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Complex Hypersurfaces in Kählerian Manifolds with Constant Holomorphic Sectional Curvature: Geometric Identities and Curvature Constraints

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Abstract:
This study introduces a new framework for analyzing complex hypersurfaces embedded in Kähler manifolds characterized by vanishing Bochner curvature tensors. Diverging from traditional approaches, we examine the intricate interplay between the second fundamental form and the shape operator under these specific curvature conditions. Our investigation reveals unique curvature identities and scalar invariants that emerge solely due to the vanishing Bochner curvature. Notably, we establish precise conditions under which such hypersurfaces become totally geodesic and constant holomorphic sectional curvature. The findings offer fresh insights into the intrinsic and extrinsic geometry of complex hypersurfaces, potentially influencing future research in complex differential geometry and theoretical physics.

1. Introduction:

1.1 Overview:

The geometry of complex hypersurfaces within Kähler manifolds has long been a subject of profound interest in differential geometry [4]. These hypersurfaces inherit rich structural properties from their ambient spaces, leading to complex interactions between intrinsic and extrinsic geometric features. A particularly intriguing scenario arises when the Bochner curvature tensor vanishes, imposing stringent constraints on the curvature and topology of the hypersurface [1,5]. This paper seeks to delve into these constraints, uncovering new geometric identities and exploring their implications.

1.2. Historical Background:

The Bochner curvature tensor, introduced in the mid-20th century, serves as a pivotal tool in understanding the curvature properties of Kähler manifolds [3]. Historically, research has focused on the implications of Bochner flatness in various contexts, including its role in characterizing Einstein manifolds and its influence on the topology of complex manifolds [2,3]. However, the specific impact of vanishing Bochner curvature on complex hypersurfaces within Kähler manifolds remains underexplored. This gap presents an opportunity to investigate new geometric phenomena arising from this condition.

1.3. Research Gap:

While substantial progress has been made in understanding the curvature properties of Kähler manifolds and their submanifolds, the literature lacks a comprehensive study of complex hypersurfaces under the constraint of vanishing Bochner curvature. Existing studies often overlook the nuanced effects this condition has on the second fundamental form, shape operator, and associated curvature invariants. Moreover, the potential for such hypersurfaces to exhibit total geodesicity or constant holomorphic sectional curvature under these conditions has not been thoroughly examined. Addressing this gap is essential for a deeper comprehension of the geometric structures involved.

1.4 Research Objectives:

This research aims to:

• Derive new curvature identities specific to complex hypersurfaces in Kähler manifolds with vanishing Bochner curvature tensors.
• Investigate the conditions under which these hypersurfaces become totally geodesic.
• Explore the implications of these conditions on the holomorphic sectional curvature of the hypersurfaces.
• Enhance the understanding of the relationship between the second fundamental form, shape operator, and curvature invariants in such contexts.
• Provide a foundation for future studies in complex differential geometry and related fields.

2. Geometric Structures on Complex Hypersurfaces:

On the hypersurface V, there exists a well-defined tensor field f_j^i, induced from the structure tensor F_μ^λ of the ambient space. This tensor field satisfies the following fundamental relations, analogous to those defining an almost complex structure on M^*[4]:

$$F_\mu^\lambda B_j^\mu = f_j^i B_i^\lambda \tag{2.1}$$

$$f_j^l f_l^i = -\delta_j^i, \tag{2.2}$$

$$f_k^j f_m^i \, g_{ji} = g_{km} \tag{2.3}$$

$$f_{ji} + f_{ij} = 0 \tag{2.4}$$

$$F_\mu^\lambda C^\mu = D^\lambda \tag{2.5}$$

where $\frac{\partial x^\lambda}{\partial v^i} = B_i^\lambda$

The Gauss and Weingarten equations for the hypersurface are given as follows [4]:

$$\nabla_j B_i^\lambda = h_{ji} C^\lambda + k_{ji} D^\lambda \tag{2.6}$$

Assuming that the second fundamental forms are proportional to the induced metric, we take:

$$k_{ji} = k_j^l \, g_{li}, \tag{2.7}$$

Differentiating equation (2.1) covariantly along the hypersurface V and employing equations (2.5) and (2.6), we get:

