field equations) that is non-spacetime-dependent (Nojiri and Odintsov, 2006).

5.1 The Standard Einstein-Hilbert Action

We have to begin with the action principle of gravity, to make changes in the strictest sense. In ordinary General Relativity, gravity dynamics are obtained by the EinsteinHilbert action. In a manifold with a metric, $g_{\mu\nu}$ and determinant g, and a matter Lagrangian L_m, the action is:

$$S_{EH} = \int d^4 x\sqrt{-g}\left[\frac{Rc^4}{16\pi G} + L_m\right]$$

The same operation, but with respect to the inverse measure $g^{\mu\nu}$ of the inverse, and the use of the principle of least action ($\delta S = 0$) , recovers exactly the conventional Einstein Field equations in the absence of a cosmological constant.

5.2 f(R) Theories of Gravity

The most widely and most basic type of gravity modification is f(R) gravity. According to this mathematical model, the standard Ricci scalar R in the Einstein-Hilbert action is substituted by a totally arbitrary and generic function of the Ricci scalar, which is denoted as f(R) (De Felice and Tsujikawa, 2010; Sotiriou and Faraoni, 2010). The action in the form of modification is:

$$S_{f(R)} = \int d^4 x\sqrt{-g}\left[\frac{f(R)c^4}{16\pi G} + L_m\right]$$

By choosing different functional forms for f(R) (for instance, $f(R) = R + \alpha R^2 \vee f(R) = R - \frac{\mu^4}{R}$), physicists are able to adapt the theory to suit certain histories of cosmological expansions.

To observe how this leads to acceleration, we have to find the modified field equations. A change in the f(R) action by rotation by the metric $g^{\mu\nu}$ gives a much more complicated system of fourth-order equations:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)F(R) = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Here, $F(R) = df(R)$ over dR is the first derivative of the function with respect to the Ricci scalar, and ∇^2 is the covariant d'Alembertian operator ($\nabla_\alpha \nabla^\alpha$). The presence of the double covariant derivatives $\nabla_\mu \nabla_\nu F(R)$ is what elevates this from a second-order to a fourth-order theory of gravity.

5.3 Geometric Dark Energy

To see the way in which f(R) gravity acts like the dark energy, it is most instructive to algebraically rewrite the modified field equations in such a way as to make them look like standard Einstein Field Equations. We may separate the standard Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ on the left hand side and extinguish all the new and higher order geometric terms to the right hand side:

$$G_{\mu\nu} = \frac{8\pi G}{c^4 F(R)} T_{\mu\nu} + T_{\mu\nu}^{(geo)}$$

in which we have determined an effective "geometric energy-momentum tensor" $T_{\mu\nu}^{(geo)}$ as:

$$T_{\mu\nu}^{(geo)} = \frac{1}{F}(R) \left[\frac{1}{2} \big(f(R) - RF(R) \big) g_{\mu\nu} + \big(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \big) F(R) \right]$$

This rearrangement of mathematical truths is very deep-hued. It shows that the changes to the gravity serve as a powerful, dynamic source of energy and momentum. The $T_{\mu\nu}^{(geo)}$ term can become large in negative pressure, even when the underlying standard matter $T_{\mu\nu}$ only contains ordinary, pressureless dust. The universe therefore accelerates, but not because it is full of an unexplained dark energy fluid, but rather because space itself is in some manner intrinsically, geometrically elastic and thus drives galaxies apart at late times (Nojiri and Odintsov, 2006; Clifton et al., 2012).

5.4 Viability Conditions

Since f(R) gravity is a modification of the universal laws of physics, the mathematical functions to use as f(R) are not all physically admissible. The theory requires an imposition of stringent mathematical constraints to guarantee that it does not lead to a nonsensical physics (this includes the presence of negative energy states, and matter decaying instantaneously, etc.) (De Felice and Tsujikawa, 2010; Sotiriou and Faraoni, 2010).

1. No Ghosts: To maintain the positive kinetic energy of the graviton, the graviton needs $F(R) > 0 For all R \% \geq R_0$ (where R_0 is the current Ricci curvature). Should this be broken, the universe will be disastrously unstable to the spontaneous creation of particles.
2. No Tachyons (Dolgov-Kawasaki instability): To get a mass squared of the new scalar degree of freedom (the "scalaron") to be positive one requires that the second derivative should be positive, $\frac{d^2 f(R)}{dR^2} > 0$. Any negative mass squared would mean tachyonic propagation, and the theory would not work at small and local scales such as the Solar System.

