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# The Expected Average Queue Length of a Feedback Queue System with Four Servers Arranged Hierarchically and A Limit on the Number of Times a Customer Can Return to Any One of the Servers

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**Abstract:**

The investigation of a feedback queueing system's mean queue length is the focus of this work. Four servers are arranged hierarchically in the Queuing system to serve customers. Depending on their needs, a customer who has received service from the first server moves on to the second, third, or fourth server. She or he may return, but only a certain number of times. It is assumed that the arrival and service patterns adhere to the Poisson process. In order to find the system's Mean Queue Length, the steady state equations were solved using the generating function technique.

**Keywords:** Feedback,Queuing System, Poisson Process, Four Server, Mean Queue Length.

## 1.Introduction

Many authors such asSingh, T.P., Kusum and Gupta, D. (2010), Zadeh, A. B. (2015), Peng (2016),Raheja et al. (2016), Reed and Zhang (2017), Sharma U and Garg K (2022) worked on queuing theory having feedback facility. Kumar and Taneja (2017), worked on the feedback queuing system comprising of three servers linked in series hierarchically in which a customer firstly join the first server, then either he/she may leave the system after getting the service or may move to the second higher ordered server for further service. From the second server either he/she may go outside the system or back to the first lower ordered server or may go to the third highest ordered server for further service depending upon the need of customer. From the third highest ordered server he/she may go outside the system or to the second server or to the first server. Here it is assumed that the customer may revisit any server atmost once and once he/she reached the third server second time, he/she will quit the system. However, they did not discuss about the situation when the number of servers are more than three. Kamal et al. (2023) worked on hierarchically structured four server feedback queueing system. But they assumed the revisit only once. There may be systems in place where services are provided in a hierarchical manner having four servers with the provision of service more than twice; as a result, the current chapter is dedicated to examining these systems. Administrative offices, medical facilities, and hierarchical organizations may all encounter this kind of circumstance.

As a result, the current paper discusses a queue system where customers can either proceed to the second higher level server based on their pleasure with the service, or they can exit the system after the first server completes their task. The customer may move to the next higher-level third server for more service after receiving care from the second server. There's also a possibility that he or she will quit the system or come back to the original server with criticism. If the customer is not satisfied, they can leave the system after being satisfied or they can go from this server to any of the lower-level servers. Every server in the system has the same set of resources. In this instance, we have taken into account the circumstance in which a client is obligated to return up to a certain number of times. Each time you come back, your chances of getting on any given server are considered to be unique. With the use of the differential-difference method, the queue lengths have been established.

**2.Notations**

$\lambda$: Mean Arrival rate at 1st server ($S_1$ )

$\mu_1$: service rate of 1st server ($S_1$ )

$\mu_2$: service rate of 2nd server ($S_2$)

$\mu_3$: service rate of 3rd server ($S_3$)

$\mu_4$: service rate of 4th server ($S_4$)

$p^i_{12}$: the probability of customer going from 1st to 2nd server ith time

$p^i_1$: the probability of exit of customer from 1st server ith time.

$p^i_2$: the probability of exit of customer from 2nd server ith time.

$p^i_{23}$: the probability of customer going from 2nd to 3rd server ith time.

$p^i_{21}$: the probability of customer going from 2nd to 1st server ith time.

$p^i_3$: the probability of exit of customer from 3rd server ith time.

$p^i_{31}$: the probability of customer going from 3rd to 1st server ith time.

$p^i_{32}$: the probability of customer going from 3rd to 2nd server ith time.

$p^i_{34}$: the probability of customer going from 3rd to 4th server ith time

$p^i_4$: the probability of exit of customer from 4th server ith time.

$p^i_{41}$ : the probability of exit of customer from 4th to 1st server ith time.

$p^i_{42}$: the probability of customer going from 4th to 2nd server ith time.

$p^i_{43}$: the probability of customer going from 4th to 3rd server ith time.

$$A_1 = \sum_{i=1}^{n} a^i p_1^i \ , \ A_{12} = \sum_{i=1}^{n} a^i p_{12}^i$$

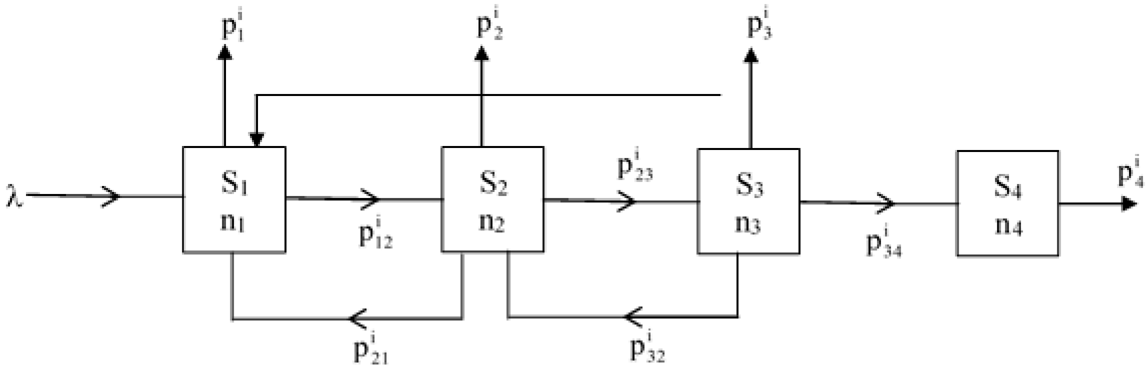
$$B_2 = \sum_{i=1}^{n} b^i p_2^i \, , B_{21} = \sum_{i=1}^{n-1} b^i p_{21}^i \, , \ B_{23} = \sum_{i=1}^{n} b^i p_{23}^i$$