$$\nabla_j f_i^k = 0 \tag{2.8}$$

The following relations hold on the hypersurface V [3]:

$$k_j^i = f_k^i h_j^k \tag{2.9}$$

$$f_l^j h_i^l + h_l^j f_i^l = 0 \tag{2.10}$$

In view of equations (2.6), (2.8) and (2.9), we observe that the complex hypersurface in a Kahlerian manifold M^* is itself a Kahlerian manifold with the induced structure ($g_{ji,} f_j^k$). Then we have $K^*_{\nu\omega\lambda\mu}$ and K_{mkjk} are the curvature tensor of M^* and V respectively.

$$K^*_{\nu\omega\lambda\mu} B_m^\nu B_k^\omega B_j^\mu B_i^\lambda = K_{mkjk} - \left(h_{mi} h_{kj} - h_{ki} h_{mj} \right) - \left(k_{mi} k_{kj} + k_{ki} k_{mj} \right) \tag{2.11}$$

The curvature tensor of M^* along directions tangent to V is given by [4]:

$$K^*_{ji} = K_{ji} - U_{jk} f_i^k \tag{2.12}$$

The Ricci tensors satisfy [3]:

$$S^*_{ji} = S_{ji} - U_{ji} \tag{2.13}$$

where $K^*_{ji} = K^*_{\mu\lambda} B_j^\mu B_i^\lambda \quad (2.14)$

Moreover,

$$S^*_{ji} = f_j^k K^*_{ki} \tag{2.15}$$

The Bochner curvature tensor $B_{\nu\omega\mu\lambda}$ of M^* is defined by[1]:

$$B^*_{\nu\omega\mu\lambda} = K^*_{\nu\omega\mu\lambda} + \frac{1}{2(n+3)} (K^*_{\nu\mu} g_{\omega\lambda} - K^*_{\omega\mu} g_{\nu\lambda} + g_{\nu\mu} K^*_{\omega\lambda} - g_{\omega\mu} K^*_{\nu\lambda}$$

$$+ S^*_{\nu\mu}f_{\omega\lambda} - S^*_{\omega\mu}f_{\nu\lambda} + f_{\nu\mu}S^*_{\omega\lambda} - f_{\omega\mu}S^*_{\nu\lambda} + 2S^*_{\nu\omega}f_{\mu\lambda} + 2f_{\nu\omega}S^*_{\mu\lambda})$$
$$- \frac{K^*}{4(n+2)(n+3)}(g_{\nu\mu}g_{\omega\lambda} - g_{\omega\mu}g_{\nu\lambda} + f_{\nu\mu}f_{\omega\lambda} - f_{\omega\mu}f_{\nu\lambda} + 2f_{\nu\omega}f_{\mu\lambda}) \quad (2.16).$$

3. Geometry of Complex Hypersurfaces with Vanishing Bochner Curvature Tensor:

Let M^* be a complex hypersurface embedded in a 2(n+1)-dimensional Kählerian manifold with constant holomorphic sectional curvature $\check{C}$. The geometry of such hypersurfaces is deeply influenced by the properties of the ambient curvature tensor and the induced structures. The vanishing of the Bochner curvature tensor on M^* imposes significant geometric constraints and leads to remarkable curvature identities.

In the ambient manifold M^*, the Riemannian curvature tensor $K^*_{\nu\omega\mu\lambda}$ is given by:

$$K^*_{\nu\omega\mu\lambda} = \frac{1}{4}\check{C}\left(G_{\nu\lambda}G_{\omega\mu} - G_{\omega\lambda}G_{\nu\mu} + F_{\nu\lambda}F_{\omega\mu} - F_{\omega\lambda}F_{\nu\mu} - 2F_{\nu\omega}F_{\mu\lambda}\right) \quad (3.1)$$

From equations (2.29) and (3.1), the induced curvature tensor on the hypersurface M^* can be written as:

$$K_{mkji} = \frac{1}{4}\check{C}\left(g_{mi}g_{kj} - g_{ki}g_{mj} + f_{mi}f_{kj} - f_{ki}f_{mj} - 2f_{mk}f_{ji}\right)$$
$$+ h_{mi}h_{kj} - h_{ki}h_{mj} + k_{mi}k_{kj} - k_{ki}k_{mj} \quad (3.2)$$

Where h_{ji} denotes the second fundamental form of the hypersurface, and k_{ji} is the corresponding shape operator.

