On Finite Topological Spaces Generated by Connected Simple Graphs

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Abstract:
In this paper, we investigate topologies produced by simple connected graphs. In particular, we will present the type of relations on the vertices set. It is converted to an adjacent matrix and through this matrix, whose elements represent the relationship between each vertex to the rest of the vertices adjacent to it, where sets are produced through which they represent the basis for topology. On the graph is the vertex set.

Keywords: Finite Topological Spaces, Connected Simple Graphs, Topologies, Adjacent Matrix, Vertices Set.

1. Introduction

Graph theory is one of the most important structures in discrete mathematics. It is a prominent mathematical tool in many subjects [4]. Graphs are mathematical structures consisting of vertices and edges. They are used to model dual relationships between objects in a particular collection. A graph includes vertices that serve as objects and edges by connecting these vertices [1]. Many researchers have used the relation between graph and topology theories to deduce a topology from a particular graph. Others put new specific models in the set of vertices in the graph, and others make it on the set of edges. Graphs can be divided into two types directed graphs and undirected graphs. The researchers' previous work to obtain the topology through the graph was associated with a graph of vertices adjacent to the vertex. Euler first proposed graph theory in 1736. Recently, the theory has been an achieved applied to places in various disciplines. Since the theory is according to relation combinations, it plays a crucial role in representing combinations of items and mathematical combinations. Simple set theory is also according to relational combinations. They are solved by using applications of graph theory. Because of the pervasive use of the theory, its topological structure has generated debate [3].

Using various techniques, some scientists have created topologies from a graph. Some scientists have generated topologies from graphs using various techniques. In 2013, M. Amiri et al. Developed a topology using the vertices of an undirected graph. In this paper, we outline several characteristics of a topological space by using a simple graph without isolated vertices. We describe some topological characteristics of the topology we produce with these graphs. We demonstrate that every basic graph may generate a topology. Obtain the topological space by converting the simple connected graph [12,13]. In this article, we discuss a new method for generating a topology on the graph using a new method, which is converting the graph into an adjacent matrix that represents the correlation of each vertex with the rest of the vertices and, depending on the shape of the matrix and the rows it contains, we get a set for each rows containing only the vertices associated with that header then we col-
lect these rows according to the sites in the set that we got. We take the highest value that represents that site. These sets represent the basis of topology.

2. Basic concepts

This section is devoted to recalling the basic definitions and preliminaries related to graph theory and topological space.

Definition (2.1): [1] A graph $G = (V(G), E(G))$ is an ordered pair consisting of two sets $V$ called the vertex set of $G$, and $E$ is a set of edges of $G$. If the vertices ($V$) and edge ($E$) are a finite set, then the graph $G$ is called a finite graph. Set ($E$) includes order pairs $e_{ij} = (v_i,v_j)$ of vertices which means an edge links these two vertices $e_{ij}$. If the graph does not have an edge $(e_i,e)$ There is no more than one edge between two vertices; it is known as a simple graph.

Definition (2.2): [1] A bipartite graph is a graph whose vertices can be partitioned into two (disjoint) sets $V_1$ and $V_2$ called bipartition sets. In such a way that every edge joins a vertex in $V_1$ and $V_2$.

Definition (2.3): [1] In a graph $(G)$, the sequence of edges joins a sequence of distinct vertices is called a path, denoted by $p$.

The following example of the path of length $n$, $p = (e_1, e_2, ..., e_n)$, where $e_i$ and $e_{i+1}$ Share the same vertex. If $e_1$ and $e_n$ Share the same vertex (i.e., it has the same initial and ending vertex); then the path is called a cycle, denoted by $C_n$.

Definition (2.4): [4] A graph which does not contain any cycle in it is called an Acyclic graph.

Definition (2.5): [1] A complete bipartite graph is a bipartite graph in which every vertex in $V_1$ is joined to every vertex in $V_2$. The complete bipartite graph on bipartition sets of $m$ vertices and $n$ vertices, respectively denoted by $K_{m,n}$.

Definition (2.6): Let $G = (V(G), E(G))$ be a simple graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_n\}$ the adjacent matrix of $G$ is defined $n \times n$ over the ring of the integer such that:

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \quad \text{and} \quad a_{ij} = \begin{cases} 1, & \text{for } (v_i,v_j) \\ 0, & \text{otherwise} \end{cases}$$

Notation (2.7):

i. The $i^{th}$ a row of a matrix $A$ will be denoted by $A^i$.

ii. The set of all rows $A^i$ at which the entry $a_{ij}$ is unity will be denoted by $A^i = \{ A^i : a_{ij} = 1 \}$

iii. The family of all $A^i$, $i = 1, 2, ..., n$ is denoted by $F(G)$.

Definition (2.8): [6] Let $X$ be an anon-empty set. A topology on $X$ is a collection of open subsets of $X$ which satisfies the following conditions:

i. $\emptyset, X \in \tau$

ii. The union of any member of $\tau$ is also a member of $\tau$.

iii. The finite intersection of any member of $\tau$ is also a member of $\tau$.

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Definition (2.9): [6] Let $X$ be a set and $\tau$ be a topology on $X$ a topology and $\beta \subseteq \tau$. We call it a base for $\tau$ if every element in $\tau$ is a union number of elements of $\beta$.

3. A topological space via a simple connected graph

Definition (3.1): Let $I = \{1, 2, \ldots, n\}$ an index set for vertex set $V(G)$ Of a simple connected graph $G$. Each index $i$ correspond to a subset of indices, which the following function can obtain.

$$\Gamma: I \rightarrow \mathcal{F}(G), \ i \rightarrow \bigoplus \mathcal{A}^i$$

Where $\bigoplus$ means a direct sum, $(a_1, a_2, \ldots, a_n) \bigoplus (b_1, b_2, \ldots, b_n) = (a_1+b_2, \ldots, a_n+b_n)$ here, $a_i$, $b_i \in \{0, 1\}$.

Notation (3.2): $J^M_i$ is the set of all positions of the element $\max\{\Gamma(i)\}$ in $\Gamma(i)$.

Example (3.3): Consider the following simple graph $G$ and its adjacent matrix $A$.

![Simple Graph](image)

The adjacent matrix of a cycle graph is as follows

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

Then

$$\mathcal{A}^1 = \{A^i = a_{i_1} = 1\} = \{A^2, A^4\}, \mathcal{A}^2 = \{A^i = a_{i_2} = 1\} = \{A^1, A^3\}, \mathcal{A}^3 = \{A^i = a_{i_1} = 1\} = \{A^2, A^4, A^5\}, \mathcal{A}^4 = \{A^i = a_{i_1} = 1\} = \{A^1, A^3\}, \mathcal{A}^5 = \{A^i = a_{i_1} = 1\} = \{A^3, A^6, A^8\}, \mathcal{A}^6 = \{A^i = a_{i_1} = 1\} = \{A^5, A^7\}$$

and

$$J^M_1 = \{1, 3\}, J^M_2 = \{2, 4\}, J^M_3 = \{3\}, J^M_4 = \{2, 4\}, J^M_5 = \{5\}, J^M_6 = \{6, 8\}, J^M_7 = \{5, 7\}, J^M_8 = \{6, 8\}.$$

$\beta = \{\{1, 3\}, \{2, 4\}, \{3\}, \{2, 4\}, \{5\}, \{6, 8\}, \{5, 7\}, \{6, 8\}\}.$

Proposition (3.4): Let $G$ is a simple connected graph. The following collection of sets $\beta = \{J^M_i, i \in I\}$ is a base for a topology on the index set $I$ of $V$. 
Proof. To prove that \( \beta \) is a base for a topology on \( V \), we prove that the union \( \bigcup J^M \) Make a cover for the set \( V \), and this is obvious since \( x \in J^M_x \text{ so } V = \bigcup J^M \) And we must show that for every \( J^M_i \), \( \forall x \in J^M \cap J^M_j \), there exists some \( J^M_k \in \beta \) such that \( x \in J^M_k \subseteq J^M_i \cap J^M_j \). Now, when \( J^M_i \cap J^M_j = \emptyset \) the proof is done. The other case, when \( J^M_i \cap J^M_j \neq \emptyset \) So there is at least one element \( x \in J^M_i \cap J^M_j \). Thus, the element \( x \) represents the position of \( \max \{ \Gamma(i) \} \) in \( \{ \Gamma(i) \} \) and the position of \( \max \{ \Gamma(j) \} \) in \( \{ \Gamma(j) \} \).

Therefore, either \( J^M_i = J^M_j \) or one subset of the other \( J^M_i \subset J^M_j \) or \( J^M_i \cap J^M_j \).

Example (3.5): The base for the topology induced by the graph given in Example 3.3 is \( \beta = \{ \emptyset, \{ 3 \}, \{ 5 \}, \{ 1, 3 \}, \{ 2, 4 \}, \{ 5, 7 \}, \{ 6, 8 \} \} \)

\[
\mathcal{T} = \{ \emptyset, 1, 3, 5, 1, 3, 2, 4, 3, 5, 7, 6, 8, 2, 3, 4, 5, 3, 5, 7, 6, 8 \} \]

\[
\mathcal{A} = \{ A^i = a_{i1} = 1 \} = \{ A^1 \}, \quad \mathcal{A}^2 = \{ A^i = a_{i2} = 1 \} = \{ A^2 \}, \quad \mathcal{A}^3 = \{ A^i = a_{i1} = 1 \} = \{ A^3 \}, \quad \mathcal{A}^4 = \{ A^i = a_{i1} = 1 \} = \{ A^4 \}, \quad \mathcal{A}^5 = \{ A^i = a_{i1} = 1 \} = \{ A^5 \}, \quad \mathcal{A}^6 = \{ A^i = a_{i1} = 1 \} = \{ A^6 \}, \quad \mathcal{A}^7 = \{ A^i = a_{i1} = 1 \} = \{ A^7 \}
\]

\[
\mathcal{J}_1 = \{ 1 \}, \quad \mathcal{J}_2 = \{ 2, 4 \}, \quad \mathcal{J}_3 = \{ 1, 3, 7 \}, \quad \mathcal{J}_4 = \{ 4 \}, \quad \mathcal{J}_5 = \{ 5, 7 \}, \quad \mathcal{J}_6 = \{ 6 \}, \quad \mathcal{J}_7 = \{ 7 \}.
\]

\( \beta = \{ \emptyset, \{ 1 \}, \{ 4 \}, \{ 6 \}, \{ 7 \}, \{ 2, 4 \}, \{ 5, 7 \}, \{ 1, 3, 7 \} \} \).

Example (3.6): Consider the following acyclic graph \( G \) and its adjacent matrix \( A \).

**Figure 1.** A cycle graph

The adjacent matrix of a cycle graph is as follows

\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Example (3.7): Consider the following example of a bipartite graph with vertex set $V = \{V_i, i = 1,2,3,\ldots,8\}$ and the edges graph: $E = \{e_{12}, e_{14}, e_{16}, e_{23}, e_{25}, e_{27}, e_{34}, e_{36}, e_{38}, e_{45}, e_{47}, e_{56}, e_{58}, e_{6,7}, e_{7,8}\}$, where $e_{ij}$ join vertices $v_i$ and $v_j$, $i=\{1, 2, ..., 8\}$.

![Figure 2. Complete bipartite graph](image)

The adjacent matrix complete bipartite graph is as follows

$$A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}$$

$\mathcal{A}^1 = \{A^i = a_{i1} = 1\} = \{A^2, A^4, A^6, A^8\}$.

$\mathcal{A}^2 = \{A^i = a_{i2} = 1\} = \{A^1, A^3, A^5, A^7\}$,

$\mathcal{A}^3 = \{A^i = a_{i3} = 1\} = \{A^2, A^4, A^6, A^8\}$,

$\mathcal{A}^4 = \{A^i = a_{i4} = 1\} = \{A^1, A^3, A^5, A^7\}$,

$\mathcal{A}^5 = \{A^i = a_{i5} = 1\} = \{A^2, A^4, A^6, A^8\}$,

$\mathcal{A}^6 = \{A^i = a_{i6} = 1\} = \{A^1, A^3, A^5, A^7\}$,

$\mathcal{A}^7 = \{A^i = a_{i7} = 1\} = \{A^2, A^4, A^6, A^8\}$,

$\mathcal{A}^8 = \{A^i = a_{i8} = 1\} = \{A^1, A^3, A^5, A^7\}$

$J^1 = \{1,3,5,7\}$, $J^2 = \{2,4,6,8\}$, $J^3 = \{1,3,5,7\}$, $J^4 = \{2,4,6,8\}$, $J^5 = \{1,3,5,7\}$, $J^6 = \{2,4,6,8\}$,

$J^7 = \{1,3,5,7\}$, $J^8 = \{2,4,6,8\}$

$\beta = \{\emptyset, \{1,3,5,7\}, \{2,4,6,8\}\}$

$\tau = \{\emptyset, V, \{1,3,5,7\}, \{2,4,6,8\}\}$.

Definition (3.8): A cycle $C_n$, $n \geq 3$, consists of $n$ vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_1\}$.

Example (3.9): The cycle graph whose vertices set is $V = \{V_i, i=1,2,3,4,5,6,7\}$ is given in figure 3.
The adjacent matrix of the cycle graph is as follows:

\[
G(c_n) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

then the set of all rows its

\[
\mathcal{A}^1 = \{A^i = a_{i1} = 1 \} = \{A^2, A^7\}, \quad \mathcal{A}^2 = \{A^i = a_{i2} = 1 \} = \{A^3, A^5\}, \quad \mathcal{A}^3 = \{A^i = a_{i1} = 1 \} = \{A^2, A^4\}, \\
\mathcal{A}^4 = \{A^i = a_{i1} = 1 \} = \{A^3, A^5\}, \quad \mathcal{A}^5 = \{A^i = a_{i1} = 1 \} = \{A^4, A^6\}, \quad \mathcal{A}^6 = \{A^i = a_{i1} = 1 \} = \{A^5, A^7\}, \\
\mathcal{A}^7 = \{A^i = a_{i1} = 1 \} = \{A^5, A^7\}, \quad \mathcal{A}^8 = \{A^i = a_{i1} = 1 \} = \{A^1, A^6\}
\]

and

\[
J_1^M = \{1\}, J_2^M = \{2\}, J_3^M = \{3\}, J_4^M = \{4\}, J_5^M = \{5\}, J_6^M = \{6\}, J_7^M = \{7\}.
\]

\[
\beta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}
\]

Therefore, the topology generated by \(C_n\) It is as follows: \(\tau_{C_n} = \mathcal{P}(V)\).

It is seen that \(\tau_{c_n}\) It is a discrete topology on \(V\).

**Definition (3.10):** A wheel graph is obtained when we add an addition as a vertex to the cycle \(C_n\), for \(n \geq 3\) and connect this new vertex to each of \(n\) vertices of \(C_n\), by the new edge. A wheel graph is represented by \(W_n\).

**Example (3.11):** Consider the following example of a wheel graph with vertex set \(V = \{v_i, i=1,2,3,\ldots,9\}\) and the set of edges, \(E = \{e_{1,9}, e_{2,9}, e_{3,9}, e_{4,9}, e_{5,9}, e_{6,9}, e_{7,9}, e_{8,9}\}\), Where \(e_{ij}\) join vertices \(v_i\) and \(v_j\), \(i=1,2,\ldots,9\) means

\[
G = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

**Figure 3.** the graph cycle

**Figure 4.** the graph wheel
\[ \mathcal{A}^1 = \{ A^i = a_{11} = 1 \} = \{ A^2, A^6, A^9 \}, \quad \mathcal{A}^2 = \{ A^i = a_{12} = 1 \} = \{ A^1, A^3, A^9 \} \]
\[ \mathcal{A}^3 = \{ A^i = a_{11} = 1 \} = \{ A^2, A^4, A^9 \}, \quad \mathcal{A}^4 = \{ A^i = a_{11} = 1 \} = \{ A^3, A^5, A^9 \} \]
\[ \mathcal{A}^5 = \{ A^i = a_{11} = 1 \} = \{ A^4, A^6, A^9 \}, \quad \mathcal{A}^6 = \{ A^i = a_{11} = 1 \} = \{ A^5, A^7, A^9 \} \]
\[ \mathcal{A}^7 = \{ A^i = a_{11} = 1 \} = \{ A^6, A^8, A^9 \}, \quad \mathcal{A}^8 = \{ A^i = a_{11} = 1 \} = \{ A^1, A^7, A^9 \} \]
\[ \mathcal{A}^9 = \{ A^i = a_{11} = 1 \} = \{ A^2, A^1, A^2, A^3, A^4, A^5, A^6, A^7, A^8 \}. \]

And
\[ J^1_1 = \{ 1 \}, J^2_2 = \{ 2 \}, J^3_3 = \{ 3 \}, J^4_4 = \{ 4 \}, J^5_5 = \{ 5 \} \]
\[ J^6_6 = \{ 6 \}, J^7_7 = \{ 7 \}, J^8_8 = \{ 8 \}, J^9_9 = \{ 9 \} \]
\[ \beta = \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \}, \{ 5 \}, \{ 6 \}, \{ 7 \}, \{ 8 \}, \{ 9 \} \} \]

Therefore, the topology generated by \( W_n \) is as follows: \( \tau_{W_n} = P(V) \).

The fact that \( \tau_{W_n} \) is a discrete topology on \( V \).

**Theorem (3.12):** Let \( C_n = (V, E) \) be a cycle graph whose vertices set is \( V = \{ v_i \, | \, i = 1, 2, \ldots, n \} \) where \( n \geq 3 \) but \( n \neq 4 \), then the cycle generated topological space is discrete topological space.

![Figure 5. The graph \( C_n \)](image)

**Proof:** The graph of the cycle is as in the figure. The adjacencies matrix of the vertices of the cycle graph is as follows:

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

let \( A^i \) a row of adjacencies matrix \( A \), and \( \mathcal{A}^i \) the set of all rows \( A^i \) at which the entry \( a_{ij} \) is unity.

\[
\mathcal{A}^1 = \{ A^2, A^n \}, \mathcal{A}^2 = \{ A^1, A^3 \}, \mathcal{A}^3 = \{ A^2, A^4 \}, \ldots, \mathcal{A}^i = \{ A^1, A^{n-1} \}.
\]

Then The set of all positions of the element \( \max \{ i \} \) is as follows:

\[ J^1_1 = \{ 1 \}, J^2_2 = \{ 2 \}, J^3_3 = \{ 3 \}, \ldots, J^i_i = \{ n \} \]

where \( i = 1, 2, 3, \ldots, n \).

Thus we have \( C_n = \{ \{ v_i \}, v_i \in V \} \).
The class $\beta_{C_n}$ is a base for a discrete topology on $V$. Thus, it is seen that the topological space generated by the graph $C_n$ is the discrete topological space on $V$.

When we assume $n=4$, the graph $C_4$ is a complete bipartite graph. The topological space generated by $C_4$ whose vertices are $V=\{v_i, i=1,2,\ldots,n\}$ and the topology cycle graph is $\{V, \emptyset, \{1\}, \{2\}\}$. The topology is not discrete topology.

**Theorem (3.13):** Let $K_n=(V,E)$ be a complete graph, where $V=\{v_i, i=1,2,\ldots,n\}$. Then topology generated by $K_n$ it is a discrete topology on $V$.

**Proof:** The set of all position of vertices $I=\{1,2,\ldots,n\}$ are as follows respectively:

$J_1^M=\{1\}, J_2^M=\{2\}, J_3^M=\{3\}, \ldots, J_n^M=\{n\}$.

Therefore,

$k_n=\{\{v_i\}, v_i \in V\}$.

And the topology generated by complete is as follows: $\mathcal{T}_{k_n} = \mathcal{P}(V)$.

It has been seen that $\tau_{k_n}$ It is a discrete topology on $V$.

**Conclusions:**

In this article, it is shown that graphs can be generated by topological space by the adjacent matrix relations on vertices. Given the fundamental step toward studying some properties of undirected graph by their corresponding topologies. The new topologies can be used to solve some problem on undirected graph. When the graph has many vertices and edges, by transforming it into an adjacency matrix, we obtain the topologies from

**References**