$C_3=\sum_{i=1}^{n}c^i p_3^i$ , $C_{34}=\sum_{i=1}^{n}c^i p_{34}^i$ , $C_{31}=\sum_{i=1}^{n-1}c^i p_{31}^i$ , $C_{32}=\sum_{i=1}^{n-1}c^i p_{32}^i$

$D_4=\sum_{i=1}^{n}d^i p_4^i$ , $D_{43}=\sum_{i=1}^{n-1}d^i p_{43}^i$ , $D_{42}=\sum_{i=1}^{n-1}d^i p_{42}^i$ , $D_{41}=\sum_{i=1}^{n-1}d^i p_{41}^i$

## 3.Formulation of Problem

The queue network consists of four service channels in hierarchical order i.e. lower level (Server 1) to higher levels (Server 2, server 3 and then server 4) if required. It is assumed that customer arrives at first server from outside the system and then goes to second, third and fourth server. The situation has been shown by the following sate transition diagram:




**Movement of the Customers from Various Servers**

If the customer gets service from first server ith time, then $p_1^i+p_{12}^i=1$. After getting service from second server ith time, we have $p_2^i+p_{23}^i+p_{21}^i=1$. Customer after getting service from the third server ith time, we have $p_3^i+p_{31}^i+p_{32}^i+p_{34}^i=1$and after from fourth server, we have $p_4^i+p_{43}^i+p_{42}^i+p_{41}^i=1$ .

Hence we can write:

$$\sum_{i=1}^{n}a^i p_1^i+\sum_{i=1}^{n}a^i p_{12}^i=1$$

$$\sum_{i=1}^{n}b^i p_2^i+\sum_{i=1}^{n-1}b^i p_{21}^i+\sum_{i=1}^{n}b^i p_{23}^i=1$$

$$\sum_{i=1}^{n}c^i p_3^i+\sum_{i=1}^{n}c^i p_{34}^i+\sum_{i=1}^{n-1}c^i p_{31}^i+\sum_{i=1}^{n-1}c^i p_{32}^i=1$$

$$\sum_{i=1}^{n}d^i p_4^i+\sum_{i=1}^{n-1}d^i p_{43}^i+\sum_{i=1}^{n-1}d^i p_{42}^i+\sum_{i=1}^{n-1}d^i p_{41}^i=1$$

Thus we have:

$A_1 + A_{12} = 1$

$B_2 + B_{23} + B_{21} = 1$

$C_3 + C_{34} + C_{31} + C_{32} = 1$

$D_4 + D_{43} + D_{42} + D_{41} = 1$

If we denote $Q_{n_1,n_2,n_3,n_4}(t)$ as the probability of $n_1$, $n_2$, $n_3$, $n_4$ customers at time t on 1st, 2nd, 3rd and 4th server $S_1$, $S_2$, $S_3$& $S_4$ resp. at time t, then we have the steady state equations as: $(\lambda+\mu_1+\mu_2+\mu_3+\mu_4)Q_{n_1,n_2,n_3,n_4} =$

$$\lambda Q_{n_1-1,n_2,n_3,n_4} + A_1\mu_1 Q_{n_1+1,n_2,n_3,n_4} + A_{12}\mu_1 Q_{n_1+1,n_2-1,n_3,n_4} + B_2\mu_2 Q_{n_1,n_2+1,n_3,n_4} +$$
$$B_{23}\mu_2 Q_{n_1,n_2+1,n_3-1,n_4} + B_{21}\mu_2 Q_{n_1-1,n_2+1,n_3,n_4} + C_3\mu_3 Q_{n_1,n_2,n_3+1,n_4} + C_{34}\mu_3 Q_{n_1,n_2,n_3+1,n_4-1} +$$
$$C_{32}\mu_3 Q_{n_1,n_2-1,n_3+1,n_4} + C_{31}\mu_3 Q_{n_1-1,n_2,n_3+1,n_4} +$$
$$D_4\mu_4 Q_{n_1,n_2,n_3,n_4+1} + D_{41}\mu_4 Q_{n_1-1,n_2,n_3,n_4+1} + D_{42}\mu_4 Q_{n_1,n_2-1,n_3,n_4+1} + D_{43}\mu_4 Q_{n_1,n_2,n_3-1,n_4+1} \quad \ldots (1)$$

$Put\, n_1 = 0$

$$(\lambda+\mu_2+\mu_3+\mu_4)Q_0, n_2, n_3, n_4 = A_1\mu_1 Q_1, n_2, n_3, n_4 +$$
$$A_{12}\mu_1 Q_{1,n_2-1,n_3,n_4} + B_2\mu_2 Q_{0,n_2+1,n_3,n_4} + B_{23}\mu_2 Q_{0,n_2+1,n_3-1,n_4} + C_3\mu_3 Q_{0,n_2,n_3+1,n_4}$$
$$C_{34}\mu_3 Q_{0,n_2,n_3+1,n_4-1} + C_{32}\mu_3 Q_{0,n_2-1,n_3+1,n_4} + D_4\mu_4 Q_{0,n_2,n_3,n_4+1} + D_{42}\mu_4 Q_{0,n_2-1,n_3,n_4+1} + D_{43}\mu_4 Q_{0,n_2,n_3-1,n_4+1}$$
$$\ldots \quad \ldots (2)$$

