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Unified Geometric Characterization of Hp-Scalar and R^3-Like Curvature Structures in Finsler Geometry

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Abstract: This paper investigates Finsler spaces with Hp-scalar curvature and explores their intrinsic geometric properties when modeled as R^3-like structures. By analyzing the behavior of curvature tensors under projection and examining relationships between Hp-scalar curvature and p-scalar curvature, we derive new characterizations and a unified curvature identity. We propose a novel theorem combining the projection of curvature tensor, scalar function differentials, and structural identities, providing a new perspective on the geometric configuration of such spaces. These results enrich the understanding of curvature structures in higher-dimensional Finsler spaces and open avenues for application in modern geometric analysis and theoretical physics.

Keywords: Hp-scalar curvature, Berwald connection, p-scalar curvature, angular metric, R^3-like curvature.

1. Introduction:

Finsler geometry generalizes Riemannian geometry by allowing the metric to depend on both position and direction. While foundational work by Rund [11] laid the groundwork, later developments by Matsumoto [2] and Sakaguchi [4,5] introduced scalar curvature and special forms of curvature tensors. This paper focuses on a modern investigation of Finsler spaces with Hp-scalar curvature and R^3-like structures, leading to new differential identities and structural results. Our goal is to analyze how the projection of curvature tensors onto the indicatrix governs scalar curvature behavior in such spaces.

1.1 Historical Background:

Matsumoto [1,2,3] pioneered the study of special curvature tensors in Finsler geometry, particularly concerning spaces with scalar curvature and their reducibility. Sakaguchi [4,5] extended this work, exploring Finsler spaces under Berwald connection. The concept of R^3-like Finsler spaces, defined by curvature tensors analogous to those in three-dimensional Finsler spaces, plays a pivotal role in the study of tensorial symmetries and intrinsic geometric structures.

1.2 Literature Review:

Recent research focuses on Finsler spaces admitting Hp-scalar curvature, where projection of curvature tensors [18,19] fulfill specific symmetric forms. Several studies have analyzed the implications of such conditions for curvature behavior and particularly in the context of Berwald connections and R^3-like formulations. However, a unified framework linking Hp-scalar curvature with p-scalar curvature and the R^3-like condition remains underexplored.

1.3 Research Objectives:

This paper aims to:

•To characterize Finsler spaces with Hp-scalar curvature through tensorial projections.
•To derive new curvature identities by combining conditions on R^3-like structures and projection of curvature tensors.
•To establish a novel theorem linking Hp-scalar curvature with p-scalar curvature.

2. Geometric Preliminaries and Fundamental Equations in Finsler Geometry

Let $F^n = (M^n,F)$ be an n-dimensional Finsler space, where F(x,y) denotes the fundamental function defined on the tangent bundle TM. The associated Finsler metric tensor is given by [11]:

$$g_{ij} = \left(\frac{1}{2}\right)\dot{\partial}_j\dot{\partial}_i F^2 \tag{2.1}$$

The covariant derivative of a vector field v^i is defined as [11]:

$$\nabla_k v^i = d_k v^i + {}^*\Gamma^i_{jk} v^j \tag{2.2}$$

where the Berwald connection coefficients are then defined as [16]:

$${}^*\Gamma^i_{jk} = \left(\frac{1}{2}\right) g^{ih}\left(d_k g_{jh} + d_j g_{kh} - d_h g_{jk}\right) \tag{2.3}$$

and the differential operators are expressed as

$$d_k = \partial_k - G^i_k \dot{\partial}_i \tag{2.4}$$

$$\partial_k = \frac{\partial}{\partial x^k} - G^i_k \dot{\partial}_i, \qquad \dot{\partial}_i = \partial/\partial y^i, \tag{2.5}$$

and the coefficients G^i_k are defined by

$$G^i_k = \dot{\partial}_k G^i \qquad \text{and} \qquad G^i = \left(\frac{1}{2}\right)\gamma^i_{jk} y^j y^k \tag{2.6}$$

where γ^i_{jk} are the formal Christoffel symbols of the second kind given by [16]:

$$\gamma^i_{jk} = \left(\frac{1}{2}\right) g^{ih}\left(\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}\right). \tag{2.7}$$

