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Injectivity of Extended Intuitionistic Fuzzy (EIF) G-Module

Ram Lakhan Pandey[1], Premlata Verma[2] and Dildar Singh Tandon[3]

[1]Dept. of Mathematics, Atal Bihari Vajpayee Vishwavidyaya Bilaspur, Chhattisgarh - 495009, India

[2]Dept. of Mathematics, Government Swami Atmanand English medium college Bilaspur Chhattisgarh - *490001,* India

[3]Dept. of Mathematics, Govt. Agrasen College Bilha, Bilaspur, Chhattisgarh Pin - 495224, India

[1]*shrirlpandey@gmail.com/*

[2]*verma.premlata@yahoo.com*

[3]*dstandon1983@gmail.com*

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Abstract*:* This article presents the concepts of injectivity and quasi-injectivity within the framework of extended intuitionistic fuzzy (EIF) G-modules. It specifically investigates the injectivity of an extended intuitionistic fuzzy (EIF) G-module in the context of being a quotient of another G-module. Several key properties related to the injectivity of such modules are analyzed, particularly in relation to the direct sum of extended intuitionistic fuzzy G-modules. Furthermore, the study establishes a connection between injectivity and quasi-injectivity in the setting of intuitionistic fuzzy G-modules.

Keywords*:* G-module homomorphism, Injective modules, Quasi-injective modules, EIF G-modules.

Mathematics Subject Classification: 03E72, 08A72, 08B30

1. INTRODUCTION

One of the fundamental structures in modern mathematics is fuzzy algebra, which has found extensive applications across a wide range of disciplines, including computer science, information technology, theoretical physics, and control engineering. Since its inception, the concept of fuzzy algebra has undergone significant extensions, particularly over the past few decades. In 1975, Ralescu and Năegoita [10] applied the theory of fuzzy sets to module theory, laying the groundwork for subsequent investigations into fuzzy submodules. Following this foundational contribution, various forms of fuzzy submodules have been rigorously examined, especially in the last two decades.

In 1989, Biswas [5] extended the framework of intuitionistic fuzzy sets to group theory by studying intuitionistic fuzzy subsets of groups. This work marked a turning point, prompting an extensive body of research on fuzzy and intuitionistic fuzzy submodules, with a particular focus on their prime and primary variants.

The classical concept of module injectivity was initially introduced by Eckmann and Schopf [6], and was later extended by Banaschewski [4], who developed essential theoretical tools related to both injective and projective modules. Building upon these foundational ideas, several generalizations of injectivity were proposed. Johnson and Wong [9] introduced the notion of quasi-injective modules, while Azumaya [3] developed the concept of M-injective modules. Singh [17] contributed further by defining pseudo-injective modules. Following Zadeh's pioneering work on fuzzy sets [21], the focus of algebraic research began to include fuzzy structures. Rosenfeld [13] was the first to define fuzzy subgroups, which catalyzed the development of fuzzy algebra. Building on this, Năegoita and Ralescu [10] introduced fuzzy submodules, and Zahedi and Ameri [20] explored the notions of fuzzy projectivity and injectivity within module theory.

As a significant extension of fuzzy set theory, Atanassov [1, 2] introduced the theory of intuitionistic fuzzy sets, thereby broadening the scope of fuzzy mathematics. Biswas [5] applied this framework to group theory, establishing the intuitionistic fuzzy analogue of subgroup structures. Subsequent studies expanded these concepts to encompass intuitionistic fuzzy subrings, submodules, and related algebraic structures [7, 8]. More recently, researchers have investigated intuitionistic fuzzy G-modules, examining their structural properties, representations, and conditions for reducibility and complete reducibility [18, 19].

In continuation of this evolving line of inquiry, the present paper introduces and examines the concept of injectivity within the framework of extended intuitionistic fuzzy G-modules. Specifically, we define and analyze the notions of injectivity and quasi-injectivity for extended intuitionistic fuzzy G-modules, and explore several of their fundamental properties. Throughout this paper, R and C denote the fields of real and complex numbers, respectively. Unless stated otherwise, all G-modules are considered over a field K, where $K \subseteq C$, and all homeomorphisms are assumed to be G-module homeomorphisms.

2. PRELIMINARIES

Definition 2.1 [3]: Let G be a group and let M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, $\exists$ a product (called the action of G on M), $gm \in M$ satisfies the following axioms:

(i) $1_G m = m, \forall m \in M$ (1_G being the identity of G).

