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K- Regular Interval Incline Matrices

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Abstract:

This paper deals the generalized regular interval incline matrices as a generalization of regular fuzzy (interval) matrices and regular (interval) incline matrices are also investigated the concept of k-right (left) regular interval incline matrix and their properties.

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1. Introduction

The concept of an incline, introduced by Cao, is an algebraic structure, a special type of a semi rings. The notion of inclines and their applications are described broadly in Cao, Kim and Roush [2]. Kim and Roush [4] have surveyed and outlined algebraic properties of inclines and incline matrices.

Inclines are additively idempotent semi rings in which products are less than or equal to each factor. An incline is a structure which has an associative, commutative addition and distributive multiplication such that $x+x=x$, $x+xy=x$ and $y+xy = y$ for all $x, y \in \mathcal{L}$

An element a in an incline is said to be regular if a solution exists for the equation axa = a, and such a solution is called a generalized inverse (or g-inverse) of a. An incline $\mathcal{L}$ is regular if and only if every element of $\mathcal{L}$ is regular. A matrix P™$\mathcal{L}_{mn}$ (the set of mxn matrices over an incline $\mathcal{L}$) is regular iff there exists A™$\mathcal{L}^{I}_{mn}$ Such that PAP = P, and Such A is called g-inverse of P. In [6] meenakshi and kaliraja have studied the regular IVFM. In [8] Meenakshi and Poongodi have introduced on generalized regular interval valued fuzzy matrices.

Recently the concept of generalized regular interval valued fuzzy matrix was introduced by Poongodi [8]. Jenita have represented an generalized regular fuzzy matrices [7].

A matrix $P(=[P_L, P_U])$ ™$\mathcal{L}_n^I$, (the set of nxn-interval incline matrices) is said to be right k-regular if there exist a matrix $A(=[A_L, A_U])$ ™$\mathcal{L}_n^I$. Such that $[P_L^k A_L P_L, P_U^k A_U P_U] = [P_L^k, P_U^k]$, for some positive integer k. A is called a right k-g-inverse of P, Let P_k-$\{1_r\}=\{[A_L, A_U]/ [P_L^k A_L P_L, P_U^k A_U P_U] = [P_L^k, P_U^k]\}$. Which is an extension of the Concept of generalized Regular Fuzzy interval incline matrices.

In this paper we investigate the k-regular interval incline matrices.

2. Preliminaries

In this section, we recall some basic definitions and results needed are given.

Definition 2.1

An interval Incline is a non-empty set of elements $\mathcal{L}^I$ with binary operations addition and multiplication denoted as (+, ·) defined on $\mathcal{L}^I$x$\mathcal{L}^I \rightarrow \mathcal{L}^I$

For all $[x_L, x_U]$, $[y_L, y_U]$, $[z_L, z_U]$ ™$\mathcal{L}^I$

(i) $[x_L, x_U]+ [y_L, y_U] = [y_L, y_U]+[x_L, x_U]$;
(ii) $[x_L, x_U] + ([y_L, y_U]+[z_L, z_U]) = ([x_L, x_U]+[y_L, y_U]) +[z_L, z_U]$;
(iii) $[x_L, x_U]([y_L, y_U]+[z_L, z_U]) = [x_L, x_U][y_L, y_U]+[x_L, x_U][z_L, z_U]$;
(iv) $[x_L, x_U]([y_L, y_U][z_L, z_U]) = ([x_L, x_U][y_L, y_U])[z_L, z_U]$;
(v) $([y_L, y_U]+[z_L, z_U])[x_L, x_U] = [y_L, y_U][x_L, x_U]+[z_L, z_U][x_L, x_U]$;
(vi) $[x_L, x_U]+ [x_L, x_U] = [x_L, x_U]$;
(vii) $[x_L, x_U]+ [x_Ly_L, x_Uy_U] = [x_L, x_U]$;
(viii) $[y_L, y_U]+ [x_Ly_L, x_Uy_U] = [y_L, y_U]$;

In an interval incline ($\mathcal{L}$, +, ·) acting a relation '≤' described as on $\mathcal{L}^I$.