Panamerican Mathematical Journal
ISSN: 1064-9735
Vol 31 No. 1 (2021)

Any modified gravity theory that is successful must be able to maneuver these rigid theoretical restrictions whilst at the same time decohering the very accelerated expansion we see with telescopes.

Table 2: Summary of Cosmological Dark Energy Frameworks

Cosmological Model	**Equation of State ($w = \frac{p}{\rho c^2}$)**	**Fundamental Mechanism**	**Key Theoretical Challenge / Feature**
Cosmological Constant (Λ)	$w = -1$ (Strictly constant)	Intrinsic vacuum energy of spacetime.	Suffers from the 120-order-of-magnitude fine-tuning problem and the coincidence problem.
Quintessence	$-1 < w < \frac{-1}{3}$ Dynamic)	A slow-rolling scalar field ϕ moving down a potential well $V(\phi)$.	Requires specific initial conditions for the scalar field to begin dominating at late times.
Phantom Energy	$w < -1$ (Dynamic)	A scalar field with negative kinetic energy.	Violates the null energy condition; predicts a catastrophic future "Big Rip" singularity.
Generalized Chaplygin Gas	$p = \frac{-A}{\rho^\alpha}$	An exotic fluid that mimics dust at early times and Λ at late times.	Unifies Dark Matter and Dark Energy into a single mathematical dark sector framework.
Modified Gravity (f(R))	Effective geometric $w_{geo} < \frac{-1}{3}$	Modifies the Einstein-Hilbert action; gravity becomes self-repulsive at large scales.	Must satisfy strict mathematical bounds (e.g., no ghosts, no tachyons) to remain viable.

6. Observational Constraints on Dark Energy Models

Dark energy theoretical models, modified gravity, and however elegant they are mathematically, have to be eventually subjected to empirical data. Under relativistic cosmology we cannot measure the scale factor a(t) of the universe and the energy density percentagerho of the universe directly. Rather, the cosmologists use measurable quantities to include observable quantities, like the redshift of the atomic spectra, the apparent brightness of the distant celestial objects and the angular size of the primordial temperature variations. This part describes the mathematical maps between the theoretical cosmological models and three main observational probes Type Ia Supernovae, Baryon Acoustic Oscillations and the Cosmic Microwave Background.

6.1 Type Ia Supernovae and Luminosity Distance

The rapid acceleration of the universe was first realized when Type Ia supernovae (SNe Ia) were observed which served as a standard candle (Riess et al., 1998; Perlmutter et al., 1999). Owing to the fact that these supernovae are produced as the result of the thermonuclear explosion of white dwarfs accreting the mass up to the Chandrasekhar limit, the intrinsic peak luminosity L of such supernovas is thought to be constant over cosmic time.

In order to mathematically in constrain the dark energy with SNe Ia, we need to find the relationship between the history of the expansion of the universe and the recorded flux F of the supernova. The observable that can be determined in cosmology is the cosmological redshift z which is connected to the scale factor a(t) by:

$$1 + z = \frac{a_0}{a}(t)$$

$a_0 = 1$ is the scale factor at the current epoch. The distance of the supernova is determined in terms of the luminosity distance d_L, where the inverse-square law of flux is to hold in an expanding universe.: $F = \frac{L}{4\pi d_L^2}$.

Luminosity distance In a spatially flat FriedmannLemaitreRobertsonWalker (FLRW) universe, the luminosity distance is calculated mathematically as an integral of the history of its expansion:

$$d_L(z) = c(1 + z)\frac{\int dz'}{H}(z')$$

In order to clearly view the model of the dark energy under test, we define normalized Hubble parameter $E(z) = H\frac{(z)}{H_0}$. In a universe that has matter and a general dark energy component whose equation of state is w(z) the Friedmann equation states that:

$$E(z) = \sqrt{\Omega_{m,0}(1 + z)^3 + \Omega_{DE,0}exp\left(3\frac{\int 1 + w(z')}{1 + z'}dz'\right)}$$

Astronomers do not measure the d_L directly, they measure m which is the apparent magnitude and deduce M the absolute magnitude. The measured quantity is the distance modulus μwhich can be theoretically represented as:

$$\mu(z) = m - M = 5log_{10}\left(\frac{d_L(z)}{10\text{ pc}}\right)$$

Cosmologists use statistical fittings in plotting the measured distance modulus against the redshift z of hundreds of supernovae to determine the integral $d_L(z)$. The process severely limits the allowed values of the dark energy density $\Omega_{DE,0}$ and its equation of state w(z) (Abbott et al.,).