Put $n_2 = 0$

$$(\lambda+\mu_1+\mu_3+\mu_4)Q_{n_1,0,n_3,n_4} = \lambda Q_{n_1-1,0,n_3,n_4} +$$
$$A_1\mu_1 Q_{n_1+1,0,n_3,n_4} + B_2\mu_2 Q_{n_1,1,n_3,n_4} + B_{23}\mu_2 Q_{n_1,1,n_3-1,n_4} + B_{21}\mu_2 Q_{n_1-1,1,n_3,n_4}$$
$$C_3\mu_3 Q_{n_1,0,n_3+1,n_4} + C_{34}\mu_3 Q_{n_1,0,n_3+1,n_4-1} + C_{31}\mu_3 Q_{n_1-1,0,n_3+1,n_4} + D_4\mu_4 Q_{n_1,0,n_3,n_4+1}$$
$$+D_{41}\mu_4 Q_{n_1-1,0,n_3,n_4+1} + D_{43}\mu_4 Q_{n_1,0,n_3-1,n_4+1} \quad \ldots.(3)$$

Put $n_3$= 0

$$(\lambda+\mu_1+\mu_2+\mu_4)Q_{n_1,n_2,0,n_4} = \lambda Q_{n_1-1,n_2,0,n_4} +$$
$$A_1\mu_1 Q_{n_1+1,n_2,0,n_4} + A_{12}\mu_1 Q_{n_1+1,n_2-1,0,n_4} + B_2\mu_2 Q_{n_1,n_2+1,0,n_4} + B_{21}\mu_2 Q_{n_1-1,n_2+1,0,n_4}$$
$$C_3\mu_3 Q_{n_1,n_2,1,n_4} + C_{34}\mu_3 Q_{n_1,n_2,1,n_4-1} + C_{32}\mu_3 Q_{n_1,n_2-1,1,n_4} + C_{31}\mu_3 Q_{n_1-1,n_2,1,n_4} +$$
$$D_4\mu_4 Q_{n_1,n_2,0,n_4+1} + D_{41}\mu_4 Q_{n_1-1,n_2,0,n_4+1} + D_{42}\mu_4 Q_{n_1,n_2-1,0,n_4+1} \quad \ldots(4)$$

Put $n_4 = 0$

$$(\lambda+\mu_1+\mu_2+\mu_3)Q_{n_1,n_2,n_3,0} = \lambda Q_{n_1-1,n_2,n_3,0} +$$
$$A_1\mu_1 Q_{n_1+1,n_2,n_3,0} + A_{12}\mu_1 Q_{n_1+1,n_2-1,n_3,0} + B_2\mu_2 Q_{n_1,n_2+1,n_3,0} + B_{23}\mu_2 Q_{n_1,n_2+1,n_3-1,0} +$$

$$B_{21}\mu_2 Q_{n_1-1,n_2+1,n_3,0} + C_3\mu_3 Q_{n_1,n_2,n_3+1,0} + C_{32}\mu_3 Q_{n_1,n_2-1,n_3+1,0} + C_{31}\mu_3 Q_{n_1-1,n_2,n_3+1,0} +$$

$$D_4\mu_4 Q_{n_1,n_2,n_3,1} + D_{41}\mu_4 Q_{n_1-1,n_2,n_3,1} + D_{42}\mu_4 Q_{n_1,n_2-1,n_3,1} + D_{43}\mu_4 Q_{n_1,n_2,n_3-1,1} \quad \ldots(5)$$

For $n_1$, $n_2 = 0$

$$(\lambda + \mu_3 + \mu_4) Q_{0,0,n_3,n_4} = A_1\mu_1 Q_{1,0,n_3,n_4} +$$

$$B_2\mu_2 Q_{0,1,n_3,n_4} + B_{23}\mu_2 Q_{0,1,n_3-1,n_4} + C_3\mu_3 Q_{0,n_3+1,n_4} + C_{34}\mu_3 Q_{0,0,n_3+1,n_4-1} +$$

$$D_4\mu_4 Q_{0,0,n_3,n_4+1} + D_{43}\mu_4 Q_{0,0,n_3-1,n_4+1} \quad \ldots(6)$$

For $n_1$, $n_3 = 0$

$$(\lambda + \mu_2 + \mu_4) Q_{0,n_2,0,n_4} = A_1\mu_1 Q_{1,n_2,0,n_4} +$$

$$A_{12}\mu_1 Q_{1,n_2-1,0,n_4} + B_2\mu_2 Q_{0,n_2+1,0,n_4} +$$

$$C_3\mu_3 Q_{0,n_2,1,n_4} + C_{34}\mu_3 Q_{0,n_2,1,n_4-1}$$

$$C_{32}\mu_3 Q_{0,n_2-1,1,n_4} + D_4\mu_4 Q_{0,n_2,0,n_4+1} + D_{42}\mu_4 Q_{0,n_2-1,0,n_4+1} \quad \ldots(7)$$

For $n_1$, $n_4 = 0$

$$(\lambda + \mu_2 + \mu_3) Q_{0,n_2,n_3,0} = A_1\mu_1 Q_{1,n_2,n_3,0} + A_{12}\mu_1 Q_{1,n_2-1,n_3,0} +$$

$$B_2\mu_2 Q_{0,n_2+1,n_3,0} + B_{23}\mu_2 Q_{0,n_2+1,n_3-1,0} + C_3\mu_3 Q_{0,n_2,n_3+1,0} +$$

$$D_4\mu_4 Q_{0,n_2,n_3,1} + D_{42}\mu_4 Q_{0,n_2-1,n_3,1} + D_{43}\mu_4 Q_{0,n_2,n_3-1,1} \quad \ldots(8)$$