Interval values fuzzy sets, Interval distributive lattices are the best example for interval inclines.

In an interval incline ($\mathcal{L}^I$, +, ·) acting a relation of ordering ≤ described as on $\mathcal{L}^I$.

For some $[x_L, x_U]$, $[y_L, y_U]$ ™$\mathcal{L}^I$

$[x_L, x_U] \leq [y_L, y_U]$ or $x_L \leq y_L$ *and* $x_U \leq y_U$ if and only if $[x_L, x_U]+ [y_L, y_U] = [y_L, y_U]$

By the incline axioms (vii) and (viii), we have

$[x_Ly_L, x_Uy_U] \leq [x_L, x_U]$ and $[x_Ly_L, x_Uy_U] \leq [y_L, y_U]$;

The following characteristics of this interval incline order relation are from (P1) and (P2).

For $[x_L, x_U]$, $[y_L, y_U]$ ™$\mathcal{L}^I$

Property 1: $[x_L+y_L, x_U+y_U] \geq [x_L, x_U]$ and $[x_L+y_L, x_U+y_U] \geq [y_L, y_U]$

Property 2: $[x_Ly_L, x_Uy_U] \leq [x_L, x_U]$ and $[x_Ly_L, x_Uy_U] \leq [y_L, y_U]$;

Definition 2.2

An Interval Incline Matrix (IIM) of order *m*x*n* is defined by $P=[P_{ijL}, P_{ijU}]_{mxn}$ the ijth element of P is an interval incline matrices containing all the elements as intervals P_L is the lower matrix of P and P_U is the upper matrix of P.

For $P=[P_{ijL}, P_{ijU}]$ and $Q=[Q_{ijL}, Q_{ijU}]$ of order mxn their addition denoted by P+Q defined as

$$P+Q= ([P_{ijL}+Q_{ijL}, P_{ijU}+Q_{ijU}) \qquad \rightarrow(1)$$

and their multiplication denoted by PQ is defined as,

$$PQ = \left(r_{ij}\right) = \left(\left[r_{ijL},\ r_{ijU}\right]\right) = \left(\sum_{k=1}^{n} P_{ikL}Q_{kjL}, P_{ikU}Q_{kjU}\right), \text{ i=1, 2, 3, . . ., m and j=1, 2, . . . , r}$$

$$= \left(\sum_{k=1}^{n} P_{ikL}Q_{kjL}, P_{ikU}Q_{kjU}\right), \text{ i=1, 2, 3, . . ., m and j=1, 2, . . . , r}$$

where $P= [P_L, P_U]_{mxn}$ and $Q= [Q_L, Q_U]_{nxs}$

Their multiplication denoted by $PQ = [P_LQ_L, P_UQ_U]_{mxs}$ $P= [P_L, P_U]$, $P_L+ P_U = P_U$ iff $P_{ijL} \geq Q_{ijL}$ and $P_{ijU}\geq Q_{ijU}$ iff $P+Q=P$

In this interval incline elements having the lower limit and the upper limit are same then the interval incline coincides with the incline.

Remark 2.3

An Interval incline matrices (IIM) is a matrix with the entries are the (interval) elements belongs to the Interval incline by ($\mathcal{L}^I$). In particular the interval incline matrix the lower incline matrix and upper incline matrix, that is P™IIM's which is defined as $P(=[P_L, P_U])=[p_{ijL}, p_{ijU}]$.

Remark 2.4

For $P(=[P_L, P_U])$™$\mathcal{L}^I_{mxn}$, P^T, P_i^*, P_j^*, $\mathcal{R}(P)$,$C(P)$, $\rho_r(P)$, $\rho_c(P)$ denotes the transpose of P, ith the row of P, jth column of P, row space of P, Column space of P, row rank of P, Column rank of P respectively.

