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A Total Chromatic Number of Comb Product of Bi-fan and Bi-Fish with Various Graphs

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Abstract:

In this paper, we investigate the total chromatic number of comb product involving various graph families. The comb product between G and H ,denoted by (G o H), is a graph obtained by taking one copy of G and |V (G)| copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G .The total chromatic number of a graph G,denoted $\chi tc(G)$ is defined as the minimum number of colors needed to color the vertices and edges of a graph in such a way that different colors for two adjacent vertices, two adjacent edges and an incident vertex and edge. We explore the total chromatic number of comb product involving graphs namely, Bi-fan with cycle, Bi-fan with star, Bi-fan with fan, Bi-fish with path, Bi-fish with star, Bi-fish with cycle, Bi-fish with fan and Bi-fish with complete graph.

Keywords: Total chromatic number, Comb product, Bi-fan, Bi-fish, path, star, cycle, fan and complete.

1.INTRODUCTION

In this paper, we investigate the total chromatic number of comb products involving various graph families. The comb product between G and H ,denoted by (G ▷o H),is a graph obtained by taking one copy of G and |V (G)| copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G .The total chromatic number of a graph G,denoted $\chi_{tc}(G)$ is defined as the minimum number of colors needed to color the vertices and edges of a graph in such a way that different color for two adjacent vertices, two adjacent edges and an incident vertex and edge. $\Delta(G)+1\leq\chi tc(G)\leq\Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G and this is called as total coloring conjecture(Tcc). If it is total coloring with $\Delta(G)+1$ colors it is called type-I. If it is total coloring with $\Delta(G)+2$ colors it is called a type-II.

2. COMB PRODUCT OF BI-FAN WITH SIMPLE GRPHS

Definition 2.1: A Bi-Fan $B(F_{1,m,m})$ is obtained by joining the centre (apex) vertices of two copies of ($F_{1,m}$)by an edge. The vertex set of $B(F_{1,m,m})$ is $V[B(F_{1,m,m})] = \{u, v, u_a , v_a/1 \leq a \leq m\}$, where u, v are apex vertices and u_a, v_a are vertices.

The edge set of ($F_{1,m,m}$) is $E[B(F_{1,m,m})] = \{uv, uu_a, vv_a/1 \leq a \leq m, u_a u_{a+1}/1 \leq a\leq m-1,$

$$v_a v_{a+1}/1 \leq a \leq m-1 \}$$

It is clear that $|V[B(F_{1,m,m})]| = 2m+2$ and $|E[B(F_{1,m,m})]| = 4m-1$.

Example 2.1: Consider the Fan graph ($F_{1,n}$)

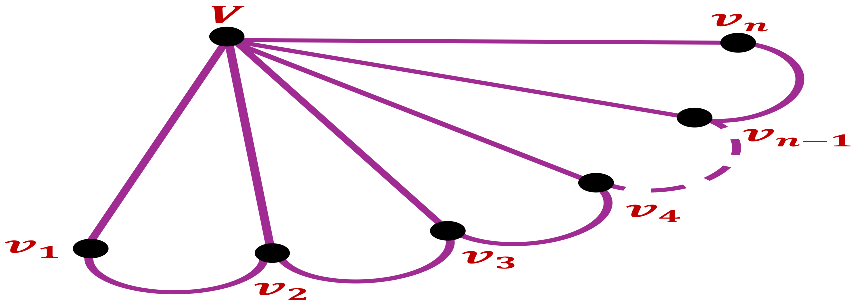

Figure:2.1

Theorem 2.1: The TCN of Bi-fan graph is m+2, for m ≥ 3

Proof: Let BFG be a Bi-fan graph. The apex of G is denoted as u, v.

Let V={u}∪{v}∪{u_a/1≤ a ≤ m}∪{v_a/1≤ a ≤ m} be the vertices and

E={uv} ∪{uu_a/ 1≤ a ≤ m}∪{vv_a/ 1≤ a ≤ m}∪{$u_a u_{a+1}$/1≤ a ≤ m-1}∪{$v_a v_{a+1}$/1≤ a ≤ m-1}

be the edges of bi-fan graph.

S = $V\left(B\left([F_{1,m,m}]\right)\right) \cup E\left(B\left([F_{1,m,m}]\right)\right)$ and C={1,2,3,…}

We now define total coloring as a function f:S→C.

First provide coloring for the vertices as follows:

$f(u)$ ={m+1}, $f(v)$ = {1}.

For 1≤ a ≤ m-1, $f(u_a)$ ={a+1}; $f(u_m)$ = {1}.

For 1≤ a ≤ m, $f(v_a)$ ={a+2}.

Next provide coloring for the edges as follows:

$f(uv)$ ={m+2}.

For 1≤ a ≤ m, $f(uu_a)$ ={a}, $f(vv_a)$ ={a+1}

For 1≤ a ≤ m-1, $f(u_a u_{a+1})= \{a+3\}$; For 1≤ a ≤ m-1, $f(v_a v_{a+1})= \{a\}$

Hence, the coloring norm above, we get TCN of Bi-fan graph is m+2.

Example 2.2:Consider the Bi-fan graph $B\left([F_{1,4,4}]\right)$

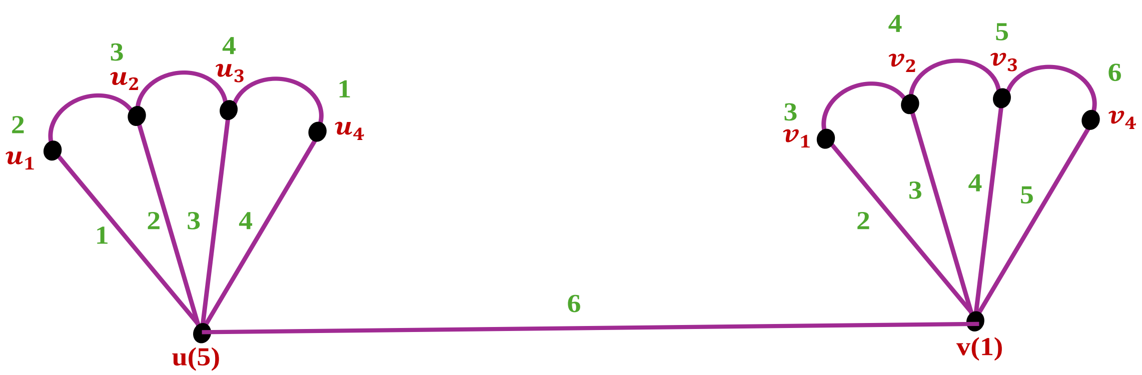

Figure:2.2

The Total Chromatic Number of the above graph is 6.