For $n_2$, $n_3 = 0$

$$(\lambda + \mu_1 + \mu_4) Q_{n_1,0,0,n_4} = \lambda Q_{n_1-1,0,0,n_4} +$$

$$A_1\mu_1 Q_{n_1+1,0,0,n_4} + B_2\mu_2 Q_{n_1,1,0,n_4} +$$

$$B_{21}\mu_2 Q_{n_1-1,1,0,n_4} + C_3\mu_3 Q_{n_1,0,1,n_4} + C_{34}\mu_3 Q_{n_1,0,1,n_4-1} + C_{31}\mu_3 Q_{n_1-1,0,1,n_4}$$

$$+ D_4\mu_4 Q_{n_1,0,0,n_4+1} + D_{41}\mu_4 Q_{n_1-1,0,0,n_4+1} \quad \ldots(9)$$

For $n_2$, $n_4 = 0$

$$(\lambda + \mu_1 + \mu_3) Q_{n_1,0,n_3,0} = \lambda Q_{n_1-1,0,n_3,0} +$$

$$A_1\mu_1 Q_{n_1+1,0,n_3,0} + B_2\mu_2 Q_{n_1,1,n_3,0} + B_{23}\mu_2 Q_{n_1,1,n_3-1,0}$$

$$B_{21}\mu_2 Q_{n_1-1,1,n_3,0} + C_3\mu_3 Q_{n_1,0,n_3+1,0} + C_{31}\mu_3 Q_{n_1-1,0,n_3+1,0} + D_4\mu_4 Q_{n_1,0,n_3,1} +$$

$$D_{41}\mu_4 Q_{n_1-1,0,n_3,1} + D_{43}\mu_4 Q_{n_1,0,n_3-1,1} \quad \ldots(10)$$

For $n_3$, $n_4 = 0$

$$\left(\lambda+\mu_1+\mu_2\right)Q_{n_1,n_2,0,0}=\lambda Q_{n_1-1,n_2,0,0}+$$
$$A_1\mu_1 Q_{n_1+1,n_2,0,0}+A_{12}\mu_1 Q_{n_1+1,n_2-1,0,0}+$$
$$B_2\mu_2 Q_{n_1,n_2+1,0,0}+B_{21}\mu_2 Q_{n_1-1,n_2+1,0,0}$$
$$+C_3\mu_3 Q_{n_1,n_2,1,0}+C_{32}\mu_3 Q_{n_1,n_2-1,1,0}+$$
$$C_{31}\mu_3 Q_{n_1-1,n_2,1,0}+D_4\mu_4 Q_{n_1,n_2,0,1}+$$
$$D_{41}\mu_4 Q_{n_1-1,n_2,0,1}+D_{42}\mu_4 Q_{n_1,n_2-1,0,1} \qquad \ldots(11)$$

For $n_1 = n_2 = n_3 = 0$

$$\left(\lambda+\mu_4\right)Q_{0,0,0,n_4}=A_1\mu_1 Q_{1,0,0,n_4}+B_2\mu_2 Q_{0,1,0,n_4}$$
$$+C_3\mu_3 Q_{0,0,1,n_4}+C_{34}\mu_3 Q_{0,0,1,n_4-1}+D_4\mu_4 Q_{0,0,0,n_4+1} \qquad \ldots(12)$$

For $n_1 = n_3 = n_4 = 0$

$$\left(\lambda+\mu_2\right)Q_{0,n_2,0,0}=A_1\mu_1 Q_{1,n_2,0,0}+$$
$$A_{12}\mu_1 Q_{1,n_2-1,0,0}+B_2\mu_2 Q_{0,n_2+1,0,0}+$$
$$C_3\mu_3 Q_{0,n_2,1,0}+C_{32}\mu_3 Q_{0,n_2-1,1,0}+$$
$$D_4\mu_4 Q_{0,n_2,0,1}+D_{42}\mu_4 Q_{0,n_2-1,0,1} \qquad \ldots(13)$$

For $n_2 = n_3 = n_4 = 0$

$$\left(\lambda+\mu_1\right)Q_{n_1,0,0,0}=\lambda Q_{n_1-1,0,0,0}+A_1\mu_1 Q_{n_1+1,0,0,0}$$
$$B_2\mu_2 Q_{n_1,1,0,0}+B_{21}\mu_2 Q_{n_1-1,1,0,0}+$$
$$C_3\mu_3 Q_{n_1,0,1,0}+C_{31}\mu_3 Q_{n_1-1,0,1,0}+$$
$$D_4\mu_4 Q_{n_1,0,0,1}+D_{41}\mu_4 Q_{n_1-1,0,0,1} \qquad \ldots(14)$$

For $n_1 = n_2 = n_4 = 0$

$$\left(\lambda+\mu_3\right)Q_{0,0,n_3,0}=A_1\mu_1 Q_{1,0,n_3,0}+$$
$$B_2\mu_2 Q_{0,1,n_3,0}+B_{23}\mu_2 Q_{0,1,n_3-1,0}+$$
$$C_3\mu_3 Q_{0,0,n_3+1,0}+D_4\mu_4 Q_{0,0,n_3,1}+D_{43}\mu_4 Q_{0,0,n_3-1,1} \qquad \ldots(15)$$

For $n_1 = n_2 = n_3 = n_4 = 0$

$$\lambda Q_{0,0,0,0}=A_1\mu_1 Q_{1,0,0,0}+B_2\mu_2 Q_{0,1,0,0}+C_3\mu_3 Q_{0,0,1,0}+D_4\mu_4 Q_{0,0,0,1} \qquad \ldots(16)$$