As a consequence of the geometric conditions imposed by the vanishing of the Bochner curvature tensor, we obtain the identity:

$$K_{ji} = \frac{1}{2}(n+1)\check{C}g_{ji} - 2h^k_j h_{ki} \quad (3.3).$$

3.1. Bochner Curvature Tensor of Complex Hypersurfaces in Kähler Manifolds:

Consider a complex hypersurface V immersed in a 2(n+1)-dimensional Kähler manifold M^* with constant holomorphic sectional curvature $\check{C}$. In such a setting, the interplay between the ambient geometry and the intrinsic geometry of the hypersurface becomes particularly rich.

From the geometric construction, we obtain the following identity involving the second fundamental form h_{ji} and the shape operator k_{ji}:

$$S_{ji} = \frac{1}{2}(n+1)\check{C}f_{ji} - 2h^k_j k_{ki} \quad (3.4)$$

and

$$\check{C} = \left\{\frac{1}{n(n+1)}\right\}(2h_{st}h^{st} + K). \quad (3.5)$$

Now, the Bochner curvature tensor B_{mkji} for the complex hypersurface V in M^* of constant holomorphic sectional curvature $\check{C}$ is given by the following expression:

$$B_{mkji} = -\left\{\frac{1}{2(n+1)(n+2)}\right\}h_{st}h^{st}\left(g_{mi}g_{kj} - g_{ki}g_{mj} + f_{mi}f_{kj} - f_{ki}f_{mj} - 2f_{mk}f_{ji}\right)$$
$$+ h_{mi}h_{kj} - h_{ki}h_{mj} + k_{mi}k_{kj} - k_{ki}k_{mj} - \frac{1}{(n+2)}(h^s_m h_{sj}g_{ki} - h^s_k g_{sj}g_{mi}$$
$$+ g_{mj}h^s_k h_{si} - g_{kj}h^s_j h_{si} + h^s_m k_{sj}f_{ci} - h^s_k k_{sj}f_{mi} + f_{mj}h^s_k k_{si} - f_{kj}h^s_m k_{si}$$
$$+ 2h^s_m k_{sk}f_{ji} + 2f_{mk}h^s_j k_{si}) \quad (3.6)$$

3.2. Implications of Vanishing Bochner Curvature in Complex Hypersurfaces:

Assuming the Bochner curvature tensor vanishes, i.e.,

$$B_{mkji} = 0 \quad (3.7)$$

Continuing from the expression of the Bochner curvature tensor B_{mkji} in equation (3.6), we now investigate the consequences of its vanishing by transvecting equation (3.6) with $h^{kj}h^{mi}$ which leads to:

$$(n^2 + 3n + 4)(h_{st}h^{st})^2 = 8(n+1)h^k_m h^j_k h^i_j h^m_i. \quad (3.8)$$

Let $N_1, N_2, N_3, \dots\dots\dots, N_{2n}$ denote the principal curvatures of the complex hypersurface V, with respect to the second fundamental form h_{ji}. Under this consideration, it follows that:

$$h_{st}h^{st} = \sum_{i=1}^{2n} N_i^2 \tag{3.9}$$

$$h_m^k h_k^j h_j^i h_i^m = \sum_{i=1}^{2n} N_i^4 \tag{3.10}$$

In view of equations (3.9) and (3.10), and using identities for symmetric bilinear forms, we derive:

$$h_m^k h_k^j h_j^i h_i^m - (h_{st}h^{st})^2 = \sum_{i=1}^{2n} N_i^4 - (\sum_{i=1}^{2n} N_i^2)^2 \le 0 \tag{3.11}$$

By incorporating the relation given in equation (3.5), the expression (3.11) transforms into the following inequality:

$$(n^2 - 5n - 4)(h_{st}h^{st})^2 \le 0,$$

i.e. $$\{n - \frac{5+\sqrt{41}}{2}\}\{n - \frac{5-\sqrt{41}}{2}\}(h_{st}h^{st})^2 \le 0.$$

Now, under the assumption that $n \geq 6$, the coefficients in the inequality above compel the vanishing of both the second fundamental form and the shape operator, i.e., $h_{st} = 0$, $k_{st} = 0$.
This result implies that for complex hypersurfaces of V, to be totally geodesic in the ambient Kähler manifold of constant holomorphic sectional curvature.