Theorem 2.2: The TCNOCP of Bi-fan graph with path graph is m+3, for $m \geq 3, n \geq 2$ and $m \neq n$

Proof: Let $B(F_{1,m,m})$ be a Bi-fan graph. V= $\{u\} \cup \{v\} \cup \{u_a, 1 \leq a \leq m\} \cup \{v_a, 1 \leq a \leq m\}$ be the vertices and E= $\{uv\} \cup \{uu_a, 1 \leq a \leq m\} \cup \{vv_a, 1 \leq a \leq m\} \cup \{u_a u_{a+1}, 1 \leq a \leq m-1\} \cup$

$\{v_a v_{a+1}/1 \leq a \leq m-1\}$ be the edges of bi-fan graph of $B(F_{1,m,m})$.

Let H be Path graph. V=$\{w_a/1 \leq a \leq n\}$ be the vertices and $\{w_a w_{a+1}/1 \leq a \leq n-1\}$ be the edges of path graph. A path P_n *is* joined with each vertex of the bi-fan graph.

Here, $\Delta[(B(F_{1,m,m})) \lhd P_n] = \{m+2\}$

Let $S=V[(B(F_{1,m,m}) \lhd P_n] \cup E[(B(F_{1,m,m}) \lhd P_n]$ and $C=\{1,2,3,4,\ldots\}$

We now define total coloring as a function f:S→C.

Total number of vertices is 2m+2n and total number of edges is 2(mn + m + n)-3

First provide coloring for the vertices as follows:

$f(u_{1,1})=\{m+2\}$, $f(v_{1,1})=\{m+3\}$.

For $1 \leq b \leq \frac{n-1}{2}$, if n is odd $\quad f(u_{1,2b}) = \{1\}$

For $1 \leq b \leq \frac{n-1}{2}$, if n is odd $\quad f(u_{1,2b+1}) = \{2\}$

For $1 \leq b \leq \frac{n-1}{2}$, if n is odd $\quad f(v_{1,2b}) = \{2\}$

For $1 \leq b \leq \frac{n-1}{2}$, if n is odd $\quad f(v_{1,2b+1}) = \{1\}$

For $1 \leq b \leq \frac{n}{2}$, if n is even $\quad f(u_{1,2b}) = \{1\}$

For $1 \leq b \leq \frac{n-2}{2}$, if n is even $\quad f(u_{1,2b+1}) = \{2\}$

For $1 \leq b \leq \frac{n}{2}$, if n is even $\quad f(v_{1,2b}) = \{2\}$

For $1 \leq b \leq \frac{n-2}{2}$, if n is even $\quad f(v_{1,2b+1}) = \{1\}$

For $2 \leq a \leq m+1$ and $1 \leq b \leq n$ $\quad f(u_{a,b})=\begin{cases} 1 \text{ If a is even and b is odd; If a is odd and b is even} \\ 2 \text{ If a is evenand b is even; If a is odd and b is odd} \end{cases}$

For $2 \leq a \leq m+1$ and $1 \leq b \leq n$ $\quad f(v_{a,b})=\begin{cases} 1 \text{ If a is even and b is odd; If a is odd and b is even} \\ 2 \text{ If a is even and b is even; If a is odd and b is odd} \end{cases}$ Next provide coloring for the edges as follows:

$f(u_{1,1}v_{1,1}) = \{1\}$

For $2 \leq a \leq m+1$, $f(u_{1,1}u_{a,1}) = \{a\}$.

For $2 \leq a \leq m+1$, $f(v_{1,1}v_{a,1}) = \{a\}$.

$f(u_{2a,1}u_{2a+1,1}) = \{m+1\}$. If m is even $1 \leq a \leq \frac{m}{2}$ and If m is odd $1 \leq a \leq \frac{m-1}{2}$

$f(u_{2a+1,b}u_{2a+2,b}) = \{m+2\}$. If m is even $1 \leq a \leq \frac{m-2}{2}$ and If m is odd $1 \leq a \leq \frac{m-1}{2}$

$f(v_{2a,b}u_{2a+1,b}) = \{m+1\}$. If m is even $1 \leq a \leq \frac{m}{2}$ and If m is odd $1 \leq a \leq \frac{m-1}{2}$

$f(v_{2a+1,b}u_{2a+2,b}) = \{m+2\}$. If m is even $1 \leq a \leq \frac{m-2}{2}$ and If m is odd $1 \leq a \leq \frac{m-1}{2}$

For $1 \leq b \leq n-1$, $f(u_{1,b}u_{1,b+1}) = \begin{cases} m+3 \text{ If b is odd} \\ m+2 \text{ If b is even} \end{cases}$

For 1≤ b ≤ n-1, $f(v_{1,b}v_{1,b+1}) = \begin{cases} m+2 & \text{If b is odd} \\ m+3 & \text{If b is even} \end{cases}$

For 1≤ b ≤ n-1, $f(u_{m+1,b}u_{m+1,b+1}) = \begin{cases} m & \text{If b is odd} \\ m-1 & \text{If b is even} \end{cases}$

For 1≤ b ≤ n-1, $f(v_{m+1,b}u_{m+1,b+1}) = \begin{cases} m & \text{If b is odd} \\ m-1 & \text{If b is even} \end{cases}$

For 2≤ a ≤ m, 1≤ b≤ n-1,

$f(v_{a,b}v_{a,b+1}) = \begin{cases} m+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ m & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

For 2≤ a ≤ m, 1≤ b≤ n-1

$f(u_{a,b}u_{a,b+1}) = \begin{cases} m+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ m & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

Therefore, from the coloring norm above, the TCNOCP of Bi-fan with path graph is m+3.

Theorem 2.3: TCNOCP of Bi-fan graph with star graph is m+n+1, for $m > 3, n \geq 2$ and $m \neq n$.

Proof: Let B($F_{1,m,m}$) be a Bi-fan graph. V= {u}∪{v}∪{u_a/1≤ a ≤ m}∪{v_a/1≤ a ≤ m} be the vertices and E={uv} ∪{uu_a/1≤ a ≤ m}∪{vv_a/1≤ a ≤ m}∪{uu_{a+1} /1≤ a ≤ m-1}∪{vv_{a+1} /1≤ a ≤ m-1} be the edges of bi-fan graph of B($F_{1,m,m}$).

Let S_n be star graph. V= {w_a /1≤ a ≤ n-1}∪ {w} be the vertices and {ww_a/1≤ a ≤ n-1} be an edges of star graph. A star S_n is joined with each vertex of the bi-fan graph.

Here Δ (B($F_{1,m,m}$▷S_n) = {m+n}.

S=V [(B($F_{1,m,m}$))◁S_n] ∪E[(B($F_{1,m,m}$))◁S_n] and C= {1,2,3, … }

We now define total coloring as a function f:S→C.

