Association Schemes for Some Finite Group Rings II

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Abstract:
Association schemes have been used in coding theory and other combinatorial problems. In this paper, we construct association schemes for the abelian groups $\mathbb{Z}_2^r$, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, set of $n \times n$ matrices over $\mathbb{Z}_m$ and for the general linear group of order 2 over $\mathbb{Z}_2, \mathbb{Z}_4$, and $\mathbb{Z}_6$. We also obtain association schemes for symmetric groups and alternating groups of degree 4 and 5 using canonical forms.

Keywords: Group ring; Association scheme.

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1. Introduction

Association schemes (AS) introduced by Bose and Shimamoto [1], play a key role in the study of algebraic combinatorics. It has applications in graph theory, coding theory, group theory and design theory [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Jørgensen’s list of non-symmetric association schemes with classes smaller than 96 in vertices in [12], inspires us to research non-symmetric association schemes for various finite groups and group rings. In our previous work [13], we have constructed non symmetric commutative AS for symmetric groups, dihedral groups, abelian groups $\mathbb{Z}_p^r$ (where $p$ is an odd prime), $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$ ($p_i$'s are distinct primes), finite group rings over $\mathbb{Z}_n$ and circulant matrices over $\mathbb{Z}_p$, for $p$ prime. In the present study, we construct association schemes for the abelian groups $\mathbb{Z}_2^r = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, the general linear group $GL(2, \mathbb{Z}_2)$, $GL(2, \mathbb{Z}_4)$, $GL(2, \mathbb{Z}_6)$, symmetric group and alternating group of order 4 and 5.

In this paper, let $\mathcal{P}$ denote the partition of $Y \times Y$, where $Y$ is a finite set, and let $\mathbb{Z}_p^r = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. Some basic literature and preliminaries on association schemes are given below.

1.1. Association Scheme

**Definition 1.** Let $\mathcal{P}$ be a partition of $Y \times Y$ where $Y$ is a finite set and let $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$ binary relations on $\mathcal{P}$. Then $\mathcal{A} = (Y, \mathcal{P})$ forms $n$-class association scheme if the subsequent conditions hold:

1. Identity relation $\mathcal{S}_0 = \{(a, a) : a \in Y\} \in \mathcal{P}$.

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(2) $S^* = \{(a, b) : (b, a) \in S\} \in \mathcal{P}$ for any relation $S \in \mathcal{P}$.

(3) If $(a, b) \in S_k$, the number of elements $c \in Y$ such that $(a, c) \in S_l$, $(c, b) \in S_m$ is a constant $p^k_{lm}$ not depending on choice of $a$ and $b$ for all integers $0 \leq k, l, m \leq n$.

The integers $\{p^k_{lm}\}_{0 \leq k, l, m \leq n}$ are called parameters or intersection numbers of $\mathcal{A}$. If each relation $S$ in $\mathcal{P}$ is a symmetric relation, that is, $S = S^*$, then $\mathcal{A}$ is called symmetric association scheme and if $p^k_{lm} = p^l_{mk} \forall 0 \leq k, l, m \leq n$, then it is called commutative association scheme. Let the set $aS = \{b \in Y \mid (a, b) \in S\}$ for $a \in Y$ and $S \in \mathcal{P}$. The elements $a$ and $b$ in $Y$ are called $k^{th}$ associates if $(a, b) \in S_k$ with $a \neq b$. Note that every symmetric association scheme is commutative. With regards to more basic association schemes results, refer [7, 14].

**Definition 2.** A finite group $G$ with the conjugacy classes $C_0, C_1, \ldots, C_d$ produces a commutative association scheme with a class of relations on $G$ defined by $S_k = \{(a, b)\mid ba^{-1} \in C_k\} \forall 0 \leq k \leq d$. This scheme is called the group association scheme of $G$.

Association schemes can be determined for all those groups whose conjugacy classes are known.

**Lemma 1.** Let $Y = \mathbb{Z}_n$ and $S_k$ defines relations on $\mathcal{P}$ by $S_k = \{(a, b)\mid a = k + b, a, b \in \mathbb{Z}_n\} \forall k \in \mathbb{Z}_n$. Then $(Y, \mathcal{P})$ is a non symmetric commutative association scheme with parameters given by:

$$p^k_{lm} = \begin{cases} 1 & \text{if } k = l + m, \\ 0 & \text{if } k \neq l + m \end{cases}$$

where $k, l, m \in \mathbb{Z}_n$.

**Proof.** Since $\mathbb{Z}_n$ is an abelian group, $(\