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Structure Of the Truncated Laurent's Series Space as An Extension of Fuhrmann's Bilinear Form

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Abstract:

The behavior of the Fuhrmann dynamic systems is a closed subspace of the truncated Laurent series space, thus it can be concluded that the stability of the dynamic systems can be related to the closed nature of the behavior system studied. Since the behavior of a dynamic systems is a closed subspace of the truncated Laurent series space, this fact provides inspiration to study the characteristics of the truncated Laurent series space, especially related to the development of the Fuhrmann's bilinear form into a k-bilinear form, the structure of the closed subspace in the truncated Laurent's series space and the condition that satisfy Riesz representation theorem in truncated Laurent's series space.

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1 Introduction

System dynamics is a method used to describe a system that changes over time consisting of input variables, vector states, output variables. Willems [5] examined dynamical systems by observing the set of all trajectories called behavior. Polderman and Willems' (1997) behavioral approach to interconnected systems theory (without input/output) can produce very satisfactory performance through the representation of higher-order polynomial matrices. In addition, discussing behavior in the context of interconnected systems theory does not rely on problems in sequels to the analysis but focuses on a limited functional input/output model approach.

Since the realization of the state space, that is, behavior has infinite dimensions, the existence of this realization can be seen in the case of a rational transfer function.

The research findings emphasize that the stability of a dynamical system is intrinsically related to its shift-invariant behavior. This means that in systems where the output at certain values is independent of the time at which input variables occur, the system's stability remains unaffected by time shifts. Furthermore, Fuhrmann [1] established a one-to-one correspondence between the collection of polynomial submodules over a polynomial ring with the behavioral collection of Willems' dynamical system. Fuhrmann clarified that system stability depends on two conditions: the system's behavior must be fully characterized, and it should exhibit invariant behavior over time.

Truncated Laurent's series space consists of the trajectories of a dynamical system that are linear, discrete, and invariant with time. Any subspace within the aforementioned space is complete if and only if the orthogonal subspace of its orthogonal subspace is related to the original subspace [3]. This leads to the principle that a subspace is complete if and only if it is closed within this space.

The behavior of the Fuhrman dynamical system is a closed subspace of the truncated Laurent's series space, thus it can be concluded that the stability of the dynamic system is related to the closed nature of the behavior of the system under study.

2 Methods

2.1 Bilinear Spaces

Let V and W are respectively vector spaces over the field F. The bilinear form $[-,-]$ is a mapping on V x W, i.e.

$$[-,-] : V \; x \; W \longrightarrow F$$
$$(v, w) \longmapsto [v, w]$$

for all v in V, w in W. that satisfies the following conditions:

1. $[\alpha v_1 + \beta v_2, w] = \alpha[v_1, w] + \beta[v_2, w]$
2. $[v, \gamma w_1 + \delta w_2] = \gamma[v, w_1] + \delta[v, w_2]$

for all v, v_1, v_2 in V ; w, w_1, w_2 in W ; $\alpha, \beta, \gamma, \delta$ in F.

These two conditions state that for any bilinear form the linear nature applies to the first and second argument [2]. A vector space over the field F in which a bilinear form is defined is called a bilinear space. Because in the inner product of positivity applies, it is necessary to add a property commensurate with that property called non-degenerated. Such properties are defined as follows:

Definition 1: *The non-degenerated bilinear form of V x W is a bilinear form with the property of only a zero vector in a bilinear space V that is bilinearly orthogonal to the bilinear space W and also only a zero vector in a bilinear space W that is bilinearly orthogonal to a linear space V [2].*

2.2 Truncated Laurent's Series

The set of truncated Laurent's series with coefficients in column space F^n is defined as

$$F^n\left((z^{-1})\right) = \left\{\sum_{j=-\infty}^{N_f} f_j \, z^j \middle| f_j \in F^n \quad , \quad N_f \in \mathbb{Z}\right\}$$

The existence of truncated Laurent's series space is guaranteed by the assumption that only a finite number of non-zero terms. The corresponding addition operation is defined as $f(z) + g(z) = \sum_{j=-\infty}^{N}(f_j + g_j)\, z^j$ for every $f(z) = \sum_{j=-\infty}^{N_f} f_j \, z^j, g(z) = \sum_{j=-\infty}^{N_g} g_j \, z^j$ in $F^n\left((z^{-1})\right)$, $N = maks\{N_f, N_g\}$ and scalar multiplication operations defined as $\alpha f(z) = \sum_{j=-\infty}^{N_f}(\alpha f_j)\, z^j$ for every $f(z) = \sum_{j=-\infty}^{N_f} f_j \, z^j \in F^n\left((z^{-1})\right)$ and $\alpha \in F$. Then the space $F^n\left((z^{-1})\right)$ forms a vector space over the field F. It can be proved that the space $F^n\left((z^{-1})\right)$ can be written as a direct sum