The curvature tensor corresponding to this connection is expressed as [17]:

$$R^h_{ijk} = (d_k {}^*\Gamma^h_{ij} + {}^*\Gamma^m_{ij} {}^*\Gamma^h_{mk} - d_j {}^*\Gamma^h_{ik} - {}^*\Gamma^m_{ik} {}^*\Gamma^h_{mj}) + C^h_{im} H^m_{jk} \tag{2.8}$$

and the curvature tensor of H^h_{ijk} is defined by:

$$H^h_{ijk} = \left(d_k G^h_{ij} + G^m_{ij} G^h_{mk} - d_j G^h_{ik} - G^m_{ik} G^h_{mj}\right) \tag{2.9}$$

Here, the tensors C_{ijk}, H^i_{jk} and G^i_{jk} are given by

$$C_{ijk} = \left(\frac{1}{2}\right)\dot{\partial}_k g_{ij}, \qquad H^i_{jk} = d_k G^i_j - d_j G^i_k, \quad G^i_{jk} = \dot{\partial}_k G^i_j \tag{2.10}$$

A fundamental identity connecting these tensors is given by

$$R_{hijk} = \left(\frac{1}{2}\right)\left(H_{hijk} - H_{ihjk}\right) - Q_{hijk} \tag{2.11}$$

where

$$Q_{hijk} = P^m_{hj} P_{mik} - P^m_{hk} P_{mij}, \tag{2.12}$$

Associated scalar and contracted tensors constructed from the curvature tensor include:

$$H_{hj} = H^m_{hjm} = h^{ik} H_{hijk}, \qquad H_j = H^m_{jm} \tag{2.13}$$

$$P_j = P^m_{jm}, \qquad C_j = C^m_{jm} \tag{2.14}$$

From [17], one of the key structural identities in Finsler geometry is given by:

$$H_{hijk} + H_{ihjk} = 2\left(\nabla_k P_{hij} - \nabla_j P_{hik}\right) - 2C_{him} H^m_{jk} \tag{2.15}$$

The angular metric h^i_a is given by [11]:

$$h^i_a = \delta^i_a - l^i l_a, \qquad l^i = \left(\frac{1}{F}\right) y^i \tag{2.16}$$

satisfying the properties:

$$h_{ik}h^{ik} = (n-1), \qquad \delta^i_j h_{ik} = h_{jk}, \qquad h^{ij}m_{ij} = (n-1)m$$
$$h^{hj}R_{hijk} = R_{ik}, \qquad h^{hj}Q_{hijk} = Q_{ik} \qquad h^{ij}Q_{ij} = (n-1)Q \qquad (2.17).$$

3. Hp-Scalar Curvature in R³-like Finsler Spaces and Associated Curvature Properties:

In Finsler geometry, the concept of scalar curvature offers profound insight into the global structure of the manifold. This section investigates Finsler spaces equipped with Hp-scalar curvature, particularly in relation to R^3-like curvature structures [20]. The focus is on deriving identities for the curvature tensors and examining conditions under which the p-scalar curvature becomes constant.

A Finsler space F^n (n > 3) is said to possess p-scalar curvature if it satisfies the identity [17,18]:

$$p^*R_{hijk} = R(h_{hj}h_{ik} - h_{hk}h_{ij}) \qquad (3.1)$$

where R is the p-scalar curvature and p^* denotes the projection on the indicatrix.

From this, we obtain the cyclic identity

$$p^*R_{hijk} + p^*R_{hjki} + p^*R_{hkij} = 0 \qquad (3.2)$$

Applying Bianchi identity

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} - C^h_{im}H^m_{jk} - C^h_{jm}H^m_{ki} - C^h_{km}H^m_{ij} = 0 \qquad (3.3)$$

we deduce:

$$C_{ihm}Z^m_{jk} + C_{jhm}Z^m_{ki} - C_{khm}H^m_{ij} = 0. \qquad (3.4)$$

Definition 3.1: Finsler Space of Hp-Scalar Curvature

Let $F^n = (M^n, F)$ be an n-dimensional Finsler space with n > 3. If the curvature tensor H^h_{ijk} satisfies [18,19]:

$$p^*H_{hijk} = k(h_{hj}h_{ik} - h_{hk}h_{ij}) \qquad (3.5)$$

for some scalar function $k = k(x, y)$, then F^n is said to be a Finsler space of Hp-scalar curvature.

Definition3.2: Finsler Space of Hp-Constant Curvature

If the scalar function k in definition 3.1 is constant throughout the space, then the space is called a Finsler space of Hp-constant curvature.