We assume here the initial condition

$$Q_{n_1}, n_2, n_3, n_4 = \begin{cases} 1 & ; \quad (n_1, n_2, n_3, n_4 \neq 0 \\ 0 & \quad otherwise \end{cases}$$

We also define the generating function as

$$F(X,Y,Z,R) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} Q_{n_1}, {}_{n_2,n_3,n_4} X^{n_1} Y^{n_2} Z^{n_3} R^{n_4}$$

Where

$|X| = |Y| = |Z| = |R| = 1 \ldots (17)$

Also we define partial generating function as

$$G_{n_2,n_3,n_4}(X) = \sum_{n_1=0}^{\infty} Q_{n_1,n_2,n_3,n_4} X^{n_1} \qquad \ldots 17(A)$$

$$G_{n_3,n_4}(X,Y) = \sum_{n_2=0}^{\infty} G_{n_2,n_3,n_4}(x).Y^{n_2} \qquad \ldots 17(B)\ G_{n_1,n_3,n_4}(Y)$$

$$= \sum_{n_2=0}^{\infty} G_{n_1,n_2,n_3,n_4} Y^{n_2} \qquad \ldots 17(B1)$$

$$G_{n_4}(X,Y,Z) = \sum_{n_3=0}^{\infty} G_{n_3,n_4}(X,Y).Z^{n_3} \qquad \ldots 17(C) G(X,Y,Z)$$

$$= \sum_{n_4=0}^{\infty} G_{n_4}(X,Y,Z).R^{n_4} \qquad \ldots 17(D)$$

On solving (1) to (16) & using (17) we have:

$$F(X,Y,Z,R) = \frac{\begin{array}{l} \mu_1 G_0(Y,Z,R)\left[1-\frac{1}{X}(A_1+A_{12}Y)\right] \\ +\mu_2 G_0\left(X,Z,R\right)\left[1-\frac{1}{Y}(B_2+B_{21}X+B_{23}Z)\right]\Big) \\ +\mu_3 G_0\left(X,Y,R\right)\left[1-\frac{1}{Z}(C_3+C_{31}X+C_{32}Y+C_{34}R)\right]\Big) \\ +\mu_4 G_0\left(X,Y,Z\right)\left[1-\frac{1}{R}(D_4+D_{41}X+D_{42}Y+D_{43}Z)\right]\Big) \end{array}}{\begin{array}{l} \lambda(1-X)+\mu_1\left[1-\frac{1}{X}(A_1+A_{12}Y)\right] \\ +\mu_2\left[1-\frac{1}{Y}(B_2+B_{21}X+B_{23}Z)\right] \\ +\mu_3\left[1-\frac{1}{Z}(C_3+C_{31}X+C_{32}Y+C_{34}R)\right] \\ +\mu_4\left[1-\frac{1}{R}(D_4+D_{41}X+D_{42}Y+D_{43}Z)\right] \end{array}} \qquad (18)$$

$$where\ f = \begin{cases} \mu_1 G_0(Y,Z,R)\left[1-\frac{1}{X}(A_1+A_{12}Y)\right] \\ +\mu_2 G_0\left(X,Z,R\right)\left[1-\frac{1}{Y}(B_2+B_{21}X+B_{23}Z)\right]\Big) \\ +\mu_3 G_0\left(X,Y,R\right)\left[1-\frac{1}{Z}(C_3+C_{31}X+C_{32}Y+C_{34}R)\right]\Big) \\ +\mu_4 G_0\left(X,Y,Z\right)\left[1-\frac{1}{R}(D_4+D_{41}X+D_{42}Y+D_{43}Z)\right]\Big) \end{cases} \qquad (18A)$$

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$$g = \begin{cases} \lambda(1-X) + \mu_1\left[1 - \frac{1}{X}(A_1 + A_{12}Y)\right] \\ +\mu_2\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right] \\ +\mu_3\left[1 - \frac{1}{Z}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right] \\ +\mu_4\left[1 - \frac{1}{R}(D_4 + D_{41}X + D_{42}Y + D_{43}Z)\right] \end{cases} \tag{18B}$$

for convenience let us defined

$$G_0(Y,Z,R) = G_1 \quad , \quad G_0(X,Z,R) = G_2$$
$$G_0(X,Y,R) = G_3 \qquad G_0(X,Y,Z) = G_4 \tag{18C}$$

Solving the linear equations obtained from (18), we have

$$G_1 = G_0(Y,Z,R) = \frac{\begin{matrix} -(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23-1})\lambda + \\ \mu_1 B_{21}A_{12}(D_{43}C_{34} - 1) + \mu_1 23(+1 - D_{43}C_{34} - D_{42}B_{23}C_{34} - C_{32}B_{23} + 1) \\ -\mu_1 D_{41}A_{12}B_{23}C_{34} - \mu_1 C_{31}A_{12}B_{23} \end{matrix}}{\begin{matrix} \mu_1(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23-1}) + \\ \mu_1 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_1 D_{41}A_{12}B_{23}C_{34} \\ +\mu_1 C_{31}A_{12}B_{23} \end{matrix}} \tag{19}$$

$$G_2 = G_0(X,Z,R) = \frac{\begin{matrix} -A_{12}(D_{43}C_{34} - 1)\lambda + \mu_2 B_{21}A_{12}(D_{43}C_{34} - 1) \\ +\mu_2(-D_{43}C_{34} - D_{42}B_{23}C_{34} - C_{32}B_{23} + 1) \\ -\mu_2 D_{41}A_{12}B_{23}C_{34} - \mu_2 C_{31}A_{12}B_{23} \end{matrix}}{\begin{matrix} \mu_2(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23} - 1) \\ +\mu_2 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_2 D_{41}A_{12}B_{23}C_{34} \\ +\mu_2 C_{31}A_{12}B_{23} \end{matrix}} \tag{20}$$