$$F^n\left((z^{-1})\right) = F^n[z] \oplus z^{-1} F^n\left[[z^{-1}]\right] \tag{1}$$

where

$$F^n[z] = \left\{\sum_{j=0}^{N_f} f_j \, z^j \middle| f_j \in F^n, \quad N_f \in \mathbb{Z}\right\}$$

is the part of the polynomial with the coefficients in space F^n and

$$z^{-1} F^n\left[[z^{-1}]\right] = \left\{\sum_{j=-\infty}^{-1} f_j \, z^j \middle| f_j \in F^n\right\}$$

is the formal series part in z^{-1}[1]. It can be proved that $F^n[z]$ and $z^{-1}F^n\left[[z^{-1}]\right]$ respectively form a subspace of the space $F^n\left((z^{-1})\right)$. In the truncated Laurent's series space, a bilinear form of Fuhrmann [1] has been defined as follows:

$$[f,g] = \sum_{j=-\infty}^{\infty} g_{-j-1}^{T} f_j \qquad (2)$$

for all $f(z) = \sum_{j=-\infty}^{N_f} f_j z^j, g(z) = \sum_{j=-\infty}^{N_g} g_j z^j$ in $F^n\left((z^{-1})\right)$, where ${g_j}^T$ is transpose of g_j for all j. Bilinear form (2) is well-defined and non-degenerated. The truncated Laurent's series space with a bilinear form (2) is a bilinear space.

2.3 Closed Subspaces

The orthogonal subspace of the subspace S of $F^n\left((z^{-1})\right)$ is written $S^{\perp}$ defined

$$S^{\perp} = \left\{ f(z) \in F^n\left((z^{-1})\right) \middle| [f,h] = 0 \quad \forall h(z) \in S \right\} \qquad (3)$$

with the bilinear form $[-,-]$ given to equation (2) [4]. According to equation (3) then in truncated Laurent series space is obtained

1. $(F^n[z])^{\perp} = F^n[z]$ $\qquad (3a)$
2. $\left(z^{-1}F^n\left[[z^{-1}]\right]\right)^{\perp} = z^{-1}F^n\left[[z^{-1}]\right]$ $\qquad (3b)$

The orthogonal subspace of the orthogonal subspace of the S subspace of $F^n\left((z^{-1})\right)$ written $S^{\perp\perp}$ is defined

$$S^{\perp\perp} = \left\{ f(z) \in F^n\left((z^{-1})\right) \middle| [f,h] = 0 \quad \forall h(z) \in S^{\perp} \right\}$$

with the bilinear form $[-,-]$ given to equation (2). The non-empty subset S of space $F^n\left((z^{-1})\right)$ is said to be closed if

$$S^{\perp\perp} = S$$

So subspace S is said to be closed if the subspace that is orthogonal with the orthogonal subspace S are all contained in subspace S [4]. According to equation (3a) and (3b) are obtained

$$(F^n[z])^{\perp\perp} = F^n[z] \qquad (3c)$$

$$\left(z^{-1}F^n\left[[z^{-1}]\right]\right)^{\perp\perp} = z^{-1}F^n\left[[z^{-1}]\right] \qquad (3d)$$

3 Results

The following are presented the results of research related to the structure of the truncated Laurent's series space as an expansion of the Fuhrmann bilinear form

A. Let

$$V = F^n\left((z^{-1})\right) = \left\{ \sum_{j=-\infty}^{N_f} f_j z^j \middle| f_j \in F^n \quad , \quad N_f \in \mathbb{Z} \right\}$$

For every k integer, define the bilinear form as follows:

$$[f,g]_k = \sum_{j=-\infty}^{\infty} g_{-j-k}^{T} f_j \qquad (4)$$

for all $f(z) = \sum_{j=-\infty}^{N_f} f_j z^j$ and $g(z) = \sum_{j=-\infty}^{N_g} g_j z^j$ in V, where ${g_j}^T$ is transpose of g_j $for\ all\ j$. Then bilinear form (4) is non-degenerated.

