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Stability of Quadratic Functional Equation in Finite Variable Using Fuzzy Normed Space

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Abstract: This paper defines a new generalized quadratic functional equation and discusses its Ulam-Hyers stability in the context of fuzzy normed spaces using both direct and fixed-point approaches. An example is also given to show the unstability of the newly defined equation. These type of functionals play a fundamental role in various mathematical models and optimization problems.

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1. Introduction:

Ulam [10] proposed the concept of FE(functional equation) stability in 1940, and Hyer [5] gave a answer for additive group in 1941 by considering a Cauchy FE. Hyers's theorem for additive mappings was expanded by Aoki [13] and Rassias [15] for linear mappings, respectively. Researchers analyzed the Hyers-Ulam-Rassias stability results for FE with several variables in ([7], [9]).

Zadeh proposed fuzzy set theory for the first time in 1965. A.K. Katsaras presents the fuzzy norm on a vector space in [1]. FE in FNS(fuzzy normed spaces) have been established recently by numerous scholars. ([2],[3],[4],[11],[14],[16]).

Our task involves presenting a QFE(quadratic functional equation) with a finite number of variables and determining its Hyer-Ulam stability.

$$\sum_{i=1}^{m} \aleph(2x_i - \sum_{1\le i\ne j}^{m} x_j) = (m-7)\sum_{1\le i<j\le m} \aleph(x_i + x_j) + \aleph(\sum_{i=1}^{m} x_i) - (m^2 - 9m + 5)\sum_{i=1}^{m} \aleph(x_i)$$

where $m \geq 5$ is finite natural number, in fuzzy normed space. The mapping which satisfies equation (1), is quadratic.

Definition 1.1 Let Ω be a real vector space. A generalized functional $\wp: \Omega \times R \to [0,1]$ is called a fuzzy norm if for arbitrary $\mu, x \in \Omega$, $\alpha, \beta \in$ R and $\wp$ satisfies

(a) $\wp(\mu, \alpha) = 0$ for $\alpha \leq 0$;

(b) $\mu = 0 \Leftrightarrow \wp(\mu, \beta) = 1$ for all $\beta > 0$;

(c) $\wp(\gamma\mu, \alpha) = \wp\left(\mu, \frac{\alpha}{|\gamma|}\right)$ if $\gamma \neq 0$;

(d) $\wp(\mu + x, \alpha + \beta) \geq \min\{\wp(\mu, \alpha), \wp(x, \beta)\}$;

(e) $\wp(\mu,.)$ is a non-decreasing function on R and $\lim_{\alpha\to\infty}\wp(\mu,\alpha)=1$;

(f) for $\mu\neq 0,\wp(\mu,.)$ is continuous on R.

Then, $(\Omega,\wp)$ is called FNS.

Definition 1.2 If $(\Omega,\wp)$ is a FNS and the sequence $\{\mu_n\}$ in Ω then

(i) $\mu_n\to\mu$ if $\wp(\mu_n-\mu,\alpha)\to 1$ as $n\to\infty(\alpha>0)$.

(ii) $\{\mu_n\}$ is known as fuzzy-Cauchy if $\wp(\mu_m-\mu_n)\to 1$ as $m,n\to\infty$.

(iii) If all fuzzy Cauchy sequences in subset $A\subseteq\Omega$ are convergent in A, then the subset A is considered as complete. Fuzzy Banach space is a complete FNS.

The following basic conclusion will be applied in a fixed point theory.

Definition 1.3 $\Lambda:\Omega\to\Omega$ be a strictly contractive mapping with the Lipschitz constant $\eta<1$, and let (Ω, d) be a generalized complete metric space. Assume that there is an integer k that is non-negative and such that $d(\Lambda^{l+1}a,\Lambda^k a)<\infty$ for a given element a. Then

(i) the sequence $\{\Lambda^n a\}_{n=1}^{\infty}$ converges to a fixed point $b\in\Omega$ of Λ;

(ii) b is the unique fixed point of Λ in the set $Y=\{y\in\Omega: d(\Lambda^n a,y)<\infty$;

(iii) $d(y,b)\leq\frac{1}{1-\eta}d(y,\Lambda)$ for all $y\in$ Y.