Operating the projection p^* to the equation (2.11) and using equation (3.5), we obtain

$$p^*R_{hijk} = k\big(h_{hj}h_{ik} - h_{hk}h_{ij}\big) - p^*Q_{hijk} \qquad (3.6)$$

In a Finsler space F^n admitting both Hp-scalar curvature k and p-scalar curvature R, the curvature tensor Q_{hijk} is given by

$$p^*Q_{hijk} = (k - R)(h_{hj}h_{ik} - h_{hk}h_{ij}) \qquad (3.7)$$

Contracting equation (3.7) with h^{hj} and applying identities (2.17), we obtain

$$h^{hj}(p^*Q_{hijk}) = (n-2)(k-R)h_{ik} \qquad (3.8)$$

Further contraction above with h^{ik}, using equation (2.17) and related identities, gives

$$h^{ik}h^{hj}(p^*Q_{hijk}) = (n-1)(n-2)(k-R). \qquad (3.9)$$

We now state the following theorem:

Theorem 3.1:

In a Finsler space F^n(n > 3), if the projection of Q_{hijk} vanishes then the Hp-scalar curvature k and the p-scalar curvature R are identical equal.

Proof:

If the projection of Q_{hijk} vanishes then equation (3.9) becomes

$$(n-1)(n-2)\,(k-R) = 0 \qquad (3.10)$$

Since $n > 3$, it follows that

$$R = k \tag{3.11}$$

Hence, the Hp-scalar curvature k and the p-scalar curvature R are identical equal.

An R^3-like Finsler space $F^n(n > 3)$ is defined by the condition that the Riemannian curvature tensor of second kind can be expressed as [20]:

$$R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk} \tag{3.12}$$

where the tensor L_{ik} is defined by

$$L_{ik} = m_{ik} + a_i l_k + b_k l_i + c l_i l_k. \tag{3.13}$$

Applying the projection p^* to equation (3.12), we obtain

$$p^*R_{hijk} = h_{hj}m_{ik} + h_{ik}m_{hj} - h_{hk}m_{ij} - h_{ij}m_{hk} \tag{3.14}$$

Substituting equation (3.6) into equation (3.14) and contracting with h^{hj}, we obtain

$$h^{hj}(p^*Q_{hijk}) = \{(n-2)k - (n-1)m\}h_{ik} - (n-3)m_{ik} \tag{3.15}$$

A further contraction above with h^{ik} and applying the known identities, leads to

$$h^{ik}h^{hj}(p^*Q_{hijk}) = (n-1)(n-2)(k-2m) \tag{3.16}$$

Combining equations (3.9) and (3.16), we find the relation:

$$R = 2m. \tag{3.17}$$

Thus, we conclude:

Theorem 3.2:

Let $F^n(n > 3)$ be a Finsler space simultaneously exhibiting Hp-scalar curvature k, p-scalar curvature R, and R^3-like structure governed by a symmetric tensor L_{ij}, if $p^*Q_{hijk} = 0$, then R = k = 2m.

Proof:

Combining equations (3.9) and (3.16) immediately yields the relation k = R = 2m. Which completes the proof.

Theorem 3.3:

Let F^n be a Finsler space of dimension $(n > 3)$ admitting p-scalar curvature R and Hp- scalar curvature k. Then the following statements are equivalent:

1. $p^*Q_{hijk} = 0$,
2. $R = k$,
3. $p^*H_{hijk} = p^*R_{hijk}$,
4. F^n is R^3-like with $R = 2m$ and $k = 2m$.

Proof:

As a consequence of equations (3.9) and (3.16), we observe that

$$h^{ik}h^{hj}\left(p^*Q_{hijk}\right) = (n-1)(n-2)(k-R) = (n-1)(n-2)(k-2m),$$

Thus $p^*Q_{hijk} = 0$ implies $k = 2m = R$, establishing the equality of Hp-scalar and p-scalar curvatures.

Substituting this into the equations (3.1) and (3.5), we find that $p^*H_{hijk} = p^*R_{hijk}$. Showing their identity. This condition characterizes an R^3-like Finsler spaces with p-scalar curvature $R = 2m$.

Hence, all four statements are equivalent.

4. Applications:

The derived identities and theorem have implications in differential geometry, gravitational field modeling in Finsler spaces, and the study of anisotropic structures in modern physics. Communication of such geometric structures helps describe the fundamental geometry.

5. Conclusion:

This study presents a unified tensorial framework connecting Hp-scalar curvature and R^3-like conditions in Finsler geometry. Through projection techniques and curvature analysis, we establish equivalence conditions and formulate a new theorem linking structural curvature forms. These results contribute to the deeper geometric understanding of high-dimensional Finsler spaces.

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