6.2 Baryon Acoustic Oscillations (BAO)

Whereas supernovae are used as standard candles, Baryon Acoustic Oscillations (BAO) are used as standard rulers. The photons and baryons were closely bound in a primordial plasma in the hot and dense early universe. The dark matter halo was compressed by gravity which drew the plasma to the center whereas the plasma was pushed out by the radiation pressure, forming spherical acoustic sound waves. At a time when the universe had cooled to allow the neutral hydrogen to condense (the epoch of recombination), photons decoupled, and the acoustic waves solidified into the distribution of matter (Alam et al., 2017).

The farthest distance that these sound waves have traveled before recombining is the comoving sound horizon r_s which is mathematically defined as:

$$r_s(z_d) = \frac{\int c_s(z)}{H}(z)dz$$

and $c_s(z)$ the velocity of sound in the primordial plasma, and z_d the redshift of the drag epoch (when baryons decoupled with photons).

This frozen sound wave is today reflected in an excess probability of finding two galaxies spaced about 150 Megaparsecs apart. Measuring this characteristic length scale at different redshifts, cosmologists would be able to map the history of the expansion. The BAO scale will be quantified along the line of sight (which puts a constraint on the Hubble parameter H(z)) and perpendicular to the line of sight, which puts a constraint on the angular diameter distance $d_A(z)$:

$$d_A(z) = \frac{d_L(z)}{(1+z)^2} = \frac{c}{1+z}\frac{\int dz'}{H}(z')$$

In surveys of large-scale galaxy surveys, e.g., a survey by the Sloan Digital Sky Survey (SDSS), BAO is used to disentangle mathematical degeneracies that occur when using the data provided by supernovae, which allow extremely impressive constraints on the matter density $\Omega_{m,0}$ (Alam et al., 2017).

6.3 The Cosmic Microwave Background (CMB)

The temperature anisotropies on the Cosmic Microwave Background are the most accurate constraints to the cosmological parameters. The CMB is the oldest electromagnetic radiation in the universe, giving a record of the universe at $z \approx 1100$.

An acoustic peaks are a series of peaks that feature in the angular power spectrum of the CMB. The point at which the first acoustic peak is located is very sensitive to the spatial curvature of the universe. Existing Planck satellite measurements strongly indicate that the location of this peak is just as predicted by math of a spatially flat universe restricting $\Omega_{k,0} \approx 0$ with extreme precision (Planck Collaboration, 2020).

Moreover, the physical matter density, per cent, $\Omega_{m,0}h^2$ is tightly constrained by the CMB data. Nevertheless, the CMB does not detect different models of dark energy at late times (low redshift). Only the mathematical combination of CMB data with SNe Ia and BAO data makes the standard model with $w = -1$ standard Λ CDM model, statistically preferred.

6.4 The Hubble Tension

Although successfully used as combined observational probes, contemporary precision cosmology has a drastic mathematical crisis called the Hubble Tension. The early universe (through the CMB with the assumptions of the Λ CDM model) value of the Hubble constant H_0 is somewhere around 67.4 km s^{-1} Mpc^{-1} (Planck Collaboration, 2020). On the other hand, direct, late-universe measurements of kinemics with SNe Ia standardized by Cepheid variable stars also provide a significantly higher value of around 73.0 km s^{-1} Mpc^{-1} (Riess et al., 1998; Di Valentino et al., 2021).

This continued discrepancy, and which is now more than a statistical significance of 5 sigma, is a strong indication that the theoretical equations which form the standard model are not complete. The need to solve the Hubble Tension is now becoming the motivating factor in the further investigation of dynamical dark energy, early dark energy injections, and modified gravity theories.

Table 3: Primary Observational Probes of Cosmic Expansion

Observational Probe	**Primary Observable Quantity**	**Mathematical Distance Metric Used**	**Cosmological Parameters Constrained**
Type Ia Supernovae (SNe Ia)	Apparent Peak Luminosity (Flux)	Luminosity Distance: $d_L(z)$	Maps the late-time expansion history H(z); constrains $\Omega_\Lambda \wedge w$.
Baryon Acoustic Oscillations (BAO)	Angular Size of Galactic Clustering	Angular Diameter Distance: $d_A(z)$	Acts as a "standard ruler"; tightly constrains matter density Ω_m.
Cosmic Microwave Background (CMB)	Temperature Anisotropies (Acoustic Peaks)	Comoving Distance to the Surface of Last Scattering	Pinpoints spatial curvature ($\Omega_k \approx 0$); anchors the early-universe physics.