Lemma 2.5 [6]

For $P=[P_L, P_U]$ ™$\mathcal{L}^I_{mxn}$ and $Q =[Q_L, Q_U] \in \mathcal{L}^I_{np}$, then the following hold:

(i) $P^T=[P_L^T, P_U^T]$

(ii) $PQ = [P_LQ_L, P_UQ_U]$

Lemma 2.6 [6]

For P, Q $\in$(IIM)$_{mn}$,

(i) $\Re(Q) \subseteq \Re(P) \Leftrightarrow Q = AP$ for some $A\in(IIM)_m$

(ii) $C(Q) \subseteq C(P) \Leftrightarrow Q = PB$ for some $B\in(IIM)_n$

Lemma 2.7 [6]

For P $\in$(IIM)$_{mn}$ and Q $\in$(IIM)$_{np}$, then the following hold:

(i) $\Re(PQ) \subseteq \Re(P)$

(ii) $C(PQ) \subseteq C(Q)$

Lemma 2.8 [6]

For P, Q $\in \mathcal{L}^I_{mn}$, if P is regular then

(i) $\Re(Q) \subseteq \Re(P)\Leftrightarrow Q = QP^-P$ for each P^- of P

(ii) $C(Q) \subseteq C(P)\Leftrightarrow Q = P^-PQ$ for each P^- of P

3. k-regular Interval incline matrices

Definition 3.1

A matrix $P=[P_L, P_U]$ ™$\mathcal{L}^I_n$ is said to be right k-regular if there exists a matrix A™$\mathcal{L}^I_n$ ($A=[A_L, A_U]$ ™$\mathcal{L}^I_n$) such that $[P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$, for some positive integer k. $A=[A_L, A_U]$ is called a right k -g-inverse of P. the set of all k-g inverse is defined as and $[P_L, P_U]k\text{-}\{1_r\}=\{[A_L, A_U]/ [P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]\}$.

Definition 3.2

A matrix $P=[P_L, P_U]$ ™$\mathcal{L}_n^I$ is said to be left k-regular if there exists a matrix $B=[B_L, B_U]$ ™$\mathcal{L}^I_n$ such that $[P_LB_LP_L^k, P_UB_UP_U^k] = [P_L^k, P_U^k]$, for some positive integer k.

$B=[B_L, B_U]$ is called a left k-g-inverse of P.

Let $[P_L, P_U]$ k-$\{1_l\}=\{[B_L, B_U]/\ [P_L B_L P_L^k, P_U B_U P_U^k] = [P_L^k, P_U^k]\}$.

Let $[P_L, P_U]$k-$\{1_l\}= [P_L, P_U]$k-$\{1_r\}$U$[P_L, P_U]$k-$\{1_l\}$

In general, right k-regular is different from left k-regular.

P™$\mathcal{L}^I_{mn}$ is said to be k-regular if it is both left and right k-regular

These are illustrated in the following examples.

Example 3.3

Let us consider $P = \begin{bmatrix} [0,0] & [0.2,0.5] & [0,0] \\ [0,0] & [0.5,1] & [0.3,0.5] \\ [0.1,0.5] & [0,0] & [0,0] \end{bmatrix} \in (IIM)_{3x3}$

For this, $P^2 = \begin{bmatrix} [0,0] & [0.2,0.5] & [0.2,0.5] \\ [0.1,0.5] & [0.5,1] & [0.3,0.5] \\ [0,0] & [0.1,0.5] & [0,0] \end{bmatrix}$

$$P^3 = \begin{bmatrix} [0.1,0.5] & [0.2,0.5] & [0.2,0.5] \\ [0.1,0.5] & [0.5,1] & [0.3,0.5] \\ [0,0] & [0.1,0.5] & [0.1,0.5] \end{bmatrix}$$

For $A = \begin{bmatrix} [0.4,0.5] & [0,0] & [0.3,0.5] \\ [0.2,0.5] & [0.5,1] & [0,0] \\ [0,0] & [0.1,0.5] & [0.4,0.5] \end{bmatrix}$

$P^3AP = P^3$. Hence P is 3-regular.