Remark 3.1:
It is to be noted that if we denote the constant holomorphic sectional curvature of V by C, then

$$K_{mkji} = \frac{1}{4}\left(g_{mi}g_{kj} - g_{ki}g_{mj} + f_{mi}f_{kj} - f_{ki}f_{mj} - 2f_{mk}f_{ji}\right) \tag{3.12}$$

The equations (3.2) and (3.6) assume the form

$$\frac{1}{4}\left(\check{C} - C\right)\left(g_{mi}g_{kj} - g_{ki}g_{mj} + f_{mi}f_{kj} - f_{ki}f_{mj} - 2f_{mk}f_{ji}\right) + h_{mi}h_{kj}$$
$$- h_{ki}h_{mj} + k_{mi}k_{kj} - k_{ki}k_{mj} = 0 \tag{3.13}$$

By transvecting equation (3.13) with $h^{kj}h^{mi}$, we obtain:

$$\left(\check{C} - C\right)h_{st}h^{st} + (h_{st}h^{st})^2 = 0. \tag{3.14}$$

Theorem 3.1:
Let V be a complex hypersurface smoothly embedded in a Kählerian manifold M^*. If V is totally geodesic with respect to the ambient connection of M^*, then V necessarily admits constant holomorphic sectional curvature. In this case, the induced geometry on V reflects a uniform holomorphic curvature structure, invariant across all tangent directions.

Proof:
From equations (3.2) and (3.12), we obtain the expression for the holomorphic sectional curvature of the hypersurface V as:

$$\check{C} = \frac{1}{n(n+1)}\left(K + h_{st}h^{st}\right) \tag{3.15}$$

Consequently, the scalar curvature C of V satisfies:

$$C = K/n(n+1) \tag{3.16}$$

From equation (3.14), it follows that:

$$(n+1)(n+2)(h_{st}h^{st})^2 = 0 \tag{3.17}$$

This identity immediately implies:

$$h_{st} = 0 \tag{3.18}$$

which shows that the second fundamental form vanishes identically. The vanishing of the second fundamental form characterizes V as a totally geodesic submanifold of M^*.
Since $h_{st} = 0$, it follows that the associated tensor k_{st} also vanishes, i.e., $k_{st} = 0$.
Substituting above into equation (3.15), we obtain:

$$C = K/n(n+1),$$

which is clearly constant.

Therefore, the complex hypersurface V is totally geodesic, and consequently, it is a Kählerian manifold of constant holomorphic sectional curvature.
From the curvature relations expressed in equations (2.12) and (3.3), we derive the following identity:

$$K^*_{ji} = \frac{(n+1)}{2}\check{C}g_{ji} - 2h_j^k h_{ki} - U_{jk}f_i^k \quad (3.20)$$

Now, substituting equation (2.14) into (3.20), we obtain:

$$K^*_{\mu\lambda}B_j^\mu B_i^\lambda = \frac{(n+1)}{2}\check{C}g_{ji} - 2h_j^k h_{ki} - U_{jk}f_i^k \quad (3.21)$$

Multiplying both sides of equation (3.20) by the inverse tensor f_m^j, we arrive at:

$$K^*_{ji}f_m^j = \frac{(n+1)}{2}\check{C}g_{ji}f_m^j - 2h_j^k h_{ki}f_m^j - U_{jk}f_i^k f_m^j \quad (3.22)$$

Utilizing equation (2.15) together with (3.22), we finally deduce:

$$S^*_{mi} = \frac{(n+1)}{2}\check{C}f_{mi} - 2h_j^k f_m^j h_{ki} - U_{jk}f_i^k f_m^j \quad (3.23)$$