Total number of vertices is 2mn+2n and total number of edges is 2(mn+m+n)-3

First provide coloring for the vertices as follows:

$f(u_{1,1})$ = {m+1}, $f(u_m)$= {1}

$f(v_{1,1})$ = {m+n}, $f(v_m)$= {1}

For 2≤ a ≤ m+1, $f(u_{a,1}) = \begin{cases} 2 & \text{If a is even} \\ 1 & \text{If a is odd} \end{cases}$

For 2≤ a ≤ m+1, $f(v_{a,1}) = \begin{cases} 2 & \text{If a is even} \\ 1 & \text{If a is odd} \end{cases}$

For 2≤ a ≤ m+1 and 2≤ b≤ n $f(u_{a,b}) = \{a-1\}$

For 2≤ a ≤ m+1 and 2≤ b≤ n $f(v_{a,b}) = \{a-1\}$

Next provide coloring the edges as follows:

$f(u_{1,1}v_{1,1})$ ={m+n+1}

For 2≤ b ≤ n-1, $f(u_{1,b}u_{1,b+1}) = \{m+n+1\}$

For 2≤ b ≤ n-1, $f(v_{1,b}u_{1,b+1})$ = {m+n+1}

For 1≤ b ≤ n-1, $f(u_{m+1,b}u_{m+1,b+1}) = \begin{cases} m & \text{If b is odd} \\ m-1 & \text{If b is even} \end{cases}$

For 1≤ b ≤ n-1, $f(v_{m+1,b}v_{m+1,b+1}) = \begin{cases} m & \text{If b is odd} \\ m-1 & \text{If b is even} \end{cases}$

For 2≤ b ≤ n , $f(u_{1,1}u_{1,b}) = \{m+n+1\}$

For $2\leq b \leq n$, $f(v_{1,1}v_{1,b})=\{m+n+1\}$

For $2\leq a \leq m+1$, $f(u_{1,1}u_{a,1})=\{a-1\}$

For $2\leq a \leq m$, $f(v_{1,1}v_{a,1})=\{a-1\}$

For $2\leq a \leq m+1$ and $2\leq b \leq n$, $f(u_{a,1}u_{a,b})=\{m+b-1\}$

For $2\leq a \leq m+1$ and $2\leq b \leq n$, $f(v_{a,1}v_{a,b})=\{m+b-1\}$

For $2\leq a \leq m$, $f(u_{a,1}u_{a+1,1})=\begin{cases} m+n & \text{If a is even} \\ m+n+1 & \text{If a is odd} \end{cases}$

For $2\leq a \leq m$, $f(v_{a,1}v_{a+1,1})=\begin{cases} m+n & \text{If a is even} \\ m+n+1 & \text{If a is odd} \end{cases}$

Therefore, from the coloring norm above, the TCNOCP of Bi-fan with star graph is m+n+1.

Theorem 2.4: The TCNOCP of Bi-fan graph with cycle graph is m+4, for $m \geq 3, n \geq 2$ and $m \neq n$

Proof: Let $B(F_{1,m,m})$ be a Bi-fan graph. V= $\{u\}\cup\{v\}\cup\{u_a/1\leq a \leq m\}\cup\{v_a/1\leq a \leq m\}$ be the vertices and E=$\{uv\}\cup\{uu_a/1\leq a \leq m\}\cup\{vv_a/1\leq a \leq m\}\cup\{uu_{a+1}/1\leq a \leq m-1\}\cup\{vv_{a+1}/1\leq a \leq m-1\}$ be the edges of bi-fan graph of $B(F_{1,m,m})$.

Let C_n be Cycle graph. V=$\{w_a/1 \leq a \leq n\}$ be the vertices and $\{w_aw_{a+1}/1 \leq a \leq n\}$ be the edges of cycle graph.

A cycle C_n *is* joined with each vertex of the bi-fan graph.

Here $\Delta (B(F_{1,m,m} \rhd C_n)=\{m+3\}$

Let S=V[$(B(F_{1,m,m}) \lhd C_n$] $\cup$E[$(B(F_{1,m,m}) \lhd C_n$] and C= {1,2,3,4,… }

We now define total coloring as a function f:S→C.

Total number of vertices is 2mn+2n and total number of edges is 2mn+4m+2n-1

First provide coloring for the vertices as follows:

Case 1: If m is odd

$f(u_{1,1})=\{m+1\}$, $f(v_{1,1})=\{m+4\}$.

For $2\leq b \leq n-1$, $f(u_{1,b})=\begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For $2\leq b \leq n-1$, $f(v_{1,b})=\begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For $2 \leq a \leq m+1$ and $1 \leq b \leq n-1$

$f(u_{a,b}) = \begin{cases} 1 \text{ If a is odd and b is odd; If a is even and b is even} \\ 2 \text{ If a is even and b is odd; If a is odd and b is even} \end{cases}$

For $2\leq a \leq m+1$ and $1\leq b\leq n-1$,

$f(v_{a,b}) = \begin{cases} 1 \text{ If a is even and b is even; If a is odd and b is odd} \\ 2 \text{ If a is even and b is even; If a is odd and b is odd} \end{cases}$

For $2\leq a \leq m-1$, $f(u_{a,n})=\{m+1\}$

For $2\leq a \leq m-1$, $f(v_{a,n})=\{m+1\}$

Next provide coloring for the edges as follows:

$f(u_{1,1}v_{1,1})=\{m+3\}$,

For $1 \le b \le n-1$ $f(u_{1,b}u_{1,b+1}) = \begin{cases} m+2 & \text{If b is odd} \\ m+3 & \text{If b is even} \end{cases}$

For $1 \le b \le n-1$ $f(v_{1,b}v_{1,b+1}) = \begin{cases} m+2 & \text{If b is odd} \\ m+3 & \text{If b is even} \end{cases}$

For $2 \le a \le m+1$, $f(u_{1,1}u_{a,1}) = \{a-1\}$

For $2 \le a \le m+1$, $f(v_{1,1}v_{a,1}) = \{a-1\}$

For $2 \le a \le m+1$, $f(u_{a,n}u_{a,1}) = \{m+4\}$

For $2 \le a \le m+1$, $f(v_{a,n}v_{a,1}) = \{m+4\}$

For $2 \le a \le m$ $f(u_{a,1}u_{a+1,a}) = \begin{cases} m+2 & \text{If a is odd} \\ m+3 & \text{If a is even} \end{cases}$

For $2 \le a \le m$, $f(v_{a,1}v_{a+1,1}) = \begin{cases} m+2 & \text{If a is odd} \\ m+3 & \text{If a is even} \end{cases}$

For $2 \le a \le m+1$, $1 \le b \le n-1$,

$f(u_{a,b}v_{a,b+1}) = \begin{cases} m+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ m+2 & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

For $2 \le a \le m+1$, $1 \le b \le n-1$,

$f(v_{a,b}v_{a,b+1}) = \begin{cases} m+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ m+2 & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

Case ii. If m is even

$f(u_{1,1}) = \{m+1\}$, $f(v_{1,1}) = \{m+4\}$.