For k=3, $P^3AP=P^3$ but $PAP^3 \neq P^3$

Hence A is a right 3-g-inverse but need not be a 3-g-inverse, and also a right k-g-inverse need not be a left k-g-inverse.

Remark 3.4

Each interval element of the set Pk-{1} is called a k-g-inverse of P. If P is k-regular then P is q-regular for all integer $q \geq k$. For k=1, Pk-{1} reduces to the set of all g-inverses of a regular interval matrix P.

Lemma 3.5

Let $P(=[P_L, P_U])$ ™$\mathcal{L}^I_{mn}$ is said to be k-regular interval incline matrix $A(=[A_L, A_U])$ be a k- g inverse of P and λ™$\mathcal{L}^I$. then

(i) $[A_L^T, A_U^T]$ ™ $[P_L^T, P_U^T]$k -{1}

(ii) $[\lambda A_L, \lambda A_U]$ ™$[\lambda P_L, \lambda P_U]$k –{1}

(iii) $[\rho(A_L), \rho(A_U)] \geq [\rho(P^k_L), \rho(P^k_U)]$

(iv) If R and S are permutation interval matrices, then

$[R_L^T P_L S_L^T, R_U^T P_U S_U^T]$ ™ $[R_L P_L S_L, R_U P_U S_U]$k –{1}.

Proof

Let $[A_L, A_U]$ be a k- g inverse of $[P_L, P_U]$ then $[P_L^k A_L P_L, P_U^k A_U P_U] = [P_L^k, P_U^k]$

Taking the transpose on both sides, we get

$[(P_L^kA_LP_L)^T, (P_U^kA_UP_U)^T] = [(P_L^k)^TA_L^TP_L^T, (P_U^k)^TA_U^TP_U^T]$

$= [(P_L^T)^kA_L^TP_L^T, (P_U^T)^kA_U^TP_U^T]$

$= [P_L^k, P_U^k].$

Thus $[A_L^T, A_U^T]$ ™ $[P_L^T, P_U^T]k$ -{1}

(ii) Let $[P_L, P_U] = [P_{ijL}, P_{ijU}]$ then $[\lambda P_L, \lambda P_U] = [P_L\lambda, P_U\lambda]$ and $\lambda\lambda=\lambda$

Hence $[\lambda(P_L^kA_LP_L), \lambda(P_U^kA_UP_U)]$

$= [\lambda P_L^k, \lambda P_U^k]$

Thus $[\lambda A_L, \lambda A_U]$ ™$[\lambda P_L, \lambda P_U]k$ –$\{1_r\}$

(iii) Since $[P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$

$\Rightarrow [\rho(P_L^kA_LP_L), \rho(P_U^kA_UP_U)]$

$= [\rho(P_L^k), \rho(P_U^k)]$

$\Rightarrow [\rho(P_L), \rho(P_U)] \leq [\rho(A_L), \rho(A_U)]$

(iv) Since $[R_L, R_U]$ and $[S_L, S_U]$ are interval permutation matrices, R and S are invertible $R^{-1} = R^T$, $S^{-1} = S^T$

Let $[P_L^kA_LP_L, P_UA_UP_U] = [P_L^k, P_U]$

$[R_LP_L^kS_L (S_L^TA_LR_L^T) R_LP_L^kS_L , R_UP_U^kS_U (S_U^TA_UR_U^T) R_UP_U^kS_U]$

$= [R_LP_L^k (S_LS_L^T)A_L(R_L^TR_L)P_L^kS_L , R_UP_U^k (S_US_U^T)A_U(R_U^TR_U)P_U^kS_U]$

$= [R_LP_L^k A_LP_LS_L , R_UP_U^k A_UP_US_U]$

$= [R_L(P_L^k A_LP_L)S_L , R_U(P_U^k A_UP_U)S_U]$

$= [R_LP_L^k S_L , R_UP_U^k S_U]$

Thus $[S_L^TA_LR_L^T, S_U^TA_UR_U^T]$™ $[R_LP_L^k S_L , R_UP_U^k S_U]k$ -{1}

Hence the proof

Theorem 3.6

Let $[P_L, P_U]$™$\mathcal{L}_n^I$ and k be a positive integer, then $[A_L, A_U]$™ Pk-$\{1_r\}$$\Leftrightarrow$$[A_L^T, A_U^T]$™Pk-$\{1_l\}$.