From equations (2.10) and (3.23), we deduce the following expression:

$$S^*_{mi} = \frac{(n+1)}{2}\check{C}f_{mi} + 2h_m^j f_j^k h_{ki} - U_{jk}f_i^k f_m^j \quad (3.24)$$

In view of equations (2.9) and (3.24), we obtain:

$$S^*_{mi} = \frac{(n+1)}{2}\check{C}f_{mi} + 2k_m^k h_{ki} - U_{jk}f_i^k f_m^j \quad (3.25)$$

Now, multiplying both sides by g_{kj}, the equation becomes:

$$S^*_{mi}g_{kj} = \frac{(n+1)}{2}\check{C}f_{mi}g_{kj} + 2k_m^k h_{ki}g_{kj} - U_{jk}f_i^k f_m^j g_{kj} \quad (3.26)$$

As a consequence of equations (2.3) and (3.26), we further get:

$$S^*_{mi}g_{kj} = \frac{(n+1)}{2}\check{C}f_{mi}g_{kj} + 2k_m^k h_{ki}g_{kj} - U_{jk}g_{im} \quad (3.27)$$

Inserting equation (2.7) in equation (3.27), we now derive:

$$S^*_{mi}g_{kj} = \frac{(n+1)}{2}\check{C}f_{mi}g_{kj} + 2k_{mj}h_{ki} - U_{jk}g_{im} \quad (3.28)$$

Utilizing the geometric relations expressed in equations (2.11), (2.12), and (2.16), the Bochner curvature tensor B_{mkji} may be written in the following compact form:

$$\begin{aligned} B_{mkji} = {} & \frac{1}{2(n+2)(n+3)}\big(K_{mj}g_{ki} - K_{kj}g_{mi} + g_{mj}K_{ki} - g_{kj}K_{mi} + S_{mj}f_{ki} - S_{kj}f_{mi}\big) \\ & + f_{mj}S_{ki} - f_{kj}S_{mi} + 2S_{mk}f_{ji} + 2f_{mk}S_{ji}) + \left[\frac{\{(n+1)K^* - (n+3)K\}}{4(n+1)(n+2)(n+3)}\right](g_{mj}g_{ki} - g_{kj}g_{mj} \\ & + f_{mj}f_{ki} - f_{kj}f_{mi} + 2f_{mk}f_{ji}) + \frac{1}{2(n+3)}(U_{me}f_j^e g_{ki} - U_{ke}f_j^e g_{mi} + g_{mj}U_{ke}f_i^e \\ & - g_{kj}U_{me}f_a^e + U_{mj}f_{ki} - U_{kj}f_{mi} + f_{mj}U_{ki} - f_{kj}U_{mi} + 2U_{mk}f_{ji} + 2f_{mk}U_{ji}) \\ & + h_{mi}h_{kj} - h_{ki}h_{mj} + k_{mi}k_{kj} - k_{ki}k_{mj} \end{aligned} \quad (3.29)$$

Since the Bochner curvature tensor satisfies the identity $B_{mkji}g^{mi} = 0$.
We proceed by contracting equation (3.29) with g^{mi}, yielding the expression:

$$\begin{aligned} & 2K_{kj} - Kg_{kj} + \frac{1}{(n+1)(n+2)}K^*g_{kj} + 2(n+2)U_{ke}f_j^e + g_{kj}U_{me}f^{me} \\ & + 4(n+3)h_k^e h_{ej} = 0 \end{aligned} \quad (3.30)$$

Further contraction of equation (3.30) with g^{kj} gives:

$$U_{st}f^{st} = \frac{(n-1)}{2(n+1)}K - \frac{n}{2(n+2)}K^* - \frac{(n+3)}{(n+1)}h_{st}h^{st} \quad (3.31)$$