For $1 \le b \le n-1$ $f(u_{1,b}) = \begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For $1 \le b \le n-1$ $f(v_{1,b}) = \begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For $2 \le a \le m+1$ and $1 \le b \le n$.

$f(u_{a,b}) = \begin{cases} 1 & \text{If a is odd and b is odd; If a is even and b is even} \\ 2 & \text{If a is even and b is odd; If a is odd and b is even} \end{cases}$

For $2 \le a \le m+1$ and $1 \le b \le n$.

$f(v_{a,b}) = \begin{cases} 1 & \text{If a is even and b is even; If a is odd and b is odd} \\ 2 & \text{If a is even and b is even; If a is odd and b is odd} \end{cases}$

Next provide the coloring for the edges as follows

$f(u_{1,1}v_{1,1}) = \{m+3\}$,

For $1 \le b \le n-1$ $f(u_{1,b}u_{1,b+1}) = \begin{cases} m+3 & \text{If b is odd} \\ m+2 & \text{If b is even} \end{cases}$

For $1 \le b \le n-1$ $f(v_{1,b}v_{1,b+1}) = \begin{cases} m+2 & \text{If b is odd} \\ m+3 & \text{If b is even} \end{cases}$

For $2 \le a \le m+1$, $f(u_{1,1}u_{a,1}) = \{a-1\}$

For $2 \le a \le m+1$ $f(v_{1,1}v_{a,1}) = \{a-1\}$

For $2 \le a \le m+1$ $f(u_{a,n}u_{a,1}) = \{m+4\}$

For $2 \le a \le m+1$ $f(v_{a,n}v_{a,1}) = \{m+4\}$

For $2 \le a \le m$ $f(u_{a,1}u_{a+1,a}) = \begin{cases} m+2 & \text{If a is odd} \\ m+3 & \text{If a is even} \end{cases}$

For 2≤ a≤ m $f(v_{a,1}v_{a+1,1})=\begin{cases} m+2 & \text{If a is odd} \\ m+3 & \text{If a is even} \end{cases}$

For 2≤ a ≤ m+1, 1≤ b≤ n-1,

$f(u_{a,b}v_{a,b+1})=\begin{cases} \mathrm{m}+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ \mathrm{m}+2 & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

For 2≤ a ≤ m+1, 1≤ b≤ n-1,

$f(v_{a,b}v_{a,b+1})=\begin{cases} \mathrm{m}+1 & \text{If a is even and b is odd; If a is odd and b is odd} \\ \mathrm{m}+2 & \text{If a is even and b is even; If a is odd and b is even} \end{cases}$

Therefore, from the coloring norm above, the TCNOCP of Bi-fan with cycle graph is m+4.

Theorem 2.5: The TCNOCP of Bi-fan graph with fan graph is m+n+2, for m≥ 3, n ≥ 2 and m ≠ n

Proof: Let B($F_{1,m,m}$) be a Bi-fan graph. V= {u}∪{v}∪{u_a/1≤ a ≤ m}∪{v_a/1≤ a ≤ m} be the vertices and {uv} ∪{uu_a/1≤ a ≤ m} ∪{vv_a/1≤ a ≤ m} ∪{uu_{a+1} /1≤ a ≤ m-1}∪{vv_{a+1} /1≤ a ≤ m-1} be the edges of Bi-fan graph of B($F_{1,m,m}$).

Let $F_{1,n}$ be a Fan graph. V= {w_a/1≤ a ≤ n}∪ {w}be the vertices and ∪ {w_aw_{a+1} /1≤ a ≤ n-1}∪

{ww_a/1≤ a ≤ n} be the edges of fan graph.

A fan $F_{1,n}$ *is* joined with each vertex of the bi-fan graph

Here, Δ[B($F_{1,m,m}$)▷ ($F_{1,n}$)]= m+ n+1

S = $V\left(B\left(\left[\mathrm{F}_{1,m,m}\mathrm{n} \triangleright \mathrm{F}_{1,n}\right]\right)\right) \cup E\left(B\left(\left[\mathrm{F}_{1,m,m}\mathrm{n} \triangleright \mathrm{F}_{1,n}\right]\right)\right)$ and C= {1, 2, 3, 4, ...}

We now define total coloring as a function f:S→C.

Total number of vertices is 2mn+2n+2m+2 and total number of edges is 4mn+2m+4n-3

First provide coloring for the vertices as follows:

$f(u_{1,1})$={m+1},$f(v_{1,1})$={m+n+1}.

For 2≤ a ≤ m+1, $f(u_{a,1})=\begin{cases} 2 & \text{If a is even} \\ 1 & \text{If a is odd} \end{cases}$

For 2≤ a ≤ m+1, $f(v_{a,1})=\begin{cases} 2 & \text{If a is even} \\ 1 & \text{If a is odd} \end{cases}$

For 2≤ b ≤ m+1, $f(u_{1,b})=\begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For 2≤ b ≤ n+1, $f(v_{1,b})=\begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For $2 \leq \mathrm{a} \leq \mathrm{m}+1,\ 1 \leq \mathrm{b} \leq \mathrm{n}-1$

$f(u_{a,b})=\begin{cases} \mathrm{m}+\mathrm{n}+1 & \text{If a is odd and b is even; If a is even and b is even} \\ \mathrm{m}+\mathrm{n}+2 & \text{If a is even and b is odd; If a is odd and b is odd} \end{cases}$

For 2≤ a ≤ m+1, 1≤ b≤ n-1,

$f(v_{a,b})=\begin{cases} \mathrm{m}+\mathrm{n}+1 & \text{If a is even and b is even; If a is odd and b is even} \\ \mathrm{m}+\mathrm{n}+2 & \text{If a is even and b is odd; If a is odd and b is odd} \end{cases}$

$f(u_{a,n})$= {m+1}, For 2≤ a ≤ m-1,

Next provide coloring for the edges as follows;

$f(u_{1,1}v_{1,1})$ ={m+n+2},

For 2≤ b ≤ n-2, $f(u_{1,b}u_{1,b+1})$ ={b+1}

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For 2≤ b ≤ n-2, $f(v_{1,b}v_{1,b+1})$ ={$b+1$}

For 2≤ a≤ m+1, $f(u_{1,1}u_{a,1}) = \{a-1\}$

For 2≤ a≤ m+1, $f(v_{1,1}v_{a,1})$ ={$a-1$}

For 2≤ b≤ n+1, $f(u_{1,1}u_{1,b}) = \{m+b-1\}$

For 2≤ b≤ n+1, $f(v_{1,1}v_{1,b})$ ={$m+b-1$}

For $2 \le a \le m+1$ and $2 \le b \le n+1$, $f(u_{a,1}u_{a,b}) = \{m+b-1\}$

For 2≤ a ≤ m+1 and 2≤ b ≤ n+1, $f(v_{a,1}v_{a,b})$ ={$m+b-1$}

For 2≤ a ≤ m+1 and 2≤ b ≤ n+1, $f(u_{a,b}v_{a,b+1}) = \begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

For 2≤ a ≤ m+1 2≤ b≤ n-2, $f(v_{a,b}v_{a,b+1}) = \begin{cases} 1 & \text{If b is even} \\ 2 & \text{If b is odd} \end{cases}$

Therefore, from the coloring norm above, the TCNOCP of Bi-fan with fan graph is m+n+2.