Proof

Let $[A_L, A_U]$™Pk-$\{1_r\}$

$\Leftrightarrow [P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$

$\Leftrightarrow [P_L^kA_LP_L, P_U^kA_UP_U]^T = [P_L^k, P_U^k]^T$

$\Leftrightarrow [(P_L^kA_LP_L)^T, (P_U^kA_UP_U)^T] = [(P_L^T)^k, (P_U^T)^k]$

$\Leftrightarrow [P_L^TA_L^T(P_L^T)^k, P_U^TA_U^T(P_U^T)^k] = [(P_L^T)^k, (P_U^T)^k]$

$\Leftrightarrow [A_L^T, A_U^T]$™$P^T$k -$\{1_l\}$

Remark 3.7

In particular for k=1, Definition (3.1) and (3.2) reduce to regular IIM poineer in [12] and in the case $P_L= P_U$, Definition (3.1) and (3.2) reduce to regular incline matrix [11].

Theorem 3.8

Let $P=[P_L, P_U]$ ™$(IIM)_n$. Then P is right k-regular interval incline matrix iff P_L and P_U ™$\mathcal{L}_n$ are right k-regular.

Proof

Let $P=[P_L, P_U]$ ™$(IIM)_n$

Since P is right k-regular IIM, there exists

$A(=[A_L, A_U])$™$(IIM)_n$, Such that $[P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$

Let $A=[A_L, A_U]$ with A_L, A_U™$\mathcal{L}^I_n$

Then by Lemma (2.5)(ii),

$$[P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$$

$$P_L^kA_LP_L = P_L^k \text{ and } P_U^kA_UP_U = P_U^k$$

Therefore P_L is right k-regular and P_U is right k-regular ™$\mathcal{L}^I_n$. Thus P is right k-regular IIM.

⇒ P_L and P_U ™$\mathcal{L}^I_n$ are right k-regular.

Conversely, Suppose P_L and P_U ™$\mathcal{L}_n$ are right k-regular, then $P_L^kA_LP_L = P_L^k$ and

$P_U^kA_UP_U = P_U^k$ for some A_L and A_U ™$\mathcal{L}^I_n$. A_L™$(P_L)k\text{-}\{1_r\}$, A_U™$(P_U)k\text{-}\{1_r\}$.

Since $P_L \leq P_U$, it is possible to choose at least one U™$(P_L)k\text{-}\{1_r\}$and V™$(P_U)k\text{-}\{1_r\}$ Such that $U \leq V$.

Let us define the interval incline matrix $Z = [U, V]$. Then by Lemma (2.5) (ii),

$$\begin{aligned} P^kZP &= [P_L^k, P_U^k]\,[U, V][P_L, P_U] \\ &= [P_L^kUP_L, P_U^kVP_U] \\ &= [P_L^k, P_U^k] \\ &= P^k \end{aligned}$$

Thus P is right k-regular IIM.

Theorem 3.9

Let $P=[P_L, P_U]$ ™$(IIM)_n$. Then P is left k-regular interval incline matrix iff P_L and P_U ™$\mathcal{L}^I_n$ are left k-regular.

Proof

This can be proof is similar to Theorem (3.8) and hence omitted.

Lemma 3.10

For $[P_L, P_U]$, $[Q_L, Q_U]$™$(IIM)_n$, and a positive integer k, the following hold.

(i) If $[P_L, P_U]$ is right k-regular and $[R(Q_L), R(Q_U)] \subseteq [R(P_L^k), R(P_U^k)]$

then, $[Q_L, Q_U]= [Q_LA_LP_L, Q_UA_UP_U]$ for each right k-g- inverse $[A_L, A_U]$ of $[P_L, P_U]$.