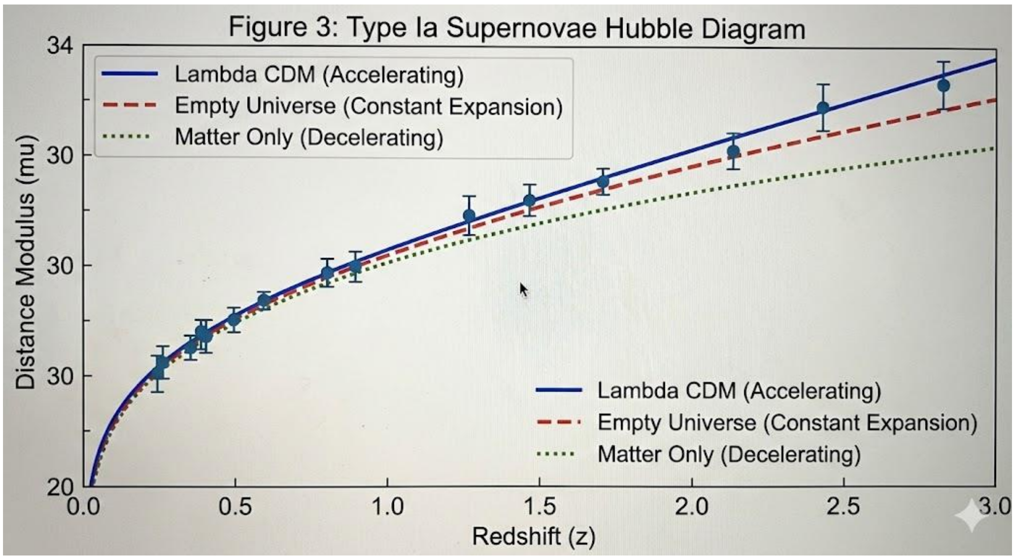

Figure 3: Type Ia Supernovae Hubble Diagram

7. Conclusion

The late-time accelerated expansion of the universe as the discovery completely disrupted the classical paradigm of relativistic cosmology. Due to the application of the Einstein Field Equations to the homogeneous and isotropic FLRW metric, which has been rigorously applied, the natural mathematical result of standard baryonic and dark matter is a decelerating universe. In order to fulfill mathematically the kinematic condition that acceleration $a\ddot{a} > 0 \vee w < \frac{-1}{3}$. the universe must have a component whose effective pressure is strongly negative.

Λ CDM model is mathematically solved by the standard model of the cosmological model of the Universe, which re-introduces the cosmological constant of Einstein, which is defined as a strict equation of state $w = -1$. Although phenomenologically preferred, it is a solution to Type Ia supernovae (Riess et al., 1998; Perlmutter et al., 1999), Cosmic Microwave Background (Planck Collaboration, 2020) and Baryon Acoustic Oscillations (Alam et al., 2017), which has the disadvantage of creating the cosmological constant problem and the coincidence problem, creating an enormous gap between quantum field theory and general relativity (Carroll, 2001; Weinberg,

To avoid the fine tuning in the context of Λ, this paper has discussed two main branches of theoretical extensions:

- Dynamical Dark Energy We have studied models in which the energy-momentum tensor $T_{\mu\nu}$ is altered through the addition of new, time-dependent scalar fields. Quintessence models use a slow-rolling field ϕ, to produce$-1 < w < \frac{-1}{3}$ at the expense of the null energy condition and forecasting a future Big Rip singularity (Caldwell, 2002). In addition, Generalized Chaplygin Gas was demonstrated as capable of mathematically uniting the dark matter and dark energy into a single perfect fluid with the $p = \frac{-A}{\rho^{\alpha}}$ (Bento, Bertolami and Sen, 2002).
- Modified Gravity: Modifying the geometric part of the Einstein-Hilber action is also a possible way to get a dynamical acceleration mechanism, without the problems of exotic fluids. The arbitrary function f(R) of f(R) gravity adds terms of higher derivative order which is effectively the same as a geometric energy-momentum term $T_{\mu\nu}^{(geo)}$ which is far more effective in triggering the process of expansion as it solely depends on the dynamics of spacetime (De Felice and Tsujikawa, 2010).