3. COMB PRODUCT OF BI-FISH WITH SIMPLE GRAPHS

Definition 3.1: A Bi-Fish graph obtained by joining the vertices u_1 , v_1 of two copies of fish graph by an edge u_1v_1.The vertex set of Bi-Fish $\{u_a, v_a/1 \le a \le 6\}$,the edge set of Bi-Fish is $\{u_1v_1\} \cup \{u_2u_4\} \cup \{u_3u_4\} \cup \{u_1u_2\} \cup \{u_1u_3\}\{u_4u_5\} \cup \{u_4u_6\} \cup \{u_5u_6\} \cup \{v_1v_3\} \cup \{v_2v_4\} \cup \{v_3v_4\} \cup \{v_1v_2\} \cup \{v_4v_5\} \cup \{v_4v_6\} \cup \{v_5v_6\}$.

Total number of vertices is 12 and total number of edges is 15

Theorem 3.1: The TCN of Bi-fish graph is 5.

Proof: Let BFG be a Bi-fish graph. The apex of BFG is denoted as u_1 and v_1. Bi-fish graph has 12 vertices and 15 edges. The vertices of BFG are denoted as $V = \{u_2, u_3, u_4, u_5, u_6\} \cup \{v_2, v_3, v_4, v_5, v_6\}$. The edges in the graph BFG are denoted as E={ u_1v_1 } ∪ $\{u_2u_4\} \cup \{u_3u_4\} \cup \{u_1u_2\} \cup$ $\{u_1u_3\}$ { u_4u_5 } ∪ $\{u_4u_6\} \cup \{u_5u_6\} \cup \{v_1v_3\} \cup \{v_2v_4\} \cup \{v_3v_4\} \cup \{v_1v_2\} \cup \{v_4v_5\} \cup \{v_4v_6\} \cup \{v_5v_6\}$.

Let S= V(BFG) ∪ E(BFG) and C ={1 ,2, 3, …}

We now define total coloring as a function f:S→C.

Provide coloring for the vertices and edges as follows:

Coloring	Vertices count	Number of edges	Verified vertices	Verified edges
1	u_2, u_3, u_6, v_1, v_4 .	$\{u_4u_5\}$.	5	1
2	u_1, u_4, v_2, v_3, v_6 .	$\{v_4v_5\}$.	5	1
3	u_5, v_5.	$\{u_1u_2\}, \{u_3u_4\}, \{v_3v_4\}, \{v_1v_2\}$.	2	4
4		$\{u_1u_3\}, \{u_2u_4\}, \{u_5u_6\}, \{v_1v_3\}$, $\{v_5v_6\}$,$\{v_2v_4\}$.		6
5		$\{u_1v_1\}, \{u_4u_6\}, \{v_4v_6\}$.		3
	TOTAL		12	15

Hence, the coloring norm shows that the TCN of Bi-fish graph is 5.

Example 3.1: Consider the Bi-fish graph

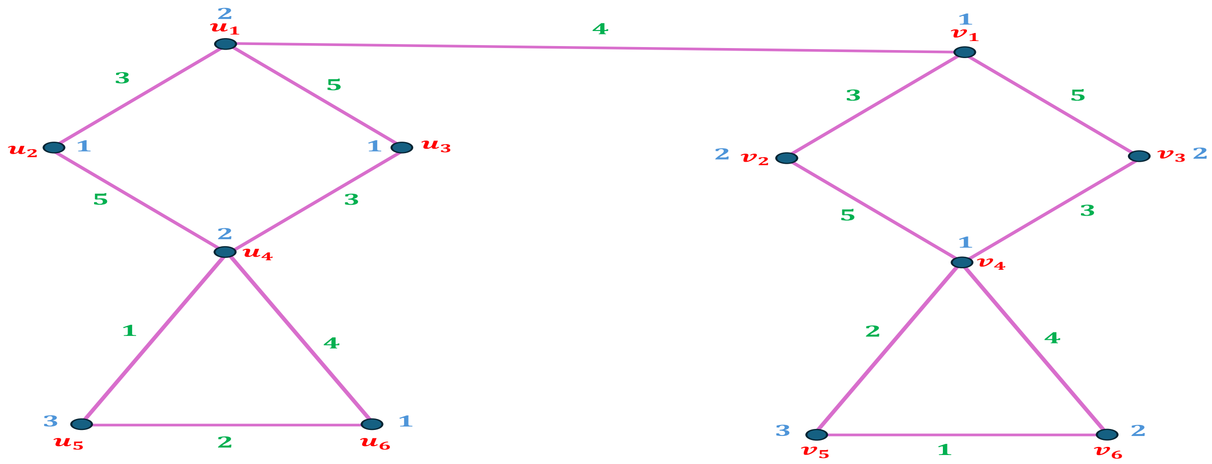

Figure:3.1

The Total Chromatic Number of the above graph is 5.

Theorem 3.2: The TCNOCP of Bi-fish with path graph is 6, for $n \geq 3$.

Proof: Let BFG be a Bi-fish graph. V= $\{u_a, v_a/1 \leq a \leq 6\}$, be the vertices and E=$\{u_1v_1\} \cup \{u_2u_4\} \cup \{u_3u_4\} \cup \{u_1u_2\} \cup \{u_1u_3\} \cup \{u_4u_5\} \cup \{u_4u_6\} \cup \{u_5u_6\} \cup \{v_1v_3\} \cup \{v_2v_4\} \cup \{v_3v_4\} \cup \{v_1v_2\} \cup \{v_4v_5\} \cup \{v_4v_6\} \cup \{v_5v_6\}$

Let P_n be Path graph. V=$\{w_a/1 \leq a \leq n\}$ be the vertices and $\{w_aw_{a+1}/1 \leq a \leq n-1\}$ be the edges of path graph. This path P_n *is* joined with each vertex of the bi-fish graph.