In view of equations (3.31), equation (3.30) follows:

$$\begin{aligned} (n+2)U_{ke}f_j^e = {} & -K_{kj} + \frac{(n+3)}{4(n+1)}Kg_{kj} - \frac{1}{4}K^*g_{kj} + \frac{(n+3)}{2(n+1)}h_{st}h^{st}g_{kj} \\ & -2(n+3)h_k^e h_{ej} \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} (n+2)U_{kj} = {} & -S_{kj} + \frac{(n+3)}{4(n+1)}Kf_{kj} - \frac{1}{4}K^*f_{kj} + \frac{(n+3)}{2(n+1)}h_{st}h^{st}f_{kj} \\ & -2(n+3)h_k^e k_{ej} \end{aligned} \quad (3.33).$$

Theorem 3.2:
Let M^n be an n-dimensional smooth semi-Riemannian manifold equipped with a Bochner curvature tensor B satisfying the condition $B_{mkji}g^{mi} = 0$, Then the associated scalar curvature invariants satisfy the following identity:

$$4(n + 1)(n + 2)U = (n + 3)\left[K + 2h_{st}h^{st} - 8(n + 1)h^{ej}k_{ej}\right] - 4(n + 1)S - (n + 2)K^*,$$

where U, K, K*, S, h_{st}, and k_{ej} are scalar quantities derived from the curvature tensor and its contractions.
Proof:
Contracting equation (3.33) by g^{kj}yields

$$U_{kj}g^{kj} = -\frac{1}{(n+2)}S_{kj}g^{kj} + \frac{(n+3)}{4(n+1)(n+2)}Kf_{kj}g^{kj} - \frac{1}{4(n+1)}K^*f_{kj}g^{kj} + \frac{(n+3)}{2(n+1)(n+2)}\}(h_{st}h^{st})f_{kj}g^{kj} - \frac{2(n+3)}{(n+2)}\, h^e_k k_{ej}g^{kj} \qquad (3.34)$$

Consequently yields

$$U = \frac{(n+3)}{4(n+1)(n+2)}\left[K + 2h_{st}h^{st} - 8(n + 1)h^{ej}k_{ej}\right] - \frac{S}{(n+2)} - \frac{K^*}{4(n+1)}$$

Multiplying through by 4(n+1)(n+2) leads directly to the theorem statement.
This yields three significant cases:
Case 1: For n = -1, the identity simplifies to:

$$K^* = 2(K + 2h_{st}h^{st}).$$

Case 2: For n = -2, the relation reduces to:

$$K = 2(h_{st}h^{st} + 4h^{ej}k_{ej} + 2S).$$

Case 3: For n = -3, we get:

$$K^* = 8(U - S).$$

3.3. Geometric Validity of Scalar Invariant U in Non-Standard Dimensions:

To assess the robustness of scalar invariants in Sasakian geometry, it becomes essential to evaluate their behavior under variations in manifold dimension. In this context, we focus on a particular scalar quantity U, defined via sectional curvature and specific contractions of the curvature and structure tensors. This invariant plays a crucial role in capturing geometric information in odd-dimensional Sasakian manifolds. However, its mathematical validity and geometric interpretability strongly depend on the dimensional parameter n. The following theorem provides a rigorous characterization of the dimensional constraints under which U remains a meaningful geometric entity.

Theorem 3.3:
Let U be a scalar curvature invariant defined on a 2(n+1)-dimensional Sasakian manifold as:

$$U = \frac{(n+3)}{4(n+1)(n+2)}\left[K + 2h_{st}h^{st} - 8(n + 1)h^{ej}k_{ej}\right] - \frac{S}{(n+2)} - \frac{K^*}{4(n+1)}$$

where K* denotes a scalar associated with the sectional curvature, and the terms $h_{st}h^{st}$, $h^{ej}k_{ej}$ represent contractions of structure and curvature tensors. Then, the scalar quantity U is algebraically meaningful and geometrically interpretable if and only if $n \in \mathbb{N}$. For non-positive integers n, the invariant becomes either undefined or lacks a coherent geometric interpretation.
Proof:
We examine three hypothetical cases to determine the behavior of the scalar invariant U under non-standard values of n:
Case I: n = -1: Substituting n = -1, the denominator 4(n+1)(n+2) becomes zero, leading to division by zero. This renders U mathematically ill-defined. Moreover, since the dimension 2n+1 = -1 does not support a differential manifold structure, this case reveals both algebraic and geometric breakdown.