(ii) If $[P_L, P_U]$ is left k-regular and $[C(Q_L), C(Q_U)] \subseteq [C(P_L^k), C(P_U^k)]$

then, $[Q_L, Q_U] = [P_LB_LQ_L, P_UB_UQ_U]$ for each left k-g- inverse $[B_L, B_U]$ of $[P_L, P_U]$.

Proof

(i) Since $[R(Q_L), R(Q_U)] \subseteq [R(P_L^k), R(P_U^k)]$, by Lemma (2.6), there exists $[Z_L, Z_U]$ such that $[Q_L, Q_U] = [Z_LP_L^k, Z_UP_U^k]$.

Since $[P_L, P_U]$ is right k-regular , by Definition (3.1),

$[P_L^kA_LP_L, P_U^kA_UP_U] = [P_L^k, P_U^k]$ for some $[A_L, A_U]$ ™$[A_L, A_U]k\text{-}\{1_r\}$

$$\begin{aligned} \text{Hence } [Q_L, Q_U] &= [Z_LP_L^k, Z_UP_U^k] \\ &= [Z_LP_L^kA_LP_L, Z_UP_U^kA_UP_U] \\ &= [Q_LA_LP_L, Q_UA_UP_U] \end{aligned}$$

Thus (i) holds.

(ii) Since $[C(Q_L), C(Q_U)] \subseteq [C(P_L^k), C(P_U^k)]$, by Lemma (2.6), there exists W such that

$[Q_L, Q_U] = [P_L^k W_L, P_U^k W_U]$

Since $[P_L, P_U]$ is left k-regular.

By Definition (3.2), $[P_L B_L P_L^k, P_U B_U P_U^k] = [P_L^k, P_U^k]$ for some $[B_L, B_U]$ ™$[P_L, P_U]k\text{-}\{1_l\}$

Hence $[Q_L, Q_U] = [P_L^k W_L, P_U^k W_U]$

$= [P_L B_L P_L^k W_L, P_U B_U P_U^k W_U]$

$= [P_L B_L Q_L, P_U B_U Q_U]$

Thus (ii) holds.

Theorem 3.11

For $[P_L, P_U], [Q_L, Q_U]$™$(IIM)_n$, with $[R(P_L), R(P_U)] = [R(Q_L), R(Q_U)]$ and $[R(P_L^k), R(P_U^k)] = [R(Q_L^k), R(Q_U^k)]$ then $[P_L, P_U]$ is right k-regular IIM iff $[Q_L, Q_U]$ is right k-regular IIM.

Proof

Let $[P_L, P_U]$ be a right k-regular IIM satisfying $[R(Q_L^k), R(Q_U^k)] \subseteq [R(P_L^k), R(P_U^k)]$ and $[R(P_L), R(P_U)] \subseteq [R(Q_L), R(Q_U)]$.

since $[R(Q_L^k), R(Q_U^k)] \subseteq [R(P_L^k), R(P_U^k)]$,

by Lemma (4.7) $[Q_L^k, Q_U^k] = [Q_L^k A_L P_L, Q_U^k A_U P_U]$ for each k-g-inverse $[A_L, A_U]$ of $[P_L, P_U]$.

since $[R(P_L), R(P_U)] \subseteq [R(Q_L), R(Q_U)]$,

by Lemma (2.6), $[P_L, P_U] = [B_L Q_L, B_U Q_U]$ for some $[B_L, B_U]$™ $(IIM)_n$.

substituting for $[P_L, P_U]$ in $[Q_L^k, Q_U^k] = [Q_L^k A_L P_L, Q_U^k A_U P_U]$,

we get, $[Q_L^k, Q_U^k] = [Q_L^k A_L P_L, Q_U^k A_U P_U]$

$= [Q_L^k A_L B_L Q_L, Q_U^k A_U B_U Q_U]$

$= [Q_L^k Z_L Q_L, Q_U^k Z_U Q_U]$ where $[A_L B_L, A_U B_U] = [Z_L, Z_U]$

Hence $[Q_L, Q_U]$ is right k-regular IIM.