Proof:

For every $f(z)=\sum_{j=-\infty}^{N_f} f_j\, z^j$ in N_V [2] satisfies

$$[f,g]_k=\sum_{j=-\infty}^{\infty} g_{-j-k}^{T} f_j=0,\quad for\ all\ g(z)\ in\ V \tag{5}$$

Equation (5) is only satisfied by $f_j=0,$ for all $j\le N_f$. On the contrary, suppose $g(z)=\sum_{j=-\infty}^{N_g} g_j\, z^j$ any series in N_V, then

$$[f,g]_k=\sum_{j=-\infty}^{\infty} g_{-j-k}^{T} f_j=0,\quad for\ all\ f(z)\ in\ V \tag{6}$$

Equation (6) is only satisfied by $g_j=0,$ for all $j\le N_g$. So obtained $N_V=\{0\}$. Thus the bilinear form (4) is non-degenerated. □

B. In accordance with the form k bilinier (4), the following statement apply:

1. $\left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp}=z^{-k}F^n\left[[z^{-1}]\right]$.
2. $(z^{-k+1}F^n[z])^{\perp}=z^{-k+1}F^n[z]$.

Proof:

1. Let k is any integer, then for every $f(z)=\sum_{j=-\infty}^{N_f} f_j z^j$ in $\left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp}$ apply

$$[f,p]_k=\sum_{j=-\infty}^{\infty} p_{-j-k}^{T} f_j=0 \tag{8}$$

for each $p(z)=\sum_{j=-\infty}^{-k} p_j z^j$ in $z^{-k}F^n\left[[z^{-1}]\right]$. Note that, for $j\ge -k+1$ and $\forall u\in F^n$ the equation (8) also applies to $p(z)=uz^{-j-k}$ in $z^{-k}F^n\left[[z^{-1}]\right]$. Substitution to equation (8) is obtained $[f,p]_k=u^T f_j=0$. Satisfies only by $f_j=0,\ \ j\ge -k+1$. So $f(z)=\sum_{j=-\infty}^{-k} f_j z^j$ is contained in $z^{-k}F^n\left[[z^{-1}]\right]$. Conversely, suppose $f(z)=\sum_{j=-\infty}^{-k} f_j z^j$ in $z^{-k}F^n\left[[z^{-1}]\right]$. View of the equation

$$[f,p]_k=\sum_{j=-\infty}^{\infty} p_{-j-k}^{T} f_j$$

for each $p(z)=\sum_{j=-\infty}^{-k} p_j z^j$ in $z^{-k}F^n\left[[z^{-1}]\right]$. Since $p_{-j-k}=0$ for $j\le -k$ then $[f,p]_k=0$. Thus $f(z)=\sum_{j=-\infty}^{-k} f_j z^j$ is contained in $\left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp}$.

2. For any $g(z)=\sum_{j=-\infty}^{N_g} g_j z^j$ in $(z^{-k+1}F^n[z])^{\perp}$ satisfies

$$[g,q]_k=\sum_{j=-\infty}^{\infty} q_{-j-k}^{T} g_j=0 \tag{9}$$

for each $q(z)=\sum_{j=-k+1}^{N_q} q_j z^j\in z^{-k+1}F^n[z]$. Note that, for $j\le -k$ and $\forall u\in F^n$ the equation (9) also applies to $q(z)=uz^{-j-k}$ in $z^{-k+1}F^n[z]$. Substitution in equation (9) is obtained $[g,q]_k=u^T g_j=$

0. Satisfies only by $g_j = 0,\ \ j \leq -k$. So $g(z) = \sum_{j=-k+1}^{N_g} g_j z^j$ is contained in $z^{-k+1}F^n[z]$. Conversely, suppose $g(z) = \sum_{j=-k+1}^{N_g} g_j z^j$ in $z^{-k+1}F^n[z]$. View of the equation

$$[g,p]_k = \sum_{j=-\infty}^{\infty} p_{-j-k}^T\, g_j$$

for each $p(z) = \sum_{j=-k+1}^{N_p} p_j z^j$ in $z^{-k+1}F^n[z]$. Therefore $p_{-j-k} = 0$ for $j \geq -k$ then $[g,p]_k = 0$. Thus $g(z) = \sum_{j=-k+1}^{N_g} g_j z^j$ is contained in $(z^{-k}F^n[z])^{\perp}$. □

C. *Subspaces* $z^{-k}F^n\left[[z^{-1}]\right]$ *and* $z^{-k}F^n[z]$ *are respectively closed subspaces of* V.

Proof:
According to point B obtained

1. $\left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp\perp} = \left(\left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp}\right)^{\perp} = \left(z^{-k}F^n\left[[z^{-1}]\right]\right)^{\perp} = z^{-k}F^n\left[[z^{-1}]\right]$