(ii) $(gh)m = g(hm), \forall m \in M, g, h \in G$.

(iii) $g(k_1m_1 + k_2m_2) = k_1(gm_1) + k_2(gm_2), \forall k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

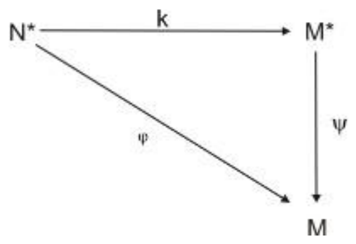
Definition 2.2 [15]: Let G be a group and let M be a G-module over the field K. Let N be a subspace of the vector space M over K. Then N is called a G-submodule of M if $an_1 + bn_2 \in N$, for all $a, b \in K$ and $n_1, n_2 \in N$.

Definition 2.3 [15]: Let M and M^* be G-modules. A mapping $f: M \rightarrow M^*$ is called a G-module homomorphism if

(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$

(ii) $f(gm) = gf(m), \forall k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$.

Definition 2.4 [3]: A G-module M is said to be injective if for any G-module M^* and for any G-submodule N^* of M^*, every monomorphism from N^* into M can be extended to a homomorphism from M^* into M.



In other words, a G-module M is said to be injective if for all homomorphism ϕ: $N^* \to M$ and injection k: $N^* \to M^*$, there exists homomorphism $\psi : M^* \to M$ such that $\psi o k = \phi$ (i.e., injection defined on submodules can be extended to the entire module. i.e., $\psi \mid N^* = \phi$).

Definition 2.5 [3]: Let M and M^* be G-modules. Then M is M^*-injective if for every G-submodule N^* of M^*, any homomorphism $\phi : N^* \to M$ can be extended to a homomorphism $\psi : M^* \to M$.

Remark 2.1 [18]:

a. Let $M = M_1 \oplus M_2$, where M_1 and M_2 are G-submodules of M. Then M is injective if and only if M_1 and M_2 are both injective.
b. A G-module M is called quasi-injective, or self-injective, when it is M- injective.

3. INJECTIVITY OF EIF G-MODULE

Definition 3.1: Let G be a group and let E be a G-module over K, which is a subfield of C. Then an extended intuitionistic fuzzy G-module (EIF-G module) on E is an extended intuitionistic fuzzy set (EIFS) $A = (\alpha_A, \beta_A)$ such that following conditions are satisfied:

i) $0 \le (\alpha_A(x))^2 + (\beta_A(x))^2 \le 1 \quad \forall\ x \in E$

ii) $\alpha_A{}^2(ax+by) \ge \alpha_A{}^2(x) \wedge \alpha_A{}^2(y)$ and $\beta_A{}^2(ax+by) \le \beta_A{}^2(x) \vee \beta_A{}^2(y) \quad \forall a, b \in K$ and $x, y \in E$

iii) $\alpha_A{}^2(gm) \ge \alpha_A{}^2(m)$ and $\beta_A{}^2(gm) \le \beta_A{}^2(m)\ \forall g \in G, m \in E$

Example 3.1: Let G={1,-1} be a group and E=R over R. Then E is a EIF G-module. Define a extended intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ on M by

$$\alpha_E = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if } x \ne 0 \end{cases} \quad ; \quad \beta_E = \begin{cases} 0 & \text{if } x = 0 \\ 0.3 & \text{if } x \ne 0 \end{cases}$$

Then A is an extended intuitionistic fuzzy G-module (EIF G-module) on E.

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Proposition 3.1: Let $A = (\alpha_A, \beta_A)$ is an EIF G-module on E. then the extended intuitionistic fuzzy set $A_{0.1} = (\alpha_{A_{0.1}}, \beta_{A_{0.1}})$, $A_{0.2} = (\alpha_{A_{0.2}}, \beta_{A_{0.2}})$, $A_{0.3} = (\alpha_{A_{0.3}}, \beta_{A_{0.3}}) \ldots\ldots A_{0.9} = (\alpha_{A_{0.9}}, \beta_{A_{0.9}})$ defined by $\alpha_{A_r}{}^2(x) = \alpha_A{}^2(x) \wedge r$ and $\beta_{A_r}{}^2(x) = \beta_A{}^2 \vee (1-r)$ $\forall x \in E$ and for each $r \in [0,1]$ are EIF G-module on E.