. Solution of FE (1)

The general solution of FE (1) is obtained here. In this instance, the real vector spaces are V and W.

Theorem 2.1 If a function $\aleph: V\to W$ satisfies the FE (1) for all $x_1,x_2,x_3,\dots,x_n\in V$, then the function $\aleph$ is quadratic.

Proof. Assume that the mapping $\aleph: V\to W$ satisfying the equation (1).

Substituting $(0,0,0,\dots,0)$ for $(x_1,x_2,x_3,\dots,x_n)$ in equation (1), we get $\aleph(0)=0$.

Also, Replacing $(x_1,x_2,x_3,\dots,x_n)$ by $(x,0,0,\dots,0)$ in equation (1), we obtain

$$\aleph(2x)=(n+3)\aleph(x)-(n-1)\aleph(-x). \tag{2}$$

Substituting $(-x)$ for (x) in equation (2), we obtain

$$\aleph(-2x)=(n+3)\aleph(-x)-(n-1)\aleph(x). \tag{3}$$

When we replace $(x_1,x_2,x_3,\dots,x_n)$ by $(x,x,0,0,0,\dots,0)$ in equation (1), then we obtain

$$(n-2)\aleph(-2x)-(n-6)\aleph(2x)=16\aleph(x). \tag{4}$$

Using equation (2) and equation (3) in equation (4), we get

$$\aleph(-x)=\aleph(x) \tag{5}$$

for all $x\in V$.

Therefore $\aleph$ is an even function. Now from equation (4) using equation (5), we have

$$\aleph(2x)=2^2\aleph(x), \tag{6}$$

for each $x\in V$ and for every $n\in Z^+$.

Substituting $(2x)$ for x, we obtain

$$\aleph(2^2x)=2^4\aleph(x) \tag{7}$$

for each $x\in V$.

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Substituting $(2x)$ for x, we get

$$\aleph(2^3 x) = 2^6 \aleph(x) \tag{8}$$

for each $x \in V$.

Similarly, for all positive integer n, we can say

$$\aleph(2^n x) = 2^{2n} \aleph(x) \tag{9}$$

for each $x \in V$.

Again, when we replace $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, x, 0,0,0, \ldots, 0)$ in equation (1), then we obtain

$$(n-3)\aleph(-3x) - \aleph(3x) = 9(n-4)\aleph(x). \tag{10}$$

Replacing x by $-x$, we get

$$(n-3)\aleph(3x) - \aleph(-3x) = 9(n-4)\aleph(-x), \tag{11}$$

$$\aleph(-3x) = (n-3)\aleph(3x) - 9(n-4)\aleph(-x). \tag{12}$$

Using equation (12) in equation (10), we get

$$(n-3)[(n-3)\aleph(3x) - 9(n-4)\aleph(-x)] - \aleph(3x) = 9(n-4)\aleph(x),$$

$$(n^2 - 6n + 9 - 1)\aleph(3x) - 9(n^2 - 7n + 12)\aleph(x) = 9(n-4)\aleph(x),$$

$$(n^2 - 6n + 8)\aleph(3x) = (9n^2 - 54n + 72)\aleph(x),$$

$$(n^2 - 6n + 8)\aleph(3v) = 9(n^2 - 6n + 8)\aleph(x),$$

$$\aleph(3x) = 3^2 \aleph(x), \tag{13}$$

for each $x \in V$.

Substituting $(3x)$ for x, we get

$$\aleph(3^2 x) = 3^4 \aleph(x) \tag{14}$$

for each $x \in V$.

Substituting $(3x)$ for x, we get

$$\aleph(3^3 x) = 3^6 \aleph(x) \tag{15}$$

for each $x \in V$.

Similarly, for all positive integer n, we can say

$$\aleph(3^n x) = 3^{2n} \aleph(x) \tag{16}$$

for every $x \in V$. Hence $\aleph$ is a quadratic function.