Let S= $V(BFG) \triangleright P_n) \cup E(BFG) \triangleright P_n)$ and C= $\{1,2,3,4,\ldots\}$

We now define total coloring as a function f:S→C.

Total number of vertices is 12n and total number of edges is 12n+3

First provide coloring for the vertices as follows:

$f(u_{a,b})= \begin{cases} 1 \text{ If b is odd} \\ 2 \text{ If b is even} \end{cases}$ If $a = 1,4$ for $1 \leq b \leq n$

$f(u_{a,b})= \begin{cases} 2 \text{ If a is odd} \\ 1 \text{ If a is even} \end{cases}$ If $a = 2,3,5$ for $1 \leq b \leq n$

$f(v_{a,b})= \begin{cases} 2 \text{ If b is odd} \\ 1 \text{ If b is even} \end{cases}$ If $a = 1,4$ for $1 \leq b \leq n$

$f(v_{a,b})= \begin{cases} 1 \text{ If a is odd} \\ 2 \text{ If a is even} \end{cases}$ If $a = 2,3,5$ for $1 \leq b \leq n$

$f(u_{6,b})= \begin{cases} 1 \text{ If b is even} \\ 2 \text{ If b is odd} \end{cases}$ for $2 \leq b \leq n$, $f(u_{6,1})= \{3\}$

$f(v_{6,b})= \begin{cases} 1 \text{ If a is odd} \\ 2 \text{ If a is even} \end{cases}$ for $2 \leq b \leq n$, $f(v_{6,1})= \{3\}$

Next provide coloring for the edges as follows:

$f(u_{1,1}v_{1,1}) = \{6\}$,$f(u_{1,1}u_{2,1}) = \{3\}$, $f(u_{1,1}u_{3,1}) = \{4\}$, $f(u_{2,1}u_{4,1}) = \{4\}$, $f(u_{3,1}u_{4,1}) = \{3\}$,

$f(u_{4,1}u_{5,1}) = \{6\}$,$f(u_{4,1}u_{6,1}) = \{2\}$,$f(u_{5,1}u_{6,1}) = \{4\}$,$f(v_{1,1}v_{2,1})= \{4\}$,$f(v_{1,1}v_{3,1})=\{3\}$,

$f(v_{2,1}v_{4,1})$={3}, $f(v_{3,1}v_{4,1})$={4}, $f(v_{4,1}v_{5,1})$={6}, $f(v_{4,1}v_{6,1})$={1}, $f(v_{5,1}v_{6,1})$={4},

For 1≤ a ≤ 6, 1≤ b≤ n-1,

$$f(u_{a,b}u_{a,b+1}) = \begin{cases} 5 \text{ If a is odd and b is odd; If a is even and b is odd} \\ 6 \text{ If a is odd and b is even; If a is even and b is even} \end{cases}$$

For 1≤ a ≤ 6, 1≤ b≤ n-1,

$$f(v_{a,b}v_{a,b+1}) = \begin{cases} 5 \text{ If a is oddand b is odd; If a is even and b is odd} \\ 6 \text{ If a is odd and b is even; If a is even and b is even} \end{cases}$$

Therefore, from the coloring norm above, the TCNOCP of Bi-fish with path graph is 6.

Theorem 3.3: The TCNOCP of Bi-fish with star graph is n+4, for n ≥ 3.

Proof: Let BGG be a Bi-fish graph. V= $\{u_a, v_a/1 \leq$ a $\leq 6\}$, be the vertices and E=$\{u_1v_1\}\cup\{u_2u_4\}\cup\{u_3u_4\}\cup\{u_1u_2\}\cup\{u_1u_3\}\cup\{u_4u_5\}\cup\{u_4u_6\}\cup\{u_5u_6\}\cup\{v_1v_3\}\cup\{v_2v_4\}\cup\{v_3v_4\}\cup\{v_1v_2\}\cup\{v_4v_5\}\cup\{v_4v_6\}\cup\{v_5v_6\}$ be the edges.

Let S_n be star graph. V= {w_a /1≤ a ≤ n-1}∪ {w} be the vertices and {ww_a/1≤ a ≤ n-1} be an edges of star graph. This star S_n is joined with each vertex of the bi-fish graph.

Let $S = V$(BFG)$\triangleright S_n$) ∪ E(BFG) $\triangleright S_n$) and C= {1,2, 3, 4, …}

We now define total coloring as a function f:S→ C.

Total number of vertices is 12n and total number of edges is 12n+3. First provide coloring for the vertices as follows:

$f(u_{1,1})$= {n+3}, $f(u_{4,1})$= {n+3}, $f(u_{6,1})$= {n+3},

$f(v_{1,1})$= {n+4}, $f(v_{4,1})$= {n+4}, $f(v_{6,1})$= {n+4},

$f(u_{2,1})$= {n+4}, $f(u_{3,1})$= {n+4}, $f(u_{5,1})$= {n+4},

$f(v_{2,1})$= {n+3}, $f(v_{3,1})$= {n+3}, $f(v_{5,1})$= {n+3},

$f(u_{a,b})$= {n+4} If a =1,4 and 6, for 2≤ b ≤ n

$f(u_{a,b})$= {n+3} If a =2,3 and 5, for 2≤ b ≤ n

$f(v_{a,b})$= {n+3} If a =1,4 and 6, for 2≤ b ≤ n

$f(v_{a,b})$= {n+4} If a =2,3 and 5, for 2≤ b ≤ n

Next provide coloring for the edges as follows:

$f(u_{1,1}v_{1,1})$ = {n+2},$f(u_{1,1}u_{2,1})$ = {n+1}, $f(u_{1,1}u_{3,1})$ = {n}, $f(u_{2,1}u_{4,1})$ = {n},

$f(u_{3,1}u_{4,1})$ = {n+1}, $f(u_{4,1}u_{5,1})$ = {n+2},$f(u_{4,1}u_{6,1})$ = {n+4},$f(u_{5,1}u_{6,1})$ = {n+1},

$f(v_{1,1}v_{2,1})$ = {n+1}, $f(v_{1,1}v_{3,1})$={n}, $f(v_{2,1}v_{4,1})$={n}, $f(v_{3,1}v_{4,1})$={n+1}, $f(v_{4,1}v_{5,1})$={n+2}, $f(v_{4,1}v_{6,1})$={n+3}, $f(v_{5,1}v_{6,1})$={n+1},

For 1≤ a ≤ 6 ,1≤ b ≤ n-1, $f(v_{a,1}v_{a,b})$ = {b-1}.

For 1≤ a ≤ 6 ,1≤ b ≤ n-1, $f(u_{a,1}u_{a,b})$ = {b-1}.

Therefore, from the coloring norm above, the TCNOCP of Bi-fish with star graph is n+4.

Theorem 3.4: The TCNOCP of Bi-fish with cycle graph is 7, for n ≥ 3.