$$G_3 = G_0(X,Y,R) = \frac{\begin{matrix} A_{12}B_{23}\lambda + \mu_3(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23} - 1) \\ \mu_3 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_3 D_{41}A_{12}B_{23}C_{34} \\ +\mu_3 C_{31}A_{12}B_{23} \end{matrix}}{\begin{matrix} \mu_3(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23} - 1) \\ +\mu_3 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_3 D_{41}A_{12}B_{23}C_{34} \\ +\mu_3 C_{31}A_{12}B_{23} \end{matrix}} \tag{21}$$

$$G_4 = G_0(X,Y,Z) = \frac{\begin{matrix} A_{12}B_{23}C_{34}\lambda + \mu_4(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23} - 1) \\ +\mu_4 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_4 D_{41}A_{12}B_{23}C_{34} \\ +\mu_4 C_{31}A_{12}B_{23} \end{matrix}}{\begin{matrix} \mu_4(D_{43}C_{34} + D_{42}B_{23}C_{34} + C_{32}B_{23} - 1) \\ +\mu_4 B_{21}A_{12}(1 - D_{43}C_{34}) + \mu_4 D_{41}A_{12}B_{23}C_{34} \\ +\mu_4 C_{31}A_{12}B_{23} \end{matrix}} \tag{22}$$

From (18)

$$\frac{\partial f}{\partial X} = \mu_1 G_0(Y,Z,R)\frac{1}{X^2}[A_1 + A_{12}Y] + \mu_2 G_0(X,Z,R)\left[\frac{-1}{Y}B_{21}\right]$$
$$+\mu_2 G_0^1\left(X,Z,R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]\right)$$
$$+\mu_3 G_0(X,Y,R)\left[\frac{-1}{Z}C_{31}\right] + \mu_3 G_0^1(X,Y,R)\left[1 - \frac{1}{Z}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right]$$
$$+\mu_4 G_0(X,Y,Z)\left(\frac{-D_{41}}{R}\right) + \mu_4 G_0^1(X,Y,Z)\left[1 - \frac{1}{R}(D_4 + D_{41}X + D_{42}Y + D_{43}Z)\right]$$

$$\left(\frac{\partial f}{\partial X}\right)_{(1,1,1,1)} = \mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4$$

$$\left(\frac{\partial g}{\partial X}\right) = \lambda(-1) + \mu_1\left(\frac{1}{X^2}(A_1 + A_{12}Y)\right) + \mu_2\left(\frac{-1}{Y}B_{21}\right) +$$
$$\mu_3\left(\frac{-1}{Z}C_{31}\right) + \mu_4\left(-\frac{-D_{41}}{R}\right)$$

$$\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)} = -\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41}$$

$$\frac{\partial^2 f}{\partial X^2} = \mu_1 G_0(Y, Z, R)\left[\frac{-2}{X^3}(A_1 + A_{12}Y)\right]\frac{-\mu_2 B_{21}}{Y}G_0^1(X, Z, R)$$
$$+\mu_2 G_0^1(X, Z, R)\left(\frac{-B_{21}}{Y}\right) + \mu_2 G_0^1(X, Z, R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]$$
$$+\mu_3 G_0^1(X, Y, R)\left(\frac{-1}{Z}C_{31}\right) + \mu_3 G_0^1(X, Y, R)\left[\frac{-1}{Z}C_{31}\right]$$
$$+\mu_3 G_0^{11}(X, Y, R)\left(1 - \frac{1}{Z}(C_3 + C_{31}X + C_{32}Y + C_{34}R\right) + \mu_3 G_0^1(X, Y, R)\left[\frac{-1}{Z}C_{31}\right]$$
$$+\mu_4 G_0^1\left(X, Y, Z()\left(\frac{-D_{41}}{R}\right)_{4_0}^{1}\left(X, Y, Z()\left[\frac{-1}{R}D_{41}\right]\right)\right)$$
$$+\mu_4 G_0^{11}\left((X, Y, Z)\left[1 - \frac{1}{R}(D_4 + D_{41}X + D_{42}Y + D_{43}Z)\right]\right)$$

$$\left(\frac{\partial^2 f}{\partial X^2}\right)_{(1,1,1,1)} = -2\mu_1 G_1 - \mu_2 B_{21} G_2^1 - \mu_2 B_{21} G_2^1 - \mu_3 C_{31} G_3^1$$
$$-\mu_3 C_{31} G_3^1 - \mu_4 D_{41} G_4^1 - \mu_4 D_{41} G_4$$
$$= -2\mu_1 G_1$$

$$\left(\frac{\partial^2 g}{\partial X^2}\right) = \mu_1\left[\frac{-2}{X^3}(A_1 + A_{12}Y)\right]$$

$$\left(\frac{\partial^2 g}{\partial X^2}\right)_{(1,1,1,1)} = -2\mu_1$$

Let $Lq_1$ denote the mean queue length at the 1st server $S_1$.

$$Lq_1 = \frac{\left(\frac{\partial f}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial X^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial X^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\right]^2}$$

By putting the values;

$$Lq_1 = \frac{\begin{array}{c}[\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4](-2\mu_1)\\ +(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})(-2\mu_1 G_1)\end{array}}{2[(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})]^2}$$

$$Lq_1 = -\mu_1 \left[\frac{(\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4)}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})^2} + \frac{G_1}{(-\lambda+\mu_1-\mu_2 B_{21}-\mu_3 C_{31}-\mu_4 D_{41})}\right] \quad (23)$$