Conversely, if $[Q_L, Q_U]$ is right k-regular IIM.

satisfying $[R(P_L^k), R(P_U^k)] \subseteq [R(Q_L^k), R(Q_U^k)]$ and $[R(Q_L), R(Q_U)] \subseteq [R(P_L), R(P_U)]$, then $[P_L, P_U]$ is right k-regular IIM can be Proved in the same manner.

Hence the Theorem.

Theorem 3.12

For $[P_L, P_U], [Q_L, Q_U]$™$(IIM)_n$, with $[C(P_L), C(P_U)] = [C(Q_L), C(Q_U)]$ and $[C(P_L^k), C(P_U^k)]$ then $[P_L, P_U]$ is left k-regular IIM iff $[Q_L, Q_U]$ is left k-regular IIM.

Proof

This is similar to Theorem (3.11) and hence omitted.

Theorem 3.13

For $[P_L, P_U], [Q_L, Q_U]$™$(IIM)_n$, with $[R(P_L), R(P_U)] = [R(Q_L), R(Q_U)]$ and $[R(P_L^k), R(P_U^k)] = [R(Q^k{}_L), R(Q^k{}_U)]$ then the following are equivalent:

(i) $[P_L, P_U]$ is right k-regular IIM

(ii) P_L and P_U are right k-regular interval incline matrices

(iii) $[Q_L, Q_U]$ is right k-regular IIM

(iv) Q_L and Q_U are right k-regular interval incline matrices

Proof

(i) ⇔ (ii) and (iii) ⇔ (iv) are precisely Theorem (3.8)

(i) ⇔ (iii) This follows Theorem (3.11).

Theorem 3.14

For $[P_L, P_U], [Q_L, Q_U]$™$(IIM)_n$, with $[C(P_L), C(P_U)] = [C(Q_L), C(Q_U)]$ and $[C(P_L^k), C(P_U^k)] = [C(Q_L^k), C(Q_U^k)]$ then the following are equivalent:

(i) $[P_L, P_U]$ is left k-regular IIM

(ii) P_L and P_U are left k-regular interval incline matrices

(iii) $[Q_L, Q_U]$ is left k-regular IIM

(iv) Q_L and Q_U are left k-regular interval incline matrices

Proof

(i) ⇔ (ii) and (iii) ⇔ (iv) are precisely Theorem (3.9)

(i) ⇔ (iii) This follows Theorem (3.12).

Theorem 3.15

Let $[P_L, P_U]$™$(IIM)_n$ and k be a positive integer, then the following hold

(i) If $A= [A_L, A_U]$ ™Pk-$\{1_r\}$, then $\rho_c(P_L^k) = \rho_c(P_L^kA_L)$, $\rho_c(P_U^k) = \rho_c(P_U^kA_U)$ and

$\rho_r(P_L^k) \le \rho_r(A_LP_L) \le \rho_r(A_L)$, $\rho_r(P_U^k) \le \rho_r(A_UP_U) \le \rho_r(A_U)$

(ii) If $[A_L, A_U]$ ™Pk-$\{1_r\}$, Then $\rho_c[(P_L^k), (P_U^k)] = \rho_c[P_L^kA_L , P_U^kA_U]$ and

$\rho_r[P_L^k, P_U^k] \le \rho_r[A_LP_L, A_UP_U] \le \rho_r[P_L, P_U]$.