Proof: Let BFG be a Bi-fish graph. V= $\{u_a, v_a/1 \leq$ a $\leq 6\}$, be the vertices and E=$\{u_1v_1\}\cup\{u_2u_4\}\cup\{u_3u_4\}\cup\{u_1u_2\}\cup\{u_1u_3\}\cup\{u_4u_5\}\cup\{u_4u_6\}\cup\{u_5u_6\}\cup\{v_1v_3\}\cup\{v_2v_4\}\cup\{v_3v_4\}\cup\{v_1v_2\}\cup\{v_4v_5\}\cup\{v_4v_6\}\cup\{v_5v_6\}$ be the edges.

Let C_n be Cycle graph. V={$w_a/1 \leq$ a $\leq$ n} be the vertices and {$w_a w_{a+1}/1 \leq$ a $\leq$ n} be the edges of cycle graph. A cycle C_n is joined with each vertex of the bi-fish graph.

Total number of vertices is 12n and total number of edges is 12n+15

Let $S = V(\text{BFG}) \triangleright C_n) \cup E(\text{BFG}) \triangleright C_n)$ and C= {1,2, 3, 4,…}

We now define total coloring as a function f:S→ C.

We provide colors when n is even .

Coloring	Vertices count	Number of edges	Vertices count	Number of edges
1	$f(u_{a,2b-1})$for 1≤ b ≤ $\frac{n}{2}$, If a=1 & 4. $f(u_{a,2b})$for 1≤ b ≤ $\frac{n}{2}$, If a=2, 3 & 5. $f(v_{a,2b-1})$for 1≤ b ≤ $\frac{n}{2}$, If a=2, 3 & 5. $f(v_{a,2b})$ for 1≤ b ≤ $\frac{n}{2}$, If a=1 & 4. $f(u_{6,2b})$ $f$or 1≤ b ≤ $\frac{n}{2}$. $f(v_{6,2b})$ for 1≤ b ≤ $\frac{n}{2}$.	$f(v_{4,1}v_{6,1})$	6n	1
2	$f(u_{a,2b})$ for 1≤ b ≤ $\frac{n}{2}$, If a=1 & 4. $f(u_{a,2b-1})$ for 1≤ b ≤ $\frac{n}{2}$, If a=2,3 & 5. $f(v_{a,2b})$ for 1≤ b ≤ $\frac{n}{2}$, If a=2,3 & 5. $f(v_{a,2b-1})$ for 1≤ b ≤ $\frac{n}{2}$, If a=1 &4.	$f(u_{4,1}u_{6,1})$	5n	1
3		$f(u_{1,1}u_{3,1})$, $f(u_{2,1}u_{4,1})$ $f(u_{5,1}u_{6,1})f(v_{11}v_{3,1})$ $f(v_{2,1}v_{4,1})$,$f(v_{5,1}v_{6,1})$		6
4	$f(u_{6,2b-1})$ for 1≤ b ≤ $\frac{n}{2}$ $f(v_{6,2b-1})$ for 1≤ b ≤ $\frac{n}{2}$	$f(u_{1,1}u_{2,1})$ $f(u_{3,1}u_{4,1})$ $f(v_{1,1}v_{2,1})$,$f(v_{3,1}v_{4,1})$	n	4
5		$f(u_{a,2b-1}u_{a,2b})$ for 1≤ a ≤ 6,		6n

		$1\leq b \leq \frac{n}{2}$ $f(v_{a,2b-1}v_{a,2b})$ for $1\leq a \leq 6$, $1\leq b \leq \frac{n}{2}$		
6		$f(u_{a,2b}u_{a,2b+1})$ for $1\leq a \leq 6$, $1\leq b \leq \frac{n-2}{2}$ $f(v_{a,2b}v_{a,2b+1})$ for $1\leq a \leq 6$, $1\leq b \leq \frac{n-2}{2}$ $f(u_{a,n}u_{a,1})$ for $1\leq a \leq 6$, $f(v_{a,n}v_{a,1})$ for $1\leq a \leq 6$,		6n
7		$f(u_{1,1}v_{1,1})f(u_{4,1}u_{5,1})$ $f(v_{4,1}v_{5,1})$		3
	TOTAL		12n	15

Therefore, from the coloring norm above, the TCNOCP of Bi-fish with cycle graph is m+n+2.

Theorem 3.5: The TCNOCP of Bi-fish with fan graph is n+5, for $n \geq 3$.

Proof: Let BFG be a Bi-fish graph. V= $\{u_a, v_a/1 \leq a \leq 6\}$, be the vertices and E=$\{u_1v_1\}\cup\{u_2u_4\}\cup\{u_3u_4\}\cup\{u_1u_2\}\cup\{u_1u_3\}\cup\{u_4u_5\}\cup\{u_4u_6\}\cup\{u_5u_6\}\cup\{v_1v_3\}\cup\{v_2v_4\}\cup\{v_3v_4\}\cup\{v_1v_2\}\cup\{v_4v_5\}\cup\{v_4v_6\}\cup\{v_5v_6\}$ be the edges.

Let $F_{1,n}$ be a Fan graph. V= $\{\{w_a/1\leq a \leq n\}\cup\{w\}$be the vertices and $\cup\{w_a w_a + 1/1\leq a \leq n-1\}\cup$

$\{\{ww_a/1\leq a \leq n\}$ be the edges of fan graph..

A fan $F_{1,n} is$ joined with each vertex of the bi-fish graph.

Let $S = V(\text{BFG}) \triangleright F_{1,n}) \cup E(\text{BFG}) \triangleright F_{1,n})$ where C= {1,2, 3, 4, …}

We now define total coloring as a function $f:S\rightarrow C$.

Total number of vertices is 12n+12 and total number of edges is 24n+3

First provide coloring for the vertices as follows:

For $1\leq a \leq 6$, $1\leq b \leq n$, $f(u_{a,b})$= {b+1},

For $1\leq a \leq 6$, $1\leq b \leq n$, $f(v_{a,b})$= {b+1}

$f(u_1)$={n+5}, $f(u_2$ ={n+4}, $f(u_3)$= {n+4} , $f(u_4)$={n+5} , $f(u_5)$ ={n+4} , $f(u_6)$ ={n+3}

$f(v_1)$= {n+4}, $f(v_2)$={n+5} , $f(v_3)$={n+5} , $f(v_4)$ ={n+4} $f(v_5)$={n+5} , $f(v_6)$ ={n+3}

Next provide coloring for the edges as follows;

$f(u_1v_1)$ = {n+1},$f(u_1u_2)$ = {n+2}, $f(u_1u_3)$ = {n+3}, $f(u_2u_4)$ = {n+3},

$f(u_3u_4)$ = {n+2}, $f(u_4u_5)$ = {n+1},$f(u_4u_6)$ = {n+4},

$f(u_5u_6)$ = {n+2}, $f(v_1v_2)$ = {n+2}, $f(v_1v_3)$={n+3}, $f(v_2v_4)$={n+3}, $f(v_3v_4)$={n+2}, $f(v_4v_5)$={n+1}, $f(v_4v_6)$={n+5}, $f(v_5v_6)$={n+2},

For 1≤ a ≤ 6 and 1≤ b ≤ n, $f(v_av_{a,b})$ = {b}.