Again from (18)

$$\left(\frac{\partial f}{\partial Y}\right) = \mu_1 G_0(Y,Z,R)\left[\frac{-1}{X} A_{12}\right] + \mu_1 G_0^1(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12} Y)\right]$$
$$+\mu_2 G_0(X,Z,R)\left[\frac{1}{Y^2}(B_2 + B_{21} X + B_{23} Z)\right]$$
$$+\mu_4 G_0\left(X,Y,Z()\left[\frac{-D_{42}}{R}\right]_{4_0}^{1}\left(X,Y,Z()\left[1 - \frac{1}{R}(D_4 + D_{41} X + D_{42} Y + D_{43} Z)\right]\right)\right)$$

$$\left(\frac{\partial f}{\partial Y}\right)_{(1,1,1,)} = -\mu_1 G_1 A_{12} + \mu_2 G_2 - \mu_4 G_4 D_{42}$$

$$\frac{\partial g}{\partial Y} = \mu_1\left(\frac{-A_{12}}{X}\right) + \mu_2\left[\frac{1}{Y^2}\left(B_2 + B_{21} X + B_{23} Z\right)\right] + \mu_3\left(\frac{-C_{32}}{Z}\right) + \mu_4\left(\frac{-D_{42}}{R}\right)$$

$$\left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)} = -A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42}$$

$$\left(\frac{\partial^2 f}{\partial Y^2}\right) = \mu_1 G_0^1(Y,Z,R)\left(\frac{-1}{X} A_{12}\right) + \mu_1 G_0^1(Y,Z,R)\left(\frac{-A_{12}}{X}\right)$$
$$+\mu_1 G_0^{11}(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12} Y)\right] + \mu_2 G_0(X,Z,R)\left[\frac{-2}{Y^3}(B_2 + B_{21} X + B_{23} Z)\right]$$
$$+\mu_4 G_0^1(X,Y,Z)\left(\frac{-D_{42}}{R}\right) + \mu_4 G_0^1(X,Y,Z)\left[\frac{-D_{42}}{R}\right]$$
$$+\mu_4 G_0^{11}(X,Y,Z)\left[1 - \frac{1}{R}(D_4 + D_{41} X + D_{42} Y + D_{43} Z)\right]$$

$$\left(\frac{\partial^2 f}{\partial Y^2}\right)_{(1,1,1,1)} = -2\mu_2 G_2$$

$$\frac{\partial^2 g}{\partial Y^2} = \frac{-2\mu_2}{Y^3}\left(B_2 + B_{21} X + B_{23} Z\right)$$

$$\left(\frac{\partial^2 g}{\partial Y^2}\right)_{(1,1,1,1)} = -2\mu_2$$

Let $Lq_2$ denote the mean queue length at the $2^{nd}$ server $S_2$.

$$Lq_2 = \frac{\left(\frac{\partial f}{\partial Y}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial Y^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial Y^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_2 = \frac{\begin{array}{c}(-\mu_1 G_1 A_{12} + \mu_2 G_2 - \mu_4 D_{42} G_4)(-2\mu_2) + \\ (-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})(-2\mu_2 G_2)\end{array}}{2[-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42}]^2}$$

$$Lq_2 = -\mu_2\left[\frac{(-\mu_1 G_1 A_{12} + \mu_2 G_2 - \mu_4 D_{42} G_4)}{(-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})^2} + \frac{G_2}{(-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})}\right] \qquad (25)$$

From (18)

$$\left(\frac{\partial f}{\partial Z}\right) = \mu_1 G_0^1(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12}Y)\right] + \mu_2 G_0(X,Z,R)\left[\frac{-B_{23}}{Y}\right]$$
$$+\mu_2 G_0^1(X,Z,R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]$$
$$+\mu_3 G_0\left(X,Y,Z()\left[\frac{1}{Z^2}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right]\right)$$

$$\left(\frac{\partial f}{\partial Z}\right)_{(1,1,1,1)} = -\mu_2 B_{23} G_2 + \mu_3 G_3$$

$$\frac{\partial^2 f}{\partial Z^2} = \mu_1 G_0^{11}(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12}Y)\right] - \mu_2\frac{B_{23}}{Y}G_0^1(\mathrm{X},\mathrm{Z},R)$$
$$+\mu_2 G_0^{11}(\mathrm{X},Z,R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]$$
$$+\mu_2 G_0^1(X,\mathrm{Z},\mathrm{R})\left(\frac{-B_{23}}{Y}\right) + \mu_3 G_0(X,Y,\mathrm{R})\left(\frac{-2}{Z^3}\right)[C_3 + C_{31}X + C_{32}Y + C_{34}R]$$

$$\left(\frac{\partial^2 f}{\partial Z^2}\right)_{(1,1,1,1)} = -2\mu_3 G_3$$

$$\frac{\partial g}{\partial Z} = \mu_2\left[\frac{-1}{Y}B_{23}\right] + \mu_3\left[\frac{1}{Z^2}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right]$$
$$+\mu_4\left(\frac{-1}{R}D_{43}\right)$$

$$\frac{\partial^2 g}{\partial Z^2} = \frac{-2\mu_3}{Z^3}(C_3 + C_{31}X + C_{32}Y + C_{34}R)$$

$$\left(\frac{\partial^2 g}{\partial Z^2}\right)_{(1,1,1,)} = -2\mu_3$$

Let $Lq_3$ denote the mean queue length at the 3rd server $S_3$.