Result 2.2. The following conclusions apply if V, W is a linear space and a function $\aleph: V \to W$ satisfies FE (1):

(1) $\aleph(q^t s) = q^{2t} \aleph(s)$ for all $s \in V$, q ∈ Q, t ∈ Z

(2) $\aleph(s) = s\aleph(1)$ for all $s \in V$ if $\aleph$ is continuous.

3 Stability of FE (1) Using FNS

In this section, we take $\Omega, (\Phi, \mathcal{G})$ and $(\Psi, \mathcal{G})$ to be linear, FNS, Fuzzy Banach space respectively. We are presenting Hyer-Ulam Stability of the FE (1). Let us denote the mapping $\aleph: \Omega \to \Psi$ such as:

$$D\aleph(x_1, x_2, \ldots \ldots x_n) = \sum_{i=1}^{m} \aleph\left(2x_i - \sum_{1 \le i \ne j}^{m} x_j\right) - (m-7) \sum_{1 \le i < j \le m} \aleph(x_i + x_j)$$

$$-\aleph(\sum_{i=1}^{m} x_i) + (m^2 - 9m + 5)\sum_{i=1}^{m} \aleph(x_i) \tag{17}$$

for all $x_1, x_2, \dots, x_n \in \Omega$.

Theorem 3.1: Let $\psi: \Omega_n \to \Phi$ be a mapping such that

$$\wp(\psi(2x, 2x, 0, \dots, 0), \alpha) \geq \wp(\rho\psi(x, x, 0, \dots, 0), \alpha) \tag{18}$$

including

$$\lim_{m\to\infty} \mathcal{G}(\psi(2^m x_1, 2^m x_2, \dots., 2^m x_n), 2^{2m}\alpha) = 1,$$

for all $x_1, x_2, \dots x_n \in \Omega$, $0 < \rho < 4$ and $\alpha > 0$. If a function $\aleph: \Omega \to \Psi$ with the condition $\aleph(0) = 0$ and such that

$$\wp(\mathrm{D}\aleph(x_1, x_2, \cdots, x_n), \beta) \geq \mathcal{G}(\psi(x_1, x_2, x_3, \dots, x_n), \beta) \tag{19}$$

for every $x_1, x_2, \cdots, x_n \in \Omega$ and $\beta > 0$, thus, a unique quadratic mapping $Q_2: \Omega \to \Psi$ exists that satisfies

$$\wp(\aleph(x) - Q_2(x), \alpha) \geq \mathcal{G}(\psi(x_1, x_2, \dots \dots, x_n), 4|2^2 - \rho|\alpha) \tag{20}$$

for all $x \in \Omega$, $0 < \rho < 4$ and $\alpha > 0$.

Proof: Replace $(x_1, x_2, \dots, x_n)$ by $(x, x, 0, \dots, 0)$ in equation (19), we obtain

$$\wp(4\aleph(2x) - 16\aleph(x), \beta) \geq \mathcal{G}(\psi(x, x, \dots, 0), \beta)$$

$$\wp\left(\aleph(x) - \frac{\aleph(2x)}{2^2}, \frac{\beta}{2^4}\right) \geq \mathcal{G}(\psi(x, x, \dots, 0), \beta). \tag{21}$$

Substituting $2^m x$ for x in equation (21), then we get

$$\wp\left(\aleph(2^m x) - \frac{\aleph(2^{m+1}x)}{4}, \frac{\beta}{16}\right) \geq \mathcal{G}(\psi(2^m x, 2^m x, \dots, 0), \beta).$$

Using equation (18) and (c) part of definition (1), we get

$$\wp\left(\frac{\aleph(2^m x)}{2^{2m}} - \frac{\aleph(2^{m+1}x)}{2^{2(m+1)}}, \frac{\beta}{4.2^{2(m+1)}}\right) \geq \mathcal{G}\left(\psi(x, x, 0, \dots, 0), \frac{\beta}{\rho^m}\right). \tag{22}$$

Replace β by $\rho^m\beta$ in equation (22), then we get

$$\wp\left(\frac{\aleph(2^m x)}{2^{2m}} - \frac{\aleph(2^{m+1}x)}{2^{2(m+1)}}, \frac{\rho^m\beta}{4.2^{2(m+1)}}\right) \geq \wp(\psi(x, x, 0, \dots, 0), \beta). \tag{23}$$

Since,

$$\aleph(x) - \frac{\aleph(2^m x)}{2^{2m}} = \sum_{i=0}^{m-1}\left(\frac{\aleph(2^i x)}{2^{2i}} - \frac{\aleph(2^{i+1}x)}{2^{2(i+1)}}\right)(24)$$

for all $x \in \Omega$.