For 1≤ a ≤ 6 and 1≤ b ≤ n, $f(u_au_{a,b})$ = {b}.

For 1≤ a ≤ 6 ,1≤ b ≤ n-1, $f(v_{a,b}v_{a,b+1}) = \begin{cases} n+2 & \text{If b is odd} \\ n+3 & \text{If b is even} \end{cases}$

For 1≤ a ≤ 6 ,1≤ b ≤ n-1, $f(u_{a,b}u_{a,b+1}) = \begin{cases} n+4 & \text{If b is odd} \\ n+5 & \text{If b is even} \end{cases}$

Therefore, from the coloring norm above, the TCNOCP of Bi-fish with fan graph is n+5.

Theorem 3.6: The TCNOCP of Bi-fish with complete graph is n+4 , for n ≥ 4.

Proof: Let BFG be a Bi-fish graph. V= $\{u_a, v_a/1 \le a \le 6\}$, be the vertices and E=$\{u_1v_1\}\cup\{u_2u_4\}\cup\{u_3u_4\}\cup\{u_1u_2\}\cup\{u_1u_3\}\cup\{u_4u_5\}\cup\{u_4u_6\}\cup\{u_5u_6\}\cup\{v_1v_3\}\cup\{v_2v_4\}\cup\{v_3v_4\}\cup\{v_1v_2\}\cup\{v_4v_5\}\cup\{v_4v_6\}\cup\{v_5v_6\}$ be the edges.

Let K_n be complete graph. V= {w_a /1≤ a ≤ n} be the vertices and {w_aw_{a+1} /1≤ a ≤ $\frac{n(n-1)}{2}$} be an edges of star graph. A Complete graph K_n *is* joined with each vertex of the bi-fish graph.

Let $S = V(\text{BFG}) \triangleright K_n) \cup E(\text{BFG}) \triangleright K_n)$ where C= {1,2, 3, 4, …}

We now define total coloring as a function f:S→ C.

Total number of vertices is 12n and total number of edges is $6n^2$-6n+15

First provide coloring for the vertices as follows:

V= $\{u_{a,b}\}$: 1≤ a ≤ 6, 1≤ b ≤ n}∪{v_{ab}: 1≤ a ≤ 6, 1≤ b ≤ n}

For If a =1 & 4. {1≤ b ≤ n}, $f(u_{a,b})$= {b}.

For If a =2,3 & 6. {2≤ b ≤ n},$f(u_{a,b})$= {b-1}.

For 2≤ b ≤ n}, $f(u_{5,b})$= {b-1}.

$f(u_{2,1})$ =$f(u_{3,1})$= $f(u_{6,1})$= {n+4}, $f(u_{5,1})$={n+2}

For If a =1 & 4. {1≤ b ≤ n}, $f(v_{a,b})$= {b-1}.

For If a =2,3 & 5. {1≤ b ≤ n}, $f(v_{a,b})$= {n+3}.

For {2≤ b ≤ n}, $f(v_{6,b})$= {b-1}.

$f(v_{1,1})$={n+3} , $f(u_{4,1})$={n+4} , $f(v_{6,1})$={n+3}.

Next provide coloring for the edges as follows:

$f(u_{1,1}v_{1,1})$ = {2},$f(u_{1,1}u_{2,1})$ = {n+2}, $f(u_{1,1}u_{3,1})$ = {n+3}, $f(u_{2,1}u_{4,1})$ = {n+3},

$f(u_{3,1}u_{4,1})$ = {n+2}, $f(u_{4,1}u_{5,1})$ = {n+4},$f(u_{4,1}u_{6,1})$ = {2},

$f(u_{5,1}u_{6,1})$ = {n+3}, $f(v_{1,1}v_{2,1})$ = {n+2}, $f(v_{1,1}v_{3,1})$={n+3}, $f(v_{2,1}v_{4,1})$={n+3}, $f(v_{3,1}v_{4,1})$={n+2}, $f(v_{4,1}v_{5,1})$={2}, $f(v_{4,1}v_{6,1})$={1}, $f(v_{5,1}v_{6,1})$={n+4},

For 1≤ a ≤ 6 and 1≤ b ≤ $\frac{n(n-1)}{2}$, $f(v_{a,b}v_{a,c})$ = {b+c}.

For 1≤ a ≤ 6 and 1≤ b ≤ $\frac{n(n-1)}{2}$, $f(u_{a,b}u_{a,c})$ = {b+c}.

Therefore, from the coloring norm above, the TCNOCP of Bi-fish with complete graph is n+4.

Conclusion:

We found the total chromatic number of comb product involving graphs namely, Bi-fan with cycle, Bi-fan with star, Bi-fan with fan, Bi-fish with path, Bi-fish with star, Bi-fish with cycle, Bi-fish with fan and Bi-fish with complete graph.

REFRENCES:

[1]. Behzad M., (1965). Graphs and their chromatic numbers, Doctoral Thesis, Michigan State University

[2]. Behzad M., Chartrand G., Cooper J.K. (1967), The color numbers of complete graphs, Journal London Math. Soc. 42, pp.226-228.

[3] Bondy, J.A and Murty, U.S.R (2008) Graph Theory, Springer international edition.

[4]. Borodin O.V. (1989), On the total coloring planar graphs, Journal fur die Reine und ngewandte Mathematik, 394, pp. 180-185.

[5]. Jonathan Jordan,(2009) Comb Graphs and Spectral Decimation, Glasgow Mathematical Journal,51, pp. 71–81.

[6]. Muthuramakrishnan Jayaraman(2018). Total Chromatic Number of Middle and Total Graph of Path and Sunlet Graph. Volume 6, Issue 4, pp.1-9 .

[7]. Rohmatulloh M.Y, Slamin Kristiana Dafik ,A.I,, and Alfarisi, R(2021), On metric chromatic number of comb product of ladder graph, ICCGANT IOP Publishing

[8]. Rosenfeld. M.(1971).On the total colouring of certain graphs, Israel J. Math., 9(3) pp.396-402.

[9]. Suhadi Wido Saputro, Novi Mardiana and Ira Apni Purwasih (2017), The Metric Dimension Of Comb Product Graphs, Matematiqki Vesnik 69, 4 , pp.248–258