$$Lq_3 = \frac{\left(\frac{\partial f}{\partial Z}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial Z^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial Z^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_3 = \frac{(-\mu_2 B_{23} G_2 + \mu_3 G_3)(-2\mu_3) + (-\mu_2 B_{23} + \mu_3 - \mu_4 D_{43})(-2\mu_3 G_3)}{2[-\mu_2 B_{23} + \mu_3 - \mu_4 D_{43}]^2}$$

$$Lq_3 = -\mu_3\left[\frac{(-\mu_2 B_{23} G_2 + \mu_3 G_3)}{(-\mu_2 B_{23} + \mu_3 - D_{43}\mu_4)^2} + \frac{G_3}{(-\mu_2 B_{23} + \mu_4 - \mu_4 D_{43})}\right] \qquad (26)$$

From (18)

$$\left(\frac{\partial f}{\partial R}\right) = \mu_1 G_0^1(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12}Y)\right]$$
$$+\mu_2 G_0^1(X,Z,R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]$$
$$+\mu_3 G_0^1\left(X,Y,R()\left[1 - \frac{1}{Z}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right]\right)$$
$$+\mu_3 G_0(X,Y,R)\left(-\frac{C_{34}}{Z}\right) + \mu_4 G_0(X,Y,Z)\left(\frac{1}{R^2}\right)[D_4 + D_{41}X + D_{42}Y + D_{43}Z]$$

$$\left(\frac{\partial f}{\partial R}\right)_{(1,1,1,1)} = -\mu_3 C_{34} G_3 + \mu_4 G_4$$

$$\frac{\partial^2 f}{\partial R^2} = \mu_4 G_0^{11}(Y,Z,R)\left[1 - \frac{1}{X}(A_1 + A_{12}Y)\right] + \mu_2 G_0^{11}(\mathrm{X},\mathrm{Z},R)\left[1 - \frac{1}{Y}(B_2 + B_{21}X + B_{23}Z)\right]$$
$$+\mu_3 G_0^{11}(\mathrm{X},\mathrm{Y},R)\left[1 - \frac{1}{Z}(C_3 + C_{31}X + C_{32}Y + C_{34}R)\right]$$
$$+\mu_3 G_0^1(X,\mathrm{Y},\mathrm{R})\left[\frac{-C_{34}}{Z}\right] - \mu_3 G_0^1\frac{(X,Y,\mathrm{R})C_{34}}{Z}$$
$$-\frac{2\mu_4}{R^3}G_0(X,Y,Z)[D_4 + D_{41}X + D_{42}Y + D_{43}Z]$$

$$\left(\frac{\partial^2 f}{\partial R^2}\right)_{(1,1,1,1)} = -2\mu_4 G_4$$

$$\frac{\partial g}{\partial R} = \mu_3\left(\frac{-C_{34}}{Z}\right) + \mu_4\left[\frac{1}{R^2}\left(D_4 + D_{41}X + D_{42}Y + D_{43}Z\right)\right]$$

$$\left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)} = -\mu_3 C_{34} + \mu_4$$

$$\frac{\partial^2 g}{\partial R^2} = \frac{-2\mu_4}{R^3}\left(D_4 + D_{41}X + D_{42}Y + D_{43}Z\right)$$

$$\left(\frac{\partial^2 g}{\partial R^2}\right)_{(1,1,1,1)} = -2\mu_4$$

Let $Lq_4$ denote the mean queue length at the 4th server $S_4$.

$$Lq_4 = \frac{\left(\frac{\partial f}{\partial R}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial R^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial R^2}\right)_{(1,1,1,1)}}{2\left(\frac{\partial g}{\partial R}\right)^2_{(1,1,1,1)}}$$

$$Lq_4 = \frac{(-\mu_3 C_{34} G_3 + \mu_4 G_4)(-2\mu_4) + (-\mu_3 C_{34} + \mu_4)(-2\mu_4 G_4)}{2[-\mu_3 C_{34} + \mu_4]^2}$$

$$Lq_4 = -\mu_4\left[\frac{(-\mu_3 C_{34} G_3 + \mu_4 G_4)}{(-\mu_3 C_{34} + \mu_4)^2} + \frac{G_4}{-\mu_3 C_{34} + \mu_4}\right] \tag{27}$$

If Lq be the mean queue length of the whole system then

$Lq = Lq_1 + Lq_2 + Lq_3 + Lq_4$

From (24), (25), (26) and (27) we have

$$Lq = -\mu_1\left[\frac{(\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4)}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})^2} + \frac{G_1}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})}\right]$$
$$- \mu_2\left[\frac{(-\mu_1 G_1 A_{12} + \mu_2 G_2 - \mu_4 D_{42} G_4)}{(-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})^2} + \frac{G_2}{(-A_{12}\mu_1 + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})}\right]$$
$$-\mu_3\left[\frac{(-\mu_2 B_{23} G_2 + \mu_3 G_3)}{(-\mu_2 B_{23} + \mu_3 - D_{43}\mu_4)^2} + \frac{G_3}{(-\mu_2 B_{23} + \mu_4 - \mu_4 D_{43})}\right]$$
$$-\mu_4\left[\frac{(-\mu_3 C_{34} G_3 + \mu_4 G_4)}{(-\mu_3 C_{34} + \mu_4)^2} + \frac{G_4}{-\mu_3 C_{34} + \mu_4}\right]$$

…(28)

Where $G_1$, $G_2$, $G_3$ and $G_4$ are given by (19), (20), (21) and (22).

**4 Conclusion**

As a result, we can calculate the system's mean queue length from (28), which allows us to use this queue model to estimate future events in reaction to certain alterations. To make accessible to clients who are waiting, adjustments could be made to the arrival pattern, the average length of service, the number of servers, or the available space.

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