In fact, at present, only a universe with w per cent. approximated as -1 admits of observational constraints. Nonetheless, the Hubble Tension, the drastic statistical inconsistency between early-universe and late-universe values of H_0 (Di Valentino et al., 2021), is a persistent phenomenon that indicates that the standard version of the model of the universe, called the standard Λ CDM model, may be a mathematical approximation of an underlying reality that is more complex.

Next generation cosmological surveys to measure the time dependence of the equation of state parameter, w(a), in unprecedented precision are specifically planned to be conducted with future next-generation instruments, like the Dark Energy Spectroscopic Instrument

(DESI) and the Euclid space telescope. Should such surveys even reveal that there is any kind of deviation at all in $w = -1$, it will lead to a direct failure of the cosmological constant paradigm, and bring dynamical scalar fields or modified gravity theories out of mathematical fancifuls and into the realm of physical law.

References

1. Alam, S. *et al.* (BOSS Collaboration) (2017) 'The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample', *Monthly Notices of the Royal Astronomical Society*, 470(3), pp. 2617–2652.
2. Bamba, K., Capozziello, S., Nojiri, S. and Odintsov, S.D. (2012) 'Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests', *Astrophysics and Space Science*, 342(1), pp. 155–228.
3. Bento, M.C., Bertolami, O. and Sen, A.A. (2002) 'Generalized Chaplygin gas, accelerated expansion and dark energy-matter unification', *Physical Review D*, 66(4), p. 043507.
4. Caldwell, R.R. (2002) 'A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state', *Physics Letters B*, 545(1-2), pp. 23–29.
5. Caldwell, R.R., Dave, R. and Steinhardt, P.J. (1998) 'Cosmological imprint of an energy component with fluid equation of state w < -1/3', *Physical Review Letters*, 80(8), pp. 1582–1585.
6. Carroll, S.M. (2001) 'The cosmological constant', *Living Reviews in Relativity*, 4(1), p. 1.
7. Clifton, T., Ferreira, P.G., Padilla, A. and Skordis, C. (2012) 'Modified gravity and cosmology', *Physics Reports*, 513(1-3), pp. 1–189.
8. Copeland, E.J., Sami, M. and Tsujikawa, S. (2006) 'Dynamics of dark energy', *International Journal of Modern Physics D*, 15(11), pp. 1753–1935.
9. De Felice, A. and Tsujikawa, S. (2010) 'f(R) theories', *Living Reviews in Relativity*, 13(1), p. 3.
10. Di Valentino, E., Mena, O., Pan, S., Visinelli, L., Yang, W., Melchiorri, A., Mota, D.F., Riess, A.G. and Silk, J. (2021) 'In the realm of the Hubble tension—a review of solutions', *Classical and Quantum Gravity*, 38(15), p. 153001.
11. Kamenshchik, A., Moschella, U. and Pasquier, V. (2001) 'An alternative to quintessence', *Physics Letters B*, 511(2-4), pp. 265–268.
12. Nojiri, S. and Odintsov, S.D. (2006) 'Introduction to modified gravity and gravitational alternative for dark energy', *International Journal of Geometric Methods in Modern Physics*, 4(01), pp. 115–145.

13. Padmanabhan, T. (2003) 'Cosmological constant—the weight of the vacuum', *Physics Reports*, 380(5-6), pp. 235–320.

14. Peebles, P.J.E. and Ratra, B. (2003) 'The cosmological constant and dark energy', *Reviews of Modern Physics*, 75(2), pp. 559–606.

15. Perlmutter, S. *et al.* (Supernova Cosmology Project) (1999) 'Measurements of Omega and Lambda from 42 high-redshift supernovae', *The Astrophysical Journal*, 517(2), pp. 565–586.

16. Planck Collaboration (2020) 'Planck 2018 results. VI. Cosmological parameters', *Astronomy & Astrophysics*, 641, p. A6.

17. Riess, A.G. *et al.* (Supernova Search Team) (1998) 'Observational evidence from supernovae for an accelerating universe and a cosmological constant', *The Astronomical Journal*, 116(3), pp. 1009–1038.

18. Sotiriou, T.P. and Faraoni, V. (2010) 'f(R) theories of gravity', *Reviews of Modern Physics*, 82(1), pp. 451–497.

19. Zlatev, I., Wang, L. and Steinhardt, P.J. (1999) 'Quintessence, cosmic coincidence, and the cosmological constant', *Physical Review Letters*, 82(5), pp. 896–899.