(iii) If $A= [A_L, A_U]$ ™Ak-$\{1_r\}$, then $\rho_r(P_L^k) = \rho_r(A_LP_L^k)$, $\rho_r(P_U^k) = \rho_r(A_UP_U^k)$ and

$\rho_c(P_L^k) \le \rho_c(P_LA_L) \le \rho_c(P_L)$, $\rho_c(P_U^k) \le \rho_c(P_UA_U) \le \rho_c(P_U)$

(iv) If $[A_L, A_U]$ ™Pk-$\{1_l\}$ then $\rho_r[P_L^k, P_U^k] = \rho_r[A_LP_L^k, A_UP_U^k]$ and

$\rho_r[P_L^k, P_U^k] \le \rho_c[P_LA_L, P_UA_U] \le \rho_c[P_L, P_U]$

Proof

Let $[P_L, P_U]$ Since $A= [A_L, A_U]$ ™Pk-$\{1_r\}$,

By Definition (3.1) and Lemma (2.5) (ii)

$$P_L^kA_LP_L = P_L^k \text{ and } P_U^kA_UP_U = P_U^k$$

By Lemma (2.7) , we get

$$C(P_L^k) = C(P_L^kA_LP_L) \subseteq C(P_L^kA_L) \subseteq C(P_L^k) \rightarrow 3.1 \text{ and}$$

$$C(P_U^k) = C(P_U^kA_UP_U) \subseteq C(P_U^kA_U) \subseteq C(P_U^k) \rightarrow 3.2$$

$$\rho_c(P_L^k) = \rho_c(P_L^kA_L) \text{ and } \rho_c(P_U^k) = \rho_c(P_U^kA_U)$$

Since $P_L^kA_LP_L= P_L^k$ and $P_U^kA_UP_U = P_U^k$

we have $P_L^k = P_L^kA_LP_L = P_L^k(A_LP_L)^2 = \ldots\ldots = P_L^k(A_LP_L)^k$

$$P_U^k = P_U^kA_UP_U = P_U^k(A_UP_U)^2 = \ldots\ldots = P_U^k(A_UP_U)^k$$

Therefore, $P_L^k = P_L^k(A_LP_L)^k$, Hence

$$R(P_L^k) = R(P_L^k(A_LP_L)^k) \subseteq R((A_LP_L)^k) \subseteq R(A_LP_L) \subseteq R(P_L) \rightarrow 3.3$$

Therefore $R(P_L^k) \subseteq R(A_LP_L) \subseteq R(P_L)$

$$\rho_r(P_L^k) \leq \rho_r(A_LP_L) \leq \rho_r(P_L)$$

Similarly, $P_U^k = P_U^k(A_UP_U)^k$,

Hence $R(P_U^k) = R(P_U^k(A_UP_U)^k) \subseteq R((A_UP_U)^k) \subseteq R(A_UP_U) \subseteq R(P_U)$

Therefore $R(P_U^k) \subseteq R(A_UP_U) \subseteq R(P_U) \rightarrow 3.4$

$$\rho_r(P_U^k) \leq \rho_r(A_UP_U) \leq \rho_r(P_U)$$

Thus (1) holds.

Since, $[P_L, P_U]$™$(IIM)_n$, From (3.1) and (3.2)

$$\begin{aligned} C(P^k) &= [C(P_L^k), C(P_U^k)] \\ &= [C(P_L^kA_L), \ C(P_U^kA_U)] \\ &= C(P^kA) \end{aligned}$$

and $\rho_c[(P_L^k), (P_U^k)] = \rho_c[P_L^kA_L , P_U^kA_U]$

Similarly From (3.3) and (3.4)

$$\begin{aligned} R(P^k) &= [R(P_L^k), R(P_U^k)] \\ &\subseteq [R(A_LP_L), \ R(A_UP_U)] \\ &\subseteq [R(P_L), R(P_U)] \\ &= R(P) \end{aligned}$$

Therefore, $[R(P_L^k), R(P_U^k)] \subseteq [R(A_LP_L), R(A_UP_U)] \subseteq [R(P_L), R(P_U)]$

$$\rho_r[(P_L^k), (P_U^k)] \leq \rho_r[A_LP_L , A_UP_U] \leq \rho_r[P_{L,} P_U].$$

Thus (ii) holds.

Proof is similar to that of (iii) and hence omitted.

proof is Similar to that of (iv) and hence omitted

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