Case II: n = -2: Here, (n+2) = 0, and the entire expression becomes undefined due to another zero denominator. Even though some tensorial components might appear algebraically computable, the singularity prevents U from being a valid scalar. Additionally, a manifold of real dimension 2n+1 = -3 contradicts the foundational principles of differentiable geometry.
Case III: n = -3: This case avoids direct singularities in the denominator. However, it implies a manifold of real dimension 2n + 1 = -5, which is not permissible under standard manifold theory, where dimensions are assumed to be non-negative. Therefore, even though U may be formally computed, it lacks any geometrical interpretation or theoretical support.
Therefore, we conclude within this analysis that the scalar invariant U is mathematically well-defined and geometrically interpretable only when $n \in \mathrm{N}$, thereby establishing the restriction necessary for the validity of the curvature identity.

4. Conclusion:

This study presents a comprehensive geometric analysis of complex hypersurfaces embedded in Kählerian manifolds with constant holomorphic sectional curvature and vanishing Bochner curvature tensor. By examining the curvature characteristics and the associated tensorial structures, we have established significant conditions under which these hypersurfaces exhibit geometric rigidity. In particular, we have shown that the vanishing of the Bochner curvature tensor imposes strong constraints on the second fundamental form and the shape operator, leading to the conclusion that such hypersurfaces are necessarily totally geodesic under certain curvature conditions.
Moreover, it is demonstrated that a totally geodesic complex hypersurface inherits the constant holomorphic sectional curvature of the ambient manifold, preserving its intrinsic curvature properties. The interplay between the ambient curvature tensor, the second fundamental form, and Bochner-flatness yields critical insight into the intrinsic and extrinsic geometry of these hypersurfaces.
We have also introduced and analyzed scalar curvature invariants, particularly focusing on the invariant U, and assessed its validity within both standard and degenerate dimensional contexts. It is observed that while such invariants offer deep geometric interpretations in higher dimensions, their definition becomes non-trivial or inadmissible in non-positive dimensions.

References:

[1] Devikhile, S. D., Pujar, S. S.: A Note on Kaehlerian Manifolds with Vanishing Bochner Curvature Tensor, Recent Research in Science and Technology, 3(7), 85-87, (2011).
[2] Kamishima, Y.: Uniformization of Kähler manifolds with vanishing Bochner tensor, (1994).
[3] Yano, K., Ishihara, S.: Kaehlerian manifolds with constant scalar curvature whose Bochner curvature tensor vanishes, Vol. 70, 337–344, North-Holland, Amsterdam (1982).
[4] Smyth, B.: Differential geometry of complex hypersurfaces, Annals of Mathematics, 85(2), 246-266, (1967).
[5] Yamaguchi, S., Sato, S.: On complex hypersurfaces with vanishing Bochner curvature tensor in Kählerian manifolds, Tensor (N.S.), 22, 77-81, (1971).
[6] Ni, L., Zheng, F.: Comparison and vanishing theorems for Kähler manifolds, Calculus of Variations and Partial Differential Equations, 57, 1-31, (2018).
[7] Goldberg, S. I.: Integrability of almost Kähler manifolds, Proceedings of the American Mathematical Society, 21(1), 96-100, (1969).
[8] Tricerri, F., Vanhecke, L.: Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc., 267, 365–398, (1981).
[9] Ganchev, G., Mihova, V.: Kähler manifolds of quasi-constant holomorphic sectional curvatures, Central European Journal of Mathematics, 6, 43-75, (2008).
[10] Lluch, A., Miquel, V.: Kähler tubes of constant radial holomorphic sectional curvature, Manuscripta Mathematica, 94(1), 445-463, (1997).

[11] Pyo, Y. S., Shin, K. H.: Kähler submanifolds with lower bounded totally Real Bisectional Curvature Tensor II, Communications of the Korean Mathematical Society, 17(2), 279-293, (2002).
[12] Bejan, C. L., Güler, S.: Kähler manifolds of quasi-constant holomorphic sectional curvature and generalized Sasakian space forms, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A, Matemáticas, 113(2), 1173-1189, (2019).