Now, form equation (23) and equation (24), we have

$$\wp\left(\aleph(x) - \frac{\aleph(2^m x)}{2^{2m}}, \sum_{i=0}^{m-1}\left(\frac{\rho^i\beta}{2^{2(i+1)}}\right)\right) \geq \min\bigcup_{i=0}^{m-1}\left\{\wp\left(\left(\frac{\aleph(2^i x)}{2^{2i}} - \frac{\aleph(2^{i+1}x)}{2^{2(i+1)}}\right), \frac{\rho^i\beta}{4.\,2^{2(i+1)}}\right)\right.$$

$$\geq \mathcal{G}(\psi(x, x, \dots, 0), \beta) \tag{25}$$

for all x in $\Omega, \beta > 0$.

Replacing x with $(2^n x)$ in equation (25), we get

$$\wp\left(\aleph(2^{n}x)-\frac{\aleph(2^{m+n}x)}{2^{2m}},\sum_{i=0}^{m-1}\left(\frac{\rho^{i}\beta}{2^{2(i+1)}}\right)\right)\geq\mathcal{G}(\psi(2^{n}x,2^{n}x,0,\dots,0),\beta)$$
$$\geq\mathcal{G}\left(\psi(x,x,0,\dots,0),\frac{\beta}{\rho^{n}}\right). \tag{26}$$

Substituting β by $\rho^{n}\beta$ in equation (26), we get

$$\wp\left(\frac{\aleph(2^{n}x)}{2^{2n}}-\frac{\aleph(2^{m+n}x)}{2^{2(m+n)}},\sum_{i=n}^{m+n-1}\left(\frac{\rho^{i}\beta}{2^{2(i+1)}}\right)\right)\geq\mathcal{G}(\psi(x,x,\dots,0),\beta). \tag{27}$$

Replacing β by $\frac{\beta}{\sum_{i=n}^{m+n-1}\left(\frac{\rho^{i}}{4.2^{2(i+1)}}\right)}$ in (27), we observe

$$\wp\left(\frac{\aleph(2^{n}x)}{2^{2n}}-\frac{\aleph(2^{m+n}x)}{2^{2(m+n)}},\beta\right)\geq\mathcal{G}\left(\psi(x,x,\dots,0),\frac{\beta}{\sum_{i=n}^{m+n-1}\left(\frac{\rho^{i}}{4.2^{2(i+1)}}\right)}\right) \tag{28}$$

for all $x\in\Omega$ and all non-negative integers $m,n\geq 0$. As $0<\rho<2^{2}$ and $\sum_{i=0}^{\infty}\left(\frac{\rho^{i}}{2^{2i}}\right)<\infty$. If $n\to\infty$ then R.H.S of equation (28) approaches to 1. Hence $\left\{\frac{\aleph(2^{m}x)}{2^{2m}}\right\}$ is a Cauchy sequence in Fuzzy Banach $(\Psi,\mathcal{G})$, so there exists a function $Q_{2}:\Omega\to\Psi$ as

$$\lim_{m\to\infty}\frac{\aleph(2^{m}x)}{2^{2m}}=Q_{2}(x) \tag{29}$$

for all $x\in\Omega$.