Proof: Let $x, y \in E$, $a, b \in K$ then, $\alpha_{A_r}{}^2(ax+by) = \alpha_A{}^2(ax+by) \wedge r \geq \alpha_A{}^2(x) \wedge \alpha_A{}^2(y) \wedge r = (\alpha_A{}^2(x) \wedge r) \wedge (\alpha_A{}^2(y) \wedge r) = \alpha_{A_r}{}^2(x) \wedge \alpha_{A_r}{}^2(y)$. Similarly, $\beta_{A_r}{}^2(ax + by) = \beta_A{}^2(ax+by) \vee (1-r) \leq \beta_A{}^2(x) \wedge \beta_A{}^2(y) \wedge (1-r) = \beta_A{}^2(x) \wedge (1-r)) \wedge \beta_A{}^2(y) \wedge (1-r)) = \beta_{A_r}{}^2(x) \wedge \beta_{A_r}{}^2(y)$ $\forall g \in G$; $m \in E$. $\alpha_{A_r}{}^2(gx) = \alpha_A{}^2(gx) \wedge r \geq \alpha_A{}^2(x) \wedge r = \alpha_{A_r}{}^2(x)$. Also, $\beta_{A_r}{}^2(gx) \leq \beta_{A_r}{}^2(x)$. Thus A_r is EIF G-module on E.

Remark 3.1: Let $A = (\alpha_A, \beta_A)$ is an EIF G-module of a G module E. and let F be a G-submodule of E. Then the restriction of A on F is an EIFS denoted by $A|_F = (\alpha_{A|_F}, \beta_{A|_F})$ and is defined by $\alpha_{A|_F}{}^2(x) = \alpha_A{}^2(x)$ and $\beta_{A|_F}{}^2(x) = \beta_A{}^2(x)$ $\forall x \in E$ is an EIF G-module on F.

Proposition 3.2: Let $A = (\alpha_A, \beta_A)$ is an EIF G-module of a G module E. and let F be a G-submodule of E. Then the EIFS $A_F = (\alpha_{A_F}, \beta_{A_F})$ of E/F defined by $\alpha_{A_F}{}^2(x + F) = \alpha_A{}^2(x)$ *and* $\beta_{A_F}{}^2(x+F) = \beta_A{}^2(x)$ $\forall x \in E$ is an EIF G-module on E/F.

Proof: For $x + F$, $y + F \in E/F$, $g \in G$ and scalar $a, b \in K$, we have $\alpha_{A_F}{}^2\{a(x+F) + b(y+F)\} = \alpha_{A_F}{}^2\{(ax+by)+F)\} = \alpha_A{}^2(ax+by) \geq \alpha_A{}^2(x) \wedge \alpha_A{}^2(y) = \alpha_{A_F}{}^2(x+F) \wedge \alpha_{A_F}{}^2(y+F)$. Similarly, $\beta_{A_F}{}^2\{a(x+F)+b(y+F)\} = \beta_{A_F}{}^2\{(ax+by)+F)\} = \beta_A{}^2(ax+by) \leq \beta_A{}^2(x) \wedge \beta_A{}^2(y) = \beta_{A_F}{}^2(x+F) \wedge \beta_{A_F}{}^2(y+F)$. Also, $\alpha_{A_F}{}^2(g(x+F)) = \alpha_{A_F}{}^2(gx+F) = \alpha_A{}^2(gx) \geq \alpha_A{}^2(x) = \alpha_{A_F}{}^2(x+F)$. Similarly, $\beta_{A_F}{}^2(g(x+F)) \leq \beta_{A_F}{}^2(x+F)$. Thus A_F is an EIF G-module on E/F.

Remark 3.2: Let E be a G-module and let A be an EIFS on E, then A is an EIF G-module on E if and only if either $C_{(\mu,\nu)}(A) = \phi$ or $C_{(\mu,\nu)}(A)$ for all $\mu, \nu \in [0,1]$ such that $\mu^2 + \nu^2 \leq 1$ is a G submodule of E, where $C_{(\mu,\nu)}(A) = \{x \in E : \alpha_A{}^2(x) \geq \mu^2$ and $\beta_A{}^2(x) \leq \nu^2\}$

Definition 3.2: If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be EIF G-module of G-module E and E^* respectively. A function $f: E \to E^*$ is said to be a function from A to B if $(\alpha_B of) = \alpha_A$ and $(\beta_B of) = \beta_A$.