2. $(z^{-k+1}F^n[z])^{\perp\perp} = ((z^{-k+1}F^n[z])^{\perp})^{\perp} = (z^{-k+1}F^n[z])^{\perp} = z^{-k+1}F^n[z]$□

D. (Riesz representasi theorem) Suppose ϕ is any linear fungsional on V, then the following statement is equivalent

1. There exists a $R_\phi(z)$ in V that satisfies

$$\phi(f) = \left[f, R_\phi\right]_k, \qquad (12)$$

for all $f(z) \in V$ and κ in $\mathbb{Z}$

2. There exists a κ in $\mathbb{Z}$ that satisfies

$$z^{-\kappa}F^n\left[[z^{-1}]\right] \subseteq ker(\phi)$$

Proof: *1* ⟹ *2*

Suppose there exists a $R_\phi(z) = \sum_{j=-\infty}^{N_{R_\phi}} \left(R_\phi\right)_j z^j$ in V that satisfies equation (12). Let k is any in $\mathbb{Z}$. Select

$\kappa = N_{R_\phi} + k + 1 \in \mathbb{Z}$ where N_{R_ϕ} is the degree of $R_\phi(z)$ so that for any $f(z) = \sum_{j=-\infty}^{-\kappa} f_j z^j$ in $z^{-\kappa}F^n\left[[z^{-1}]\right]$ will satisfy

$$\phi(f) = \left[f, R_\phi\right]_k = \sum_{j=-\infty}^{\infty} \left(R_\phi\right)_{-j-k}^T f_j = 0$$

2 ⟹ *1*

Suppose there exists a κ in $\mathbb{Z}$ so that $z^{-\kappa}F^n\left[[z^{-1}]\right] \subseteq ker(\phi)$. For any $j \in \mathbb{Z}$, define the linear fungsional

$\psi_j : F^n \longrightarrow F$ as follows

$$\psi_j(v) = \begin{cases} \phi(vz^j), & j > -\kappa, \\ 0 \quad , & j \le -\kappa \end{cases}$$

for all $v \in F^n$. So for each $j > -\kappa$ there exists a $R_{\psi_{-j-k}}$; $k = 1,2,\cdots$ in F^n that satisfies

$$\psi_j(v) = v \cdot R_{\psi_{-j-k}} = (R_\psi)^T_{-j-k} v$$

for all $v \in F^n$. Define

$$R_\phi(z) = \sum_{j=-\infty}^{\kappa-2} R_{\psi_j} z^j \in V \qquad (13)$$

It will be shown that for every $f(z)$ in V satisfies equation (*12*). Suppose $f(z) = \sum_{j=-\infty}^{N_f} f_j z^j$any series in V.

a) If $N_f \le -\kappa$

Since $f(z)$ in $z^{-\kappa} F^n \left[[z^{-1}]\right] \subseteq Ker(\phi)$ then

$$\phi(f) = 0$$

Conversely, according to equation (13)

$$[f, R_\phi]_k = \sum_{j=-\infty}^{\infty} (R_\psi)^T_{-j-k} f_j = 0$$

b) If $N_f > -\kappa$

Write $f(z) = u(z) + v(z)$ where $g(z) = \sum_{j=-\kappa+1}^{N_f} f_j z^j$ in $z^{-\kappa+1} F^n[z]$ and $v(z) = \sum_{j=-\infty}^{-\kappa} f_j z^j$ in $z^{-\kappa} F^n \left[[z^{-1}]\right]$. Then

$$\phi(f) = \phi(u) = \sum_{j=-\kappa+1}^{N_f} \phi(f_j z^j) = \sum_{j=-\kappa+1}^{N_f} \psi_j(f_j) = \sum_{j=-\kappa+1}^{N_f} R_{\psi}{}^T_{-j-k} f_j$$

Conversely, according to equation (13)

$$[f, R_\phi]_k = [u, R_\phi]_k + [v, R_\phi]_k = [u, R_\phi]_k$$
$$= \sum_{j=-\kappa+1}^{N_f} \left[f_j, R_{\psi_{-j-k}}\right] = \sum_{j=-\kappa+1}^{N_f} R_{\psi}{}^T_{-j-k} f_j$$

Proof of singularity, e.g. there exists $R^1_\phi(z)$, $R^2_\phi(z)$ in V that satisfy

$$\phi(f) = [f, R^1_\phi]_k = [f, R^2_\phi]_k \qquad (14)$$

For all $f(z) \in V$. Since the bilinear form $[-,-]_k$ is *non-degenerated*, equation (14) is only satisfied by $R^1_\phi(z) = R^2_\phi(z)$.

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