Putting $n=0$ and taking limit $m\to\infty$ in equation (28),

$$\wp(\aleph(x)-Q_{2}(x),\beta)\geq\mathcal{G}(\psi(x,x,\dots,0),4.|2^{2}-\rho|\beta). \tag{30}$$

Replacing $(x_{1},x_{2},\cdots,x_{n})$ by $(2^{m}x_{1},2^{m}x_{2},\cdots,2^{m}x_{n})$ in equation (19), we get

$$\wp(\mathrm{D}\aleph(x_{1},x_{2},\cdots,x_{n}),\beta)\geq\wp(\psi(x_{1},x_{2},\dots,x_{n}),\beta) \tag{31}$$

$$\wp\left(\frac{1}{2^{2m}}\mathrm{D}\aleph(2^{m}x_{1},2^{m}x_{2},\cdots,2^{m}x_{n}),\beta\right)\geq\mathcal{G}\left(\frac{1}{2^{2m}}\psi(2^{m}x_{1},2^{m}x_{2},\dots\dots,2^{m}x_{n}),\beta\right)$$
$$\geq\mathcal{G}(\psi(2^{m}x_{1},2^{m}x_{2},\dots,2^{m}x_{n}),2^{2m}\beta)$$

for all $x_{1},x_{2},\cdots,x_{n}\in\Omega,\beta>0$. Also

$$\lim_{m\to\infty}\mathcal{G}(\psi(2^{m}x_{1},2^{m}x_{2},\dots,2^{n}x_{n}),2^{2m}\beta)=1. \tag{32}$$

Therefore, taking the limit $m\to\infty$, we get

$$\mathrm{D}Q_{2}(x_{1},x_{2},\cdots,x_{n})=0 \tag{33}$$

for every $x_{1},x_{2},\cdots,x_{n}\in\Omega$

To demonstrate the uniqueness of Q_{2}, consider another quadratic mapping that satisfies equation (20) in the form of $\mathrm{T}_{2}:\Omega\to\Psi$.

$$\wp(Q_{2}(x)-T_{2}(x),\beta)\geq\min\left\{\wp\left(\left(\frac{Q_{2}(2^{m}x)}{2^{2m}}-\frac{\aleph(2^{m}x)}{2^{2m}}\right),\frac{\beta}{2}\right),\wp\left(\left(\frac{\aleph(2^{m}x)}{2^{2m}}-\frac{T_{2}(2^{m}x)}{2^{2m}}\right),\frac{\beta}{2}\right)\right\}$$
$$\geq\mathcal{G}\left(\psi(2^{m}x,2^{m}x,0,\dots\dots,0),\frac{4.|2^{2}-\rho|2^{2m}\beta}{2}\right)$$
$$\geq\mathcal{G}\left(\psi(x,x,\dots,0),\frac{4.|2^{2}-\rho|2^{2m}\beta}{2\rho^{m}}\right). \tag{34}$$

By taking $m\to\infty$, $\left(\frac{4.|2^{2}-\rho|2^{2m}\beta}{2\rho^{m}}\right)=\infty$, Hence $\wp\left(\psi(x,x,0,\dots,0),\frac{4.|2^2-\rho|2^{2m}\beta}{2\rho^m}\right)=1$ as $m\to\infty$, we have $Q_2=T_2$

Therefore, the function Q_2 is unique.

Theorem 3.2 Let $\psi:\Omega^2\to D$ be a function such that

$$\wp\left(\psi\left(\frac{x}{2},\frac{x}{2},0,\dots,0\right),\alpha\right)\geq\wp\left(\frac{1}{\rho}\psi(x,x,0,\dots,0),\alpha\right) \tag{35}$$

and

$$\lim_{m\to\infty}\wp\left(\psi\left(\frac{x_1}{2^m},\frac{x_2}{2^m},\dots\dots,\frac{x_n}{2^m}\right),\frac{\alpha}{2^{2m}}\right)=1 \tag{36}$$

for all $x_1,x_2,\cdots,x_n\in\Omega,\alpha>0,\ \rho>2^2$. If a function $\aleph:\Omega\to\Psi$ satisfies the condition $\aleph(0)=0$ such that

$$\wp(D\aleph(x_1,x_2,\dots\dots.x_n),\beta)\geq\mathcal{G}(\psi(x_1,x_2,\dots,x_n),\beta) \tag{37}$$

for every $x_1,x_2,\dots,x_n\in\Omega$ and $\beta>0$, then, a unique quadratic mapping $Q_2:\Omega\to\Psi$ exists, such that

$$\wp(\aleph(x)-Q_2(x),\alpha)\geq\mathcal{G}(\psi(x_1,x_2,\dots,x_n),4.|2^2-\rho|\alpha) \tag{38}$$

for all $x\in\Omega$ and $\alpha>0$.

Proof: The method of proof is same as in above theorem 3.1.

Corollary 3.3: If a mapping $\aleph:\Omega\to\Psi$ with $\aleph(0)=0$ and such that

$$\wp(D\aleph(x_1,x_2,\dots\dots,x_n),\beta)\geq\mathcal{G}(\textstyle\sum_{i=1}^{n}\|x_i\|^p,\beta) \tag{39}$$

for all $x_1,x_2,\dots,x_n\in\Omega$.

Then, there exists a unique quadratic mapping $Q_2:\Omega\to\Psi$

$$\wp(\aleph(x)-Q_2(x),\beta)\geq\mathcal{G}(\|x\|^p,4.(2^2-2^p)\beta) \tag{40}$$

for all $x\in\Omega$.

4. Counter Example:

For the FE (1) in real normed space, we give a counterexample that demonstrates its non-stability as:

Example 4.1 Let a function $\aleph:R\to R$ be defined as

$$\aleph(x)=\sum_{t=0}^{\infty}\frac{g(2^tx)}{2^{2t}} \tag{41}$$

where $g(x)=\begin{cases}\theta x^2, & |x|<1\\ \theta, & \text{else}\end{cases}$

then the function $\aleph:R\to R$ satisfies the inequality

$$|D\aleph(x_1,x_2,\cdots,x_n)|\leq\frac{16(3n^3-13n^2+6n+6)}{3}\theta\sum_{i=1}^{n}|x_i|^2. \tag{42}$$

for every $x_1,x_2,\cdots,x_n\in R, n\geq 8$, but there does not exist a quadratic function $Q_2:R\to R$ satisfies

$$|\aleph(x)-Q_2(x)|\leq\varepsilon|x|^2 \tag{43}$$

for all $x\in R$.

5. Stability of FE (1) using Fixed Point Method

Radu suggested a new approach to research the issue of FE's stability based on fixed point alternative. Many authors have recently employed this method. Here fixed-point technique is used to examine the Ulam-Hyers stability of generalized FE. Firstly, we define a constant ξ_{ϖ} such that

$$\xi_{\varpi} = \begin{cases} 2 & \text{if} \quad \varpi = 0 \\ \frac{1}{2} & \text{if} \quad \varpi = 1 \end{cases}$$

and we assume that ⊔.

Theorem 5.1 Let $\varphi: \hat{X} \to \hat{Y}$ be an even function with condition $\varphi(0) = 0$ for which there exists a mapping $\zeta: \hat{X}^m \to Z$ with condition

$$\lim_{n\to\infty} M(\zeta(\xi_{\varpi}^n r_1, \xi_{\varpi}^n r_2, \dots, \xi_{\varpi}^n r_n), \xi_{\varpi}^{2n}\delta) = 1 \tag{44}$$

for $r_1, r_2, \dots, r_m \in \hat{X}, \delta > 0,$ and satisfying the inequality

$$N(D\varphi(r_1, r_2, r_3, \dots, r_m), \delta) \geq M(\zeta(r_1, r_2, r_3, \dots, r_m), \delta), \tag{45}$$

for $r_1, r_2, \dots, r_m \in \hat{X}, \quad \delta > 0,$

Let $\aleph(r) = \zeta(r, r, 0, \dots, 0)$ for all $r \in \hat{X}$. If there exists $\eta = \eta_{\varpi} \in (0,1)$ such that

$$M\left(\frac{1}{\xi_{\varpi}^2}\aleph(\xi r), \delta\right) \geq M(\eta\aleph(r), \delta) \tag{46}$$

for $r \in \hat{X}$ and $\delta > 0$, then there exists a unique quadratic mapping $Q: \hat{X} \to \hat{Y}$ fulfilling

$$N(\varphi(r) - Q(r), \delta) \geq M\left(\frac{\eta^{1-\varpi}}{1-\eta}\aleph(r), \delta\right), \tag{47}$$

for $r \in \hat{X}$ and $\delta > 0$.

Proof: Let γ be a generalized metric space on ⊔ :

$$\gamma(\hat{g}, h) = \inf\{w \in (0, \infty: N(\hat{g}(r) - h(r), \delta) \geq M(w\aleph(r), \delta), \quad r \in \hat{X}, \delta > 0\},$$

and we use, $\inf\aleph = +\infty$ as usual. It is demonstrated that (⊔ is complete generalized metric space using a similar justification in ([8], Lemma 2.1). Describe ψ_{ϖ}:⊔ by $\psi_{\varpi}\hat{g}(r) = \frac{1}{\xi_{\varpi}^2}\hat{g}(\xi_{\varpi} r)$ for all $r \in \hat{X}$.

Suppose that $\hat{g}, h \in$⊔ be given such that $\gamma(\hat{g}, h) \leq \epsilon$. Then

$$N(\hat{g}(r) - h(r), \delta) \geq M(\epsilon\aleph(r), s) \tag{48}$$

for each $r \in \hat{X}$ and, $\delta > 0$, When

$$N(\psi_{\varpi}\hat{g}(r) - \psi_{\varpi}h(r), \delta) \geq M\left(\frac{\epsilon}{\xi_{\varpi}^2}\aleph(\xi_{\varpi} r), \delta\right) \tag{49}$$

for each $r \in \hat{X}$ and $\delta > 0$.

It follows from equation (46) that

$$N(\psi_{\varpi}\hat{g}(r) - \psi_{\varpi}h(r), \delta) \geq M(\epsilon\eta\aleph(r), \delta) \tag{50}$$

for $r \in \hat{X}$, $\delta > 0$. So, we get $\gamma(\psi_{\varpi}\hat{g}, \psi_{\varpi}h) \leq \epsilon\eta$. This shows that $\gamma(\psi_{\varpi}\hat{g}, \psi_{\varpi}h) \leq \eta\gamma(\hat{g}, h)$, that is, ψ_{ϖ} is strictly contractive mapping on ⊔ with Lipschitz constant η.

Replacing $(r_1, r_2, r_3, \dots, r_m)$ by $(r, r, \dots, 0)$ in (45) and using result (c) of definition (1.1), we obtain

$$N\left(\frac{\varphi(2r)}{2^2} - \varphi(r), \delta\right) \geq M\left(\frac{\xi(r, r, \dots, 0)}{16}, \delta\right) \tag{51}$$

for $r \in \hat{X}$ and $\delta > 0$. Using (46) when $\varpi = 0$, it follows from (51) that

$$N\left(\frac{\varphi(2r)}{2^2} - \varphi(r), \delta\right) \geq M(\eta\aleph(r), \delta) \tag{52}$$

where $r \in \hat{X}$ and $\delta > 0$. Therefore

$$\gamma(\psi_0\varphi, \varphi) \leq \eta = l^{1-\varpi}. \tag{53}$$

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Replacing r by $\frac{r}{2}$ in (51), we get,

$$N\left(\varphi(r) - 4\varphi\left(\frac{r}{2}\right), 16\delta\right) \geq M\left(\zeta\left(\frac{r}{2}, \frac{r}{2}, 0, \ldots, 0\right), 16\delta\right),$$

$$= M(\aleph(r), 16\delta) \tag{54}$$

for $r \in \hat{X}$ and $\delta > 0$. Therefore

$$\gamma(\psi_1\varphi, \varphi) \leq 1 = l^{1-\varpi}. \tag{55}$$

Then from (51) and (55), we obtain

$$\gamma(\psi_\varpi\varphi, \varphi) \leq l^{1-\varpi} < \infty.$$

Now, It follows from the fixed point alternative theorem that a fixed point Q of ψ_ϖ in $\sqcup$ such that

(i) $\psi_\varpi Q = Q$ and $\lim_{n\to\infty}\gamma(\psi_\varpi^n\varphi, Q) = 1$;

(ii) In the set $\varepsilon = \{\hat{g} \in \sqcup$ is the unique fixed point of ψ;

(iii) $\gamma(\varphi, Q) \leq \frac{1}{1-\eta}\gamma(\varphi, \psi_\varpi\varphi)$.

Assume $\gamma(\psi_\varpi^n\varphi, Q) = \epsilon_n$, we obtain

$$N(\psi_\varpi^n\varphi(r) - Q(r), \delta) \geq M(\epsilon_n\aleph(r), \delta) \tag{56}$$

for each $\delta > 0$.

Since $\lim_{n\to\infty}\epsilon_n = 0$, we obtain

$$Q(r) = N - \lim_{n\to\infty}\frac{\varphi(\xi_\varpi^n r)}{\xi_\varpi^{2n}}, r \in \hat{X}.$$

Replacing $(r_1, r_2, r_3, \ldots, r_m)$ by $(\xi_\varpi^n r_1, \xi_\varpi^n r_2, \xi_\varpi^n r_3, \ldots, \xi_\varpi^n r_m)$ in (45), we get

$$N\left(\frac{1}{\xi_\varpi^{2n}} D\varphi(\xi_\varpi^n r_1, \xi_\varpi^n r_2, \xi_\varpi^n r_3, \ldots, \xi_\varpi^n r_m), \delta\right) \geq M(\zeta(\xi_\varpi^n r_1, \xi_\varpi^n r_2, \xi_\varpi^n r_3, \ldots, \xi_\varpi^n r_m), \xi_\varpi^{2n}\delta)$$

for each $\delta > 0$ and each $r_1, r_2, r_3, \ldots, r_m \in \hat{X}$.

We may demonstrate that function $Q: \hat{X} \to \hat{Y}$ is quadratic using the same justification as in the proof of theorem (3.1).

Given that $\gamma(\psi_\varpi^n\varphi, \varphi) \leq l^{1-\varpi}$, it follows from (iii) $\gamma(\varphi, Q) \leq \frac{l^{1-\varpi}}{1-\eta}$ which means (47).

Let T be another quadratic mapping that satisfies (47) in order to demonstrate $Q's$ uniqueness.

We have $Q(2^n r) = 4^n Q(r)$ and $T(2^n r) = 4^n T(r)$ for all $r \in P$ and all $n \in N$, we have

$$N(Q(r) - T(r), \delta) = N\left(\frac{Q(2^n r)}{4^n} - \frac{T(2^n r)}{4^n}, \delta\right)$$

$$\geq \min\left\{N\left(\frac{Q(2^n r)}{4^n} - \frac{\varphi(2^n r)}{4^n}, \frac{\delta}{2}\right), N\left(\frac{\varphi(2^n r)}{4^n} - \frac{T(2^n r)}{4^n}, \frac{\delta}{2}\right)\right\}$$

$$\geq M\left(\frac{\eta^{1-\varpi}}{1-\eta}\aleph(2^n r), \frac{4^n\delta}{2}\right) \tag{57}$$

From (44), we have

$$\lim_{n\to\infty} M\left(\frac{\eta^{1-\varpi}}{1-\eta}\aleph(2^n r), \frac{4^n\delta}{2}\right) = 1.$$

Consequently, $N(Q(r) - T(r), \delta) = 1$ for all $r \in \hat{X}$ and each $\delta > 0$. So $Q(r) = T(r)$ for each $r \in \hat{X}$. This complete the proof.

6 Conclusions

We have demonstrated the Hyers-Ulam stability of the following generalized QFE:

$$\sum_{i=1}^{m} \wp\left(2x_i - \sum_{1\leq i\neq j}^{m} x_j\right) = (m-7)\sum_{1\leq i<j\leq m} \wp(x_i + x_j) + \wp\left(\sum_{i=1}^{m} x_i\right) - (m^2 - 9m + 5)\sum_{i=1}^{m} \wp(x_i)$$

using direct and fixed-point methods in FNS. Also provide an example for non-stability of given QFE of m variable.

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