Remark 3.3: If f is a G-module homomorphism or G-epimorphism or G-isomorphism, from E to E^*, then f is said to be EIF G-module homomorphism or G-epimorphism or G-isomorphism from A to B. Suppose E and E^* be G-modules and let E be E^*-injective. Then for every monomorphism $f_3: F^* \to E$ and injection $f_1: F^* \to E^*$, there exists homomorphism $f_2: E^* \to E$ such that $f_2 o f_1 = f_3$, where F^* is a G-submodule of G-moduleE^*. In other words the map f_3 extends to f_2, i.e., $f_2|_{F^*} = f_3$ If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be EIF G-module of G-module E and E^* respectively and $(\alpha_{B|_{F^*}}, \beta_{B|_{F^*}})$ be the EIF G-module of F^*. Then A is said to be B-injective if the following figure 2 is commutative.

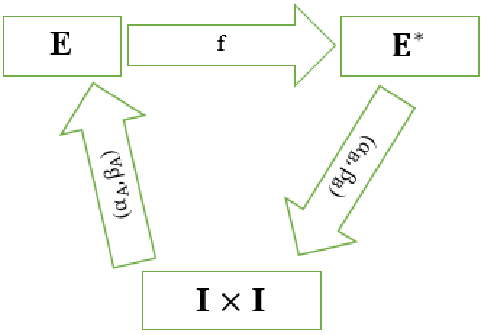

Figure -1

That is,

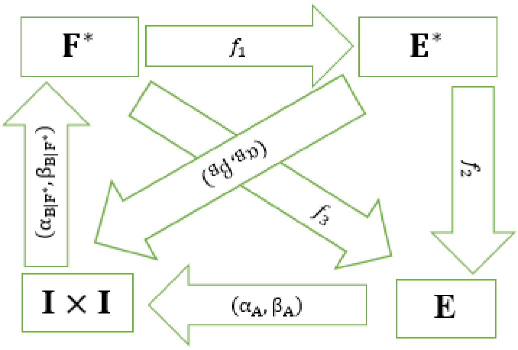

Figure -2

$\alpha_{B|_{F^*}} = \alpha_B o f_1$ and $\beta_{B|_{F^*}} = \beta_B o f_1$; $\alpha_B = \alpha_A o f_2$ and $\beta_B = \beta_A o f_2$, $\alpha_{B|_{F^*}} = \alpha_A o f_3$ and $\beta_{B|_{F^*}} = \beta_A o f_3$. Thus, $\alpha_B(f_1(m)) = \alpha_B o f_1(m) = \alpha_{B|_{F^*}}(m) = \alpha_A o f_3(m) = \alpha_A(f_3(m)) = \alpha_A((f_2 o f_1)(m)) = \alpha_A(f_2(f_1(m)))$ and $\beta_B(f_1(m)) = \beta_B o f_1(m) = \beta_{B|_{F^*}}(m) = \beta_A o f_3(m) = \beta_A(f_3(m)) = \beta_A((f_2 o f_1)(m)) = \beta_A(f_2(f_1(m)))$. If $m \in F*$, then $f_1(m) = m$, $\alpha_B(m) = \alpha_A(f_2(m))$ and $\beta_B(m) = \beta_A(f_2(m))$. If $m \in E^{*/} F^*$ then $\alpha_B(f_1(m)) = 0 \leq \alpha_A(f_2(m))$ and $\beta_B(f_1(m)) = 1 \geq \beta_A(f_2(f_1(m)))$. Hence $\forall f_2 \in Hom(E*, E)$ and $m \in E*$ $\alpha_B(m) \leq \alpha_A(f_2(m))$ and $\beta_B(m) \geq \beta_A(f_2(m))$

Definition 3.3:. Let E and E^* be G-modules. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be EIF G-module of G-module E and E^* respectively. Then A is B-injective if

i) E is E^*- injective and

ii) $\alpha_B{}^2(m) \leq \alpha_A{}^2(f_2(m))$ and $\beta_B{}^2(m) \geq \beta_A{}^2(f_2(m))$ $\forall$ $f \in Hom(E*, E)$ and $m \in E^*$

Proposition 3.3: Let E and E^* be G-modules and A, B EIF G-module on E and E^* respectively such that A is B-injective. If F^* is a G-submodule of E^* and C is EIF G-module on F^*, then A is C-injective if $C \subseteq B|_{F^*}$.

Proof: Since E and E^* –injective and F^* is a G-submodule of E^*, then, E is F^*-injective. Let $f_1 \in Hom(F^*, E)$. Since E is E^*-injective, there exist an extension homomorphism $f_2: E^* \to E$. Thus $f_1 = f_2|_{F^*}$. Since A is B-injective, we have $\forall n \in F^*, \alpha_B{}^2(n) \leq \alpha_A{}^2(f_2(m)) = \alpha_A{}^2(f_1(n))$ and $\beta_B{}^2(n) \geq \beta_A{}^2(f_2(m)) = \beta_A{}^2(f_1(n))$. Since $C \subseteq B|_{F^*}$, $\alpha_C{}^2(n) \leq \alpha_B{}^2(n)$ and $\beta_C{}^2(n) \geq \beta_A{}^2(f_1(n))$ $\forall n \in F^*$. Hence we have, $\alpha_C{}^2(n) \leq \alpha_A{}^2(f_1(n))$ and $\beta_C{}^2(n) \geq \beta_A{}^2(f_1(n))$ $\forall f_1 \in Hom(F^*, E)$ $\forall n \in F^*$. Therefore A is C-injective.

Remark 3.4: Let E and E^* be G-modules and let A and B be EIF G-modules on E and E^* respectively such that A is B-injective. Then for every r ∈ [0, 1], the EIFS $B_r = (\alpha_{B_r}, \beta_{B_r})$ where $\alpha_{B_r}{}^2(m) = \alpha_B{}^2(m) \wedge r$ and $\beta_{B_r}{}^2(m) = \beta_B{}^2(m) \vee (1 - r)$, $\forall\, m \in E^*$, is an EIF G-module on E^* and A is B_r-injective.

Proposition 3.4: Let A and B be EIF G-modules on the G-modules E and E^* respectively such that A is B-injective. For any G-submodule F^* of E^*, define the EIFS B_{F^*} on E^*/F^* by, $\alpha_{B_{F^*}}{}^2(m + F^*) = \alpha_B{}^2(m)$ and $\beta_{B_{F^*}}{}^2(m + F^*) = \beta_B{}^2(m)\ \forall\ m \in E^*$. Then B_{F^*} is an EIF G-module on E^*/F^* and A is B_{F^*} -injective.

Proof: It follows from previous results that B_{F^*} is an EIF G-module on E^*/F^*. Since F^* is a G-submodule of E^*, then E is E^*/F^*-injective. Let f_1 $\mathrm{Hom}(E^*/F^*, E)$. Since E is E^*-injective, there exist an extension $f_2 \in \mathrm{Hom}(E^*, E)$. Since A is B-injective and $f_2 \in \mathrm{Hom}(E^*, E)$, we have $\alpha_B{}^2(m) \le \alpha_A{}^2(f_2(m))$ and $\beta_B{}^2(m) \ge \beta_A{}^2(f_2(m))$ $\forall m \in E^*$. For any $m \in E^*, m + F^* \in E^*/F^*$, we have, $\alpha_A{}^2(f_2(m + F^*)) = \alpha_A{}^2(f_2(m) + 0)) = \alpha_A{}^2(1.f_2(m) + 1.0)) \ge \alpha_A{}^2(f_2(m)) \wedge \alpha_A{}^2(0) \ge \alpha_A{}^2(f_2(m))$ and $\beta_A{}^2(f_2(m + F^*)) = \beta_A{}^2(f_2(m) + 0)) = \beta_A{}^2(1.f_2(m) + 1.0)) \le \beta_A{}^2(f_2(m)) \wedge \beta_A{}^2(0) \le \beta_A{}^2(f_2(m))$. Thus, $\alpha_A{}^2(f_2(m + F^*)) \ge \alpha_A{}^2(f_2(m))$ and $\beta_A{}^2(f_2(m + F^*)) \le \beta_A{}^2(f_2(m))$. Also $\alpha_{B_{F^*}}{}^2(m + F^*) = \alpha_B{}^2(m) \le \alpha_A{}^2(f_2(m)) \le \alpha_A{}^2(f_2(m + F^*)) \le \alpha_A{}^2(f_1(m + F^*))$ $\forall\, f_1 \in \mathrm{Hom}(E^*/F^*, E)$ and $\beta_{B_{F^*}}{}^2(m + F^*) = \beta_B{}^2(m) \ge \beta_A{}^2(f_2(m)) \ge \beta_A{}^2(f_2(m + F^*)) \ge \beta_A{}^2(f_1(m + F^*))$ $\forall\, f_1 \in \mathrm{Hom}(E^*/F^*, E)$. Therefore, $\forall\, f_1 \in \mathrm{Hom}(E^*/F^*, E)$, $\alpha_{B_{F^*}}{}^2(m + F^*) \le \alpha_A{}^2(f_1(m + F^*))$ and $\beta_{B_{F^*}}{}^2(m + F^*) \ge \beta_A{}^2(f_1(m + F^*))$. Hence A is $B_{F^*} -$ injective.

Definition 3.4: Let E be a G-module and let A be an EIF G-module on E. Then A is quasi-injective if

i) E is quasi-injective and
ii) $\alpha_A{}^2(m) \le \alpha_A{}^2(f(m))$ and $\beta_A{}^2(m) \ge \beta_A{}^2(f(m))$ $\forall\, f \in \mathrm{Hom}(E, E)$ and $m \in E$

Example 3.2: Every constant EIFS defined on a quasi-injective G-module E is always quasi-injective module i.e., if E is a quasi-injective G-module. Then an EIFS $A = (\alpha_A, \beta_A)$ on E defined by $\alpha_A{}^2(x) = r$ and $\beta_A{}^2(x) = r', \forall\, x \in E$ where $r, r' \in [0,1]$ such that $r + r' \le 1$ is quasi-injective.

Remark 3.5: Let E be a G-module and let $E = \bigoplus_{i=1}^{n} E_i$ where E_i 's are G-submodules of E. If A_i's $(1 \le i \le n)$ are EIF G- modules on E_i's ,then an EIFS A of E defined by $\alpha_A{}^2(m) = \wedge \{\alpha_{A_i}{}^2(m_i) : i = 1, 2, \dots\dots, n\}$ and $\beta_A{}^2(m) = \vee \{\beta_{A_i}{}^2(m_i) : i = 1, 2, \dots\dots, n\}$. Where $m = \sum_{i=1}^{n} m_i \in E$ is an EIF G-module on E.

Definition 3.5: An EIF G-module A on $E = \bigoplus_{i=1}^{n} E_i$. In Remark 3.5 with ${\alpha_A}^2(0) = {\alpha_{A_i}}^2(0)$ and ${\beta_A}^2(0) = {\beta_{A_i}}^2(0)$ $\forall$ I is called the direct sum of A_i and it is written as $A = \bigoplus_{i=1}^{n} A_i$

Remark 3.6: Let E be a G-module and $E = \bigoplus_{i=1}^{n} E_i$ where E_i's are G-submodules of E. Let B_i's be EIF G-modules on E_i and let $B = \bigoplus_{i=1}^{n} B_i$. Let A be any EIF G-modules on E. Then A is B-injective if and only if A is B_i-injective for all i ($1 \leq i \leq n$)

Remark 3.7: Let E_1 and E_2 be two G-submodules of a G-module E such that $E = E_1 \oplus E_2$. If E is a quasi-injective, then E_i is E_j -injective for i, j $\in \{1,2\}$. Further, if B_i's are EIF G-modules on E_i (i = 1, 2) such that $E = B_1 \oplus B_2$ and if B is quasi-injective, then B_i is B_j -injective for i, j $\in \{1,2\}$.

4. CONCLUSION

This study has explored the foundational and structural properties of injectivity and quasi-injectivity within the framework of extended intuitionistic fuzzy (EIF) G-modules. By examining injective behavior in the context of quotient constructions, it offers a deeper understanding of how such properties operate within fuzzy algebraic environments. The investigation also established significant results regarding the preservation of injectivity under direct sums, thereby enriching the broader theoretical landscape of module theory in the extended intuitionistic fuzzy setting.

Furthermore, the research underscores the foundational role of various homomorphisms—homomorphisms, epimorphisms, and isomorphisms—within G-modules, which serve as essential tools in analyzing structural transformations. The notions of EIF G-modules and B-injectivity were examined in relation to their ability to maintain injectivity under specific conditions and within commutative diagrams involving injections, respectively. These conceptual foundations provide a robust basis for advancing the theory of fuzzy modules and their homomorphic properties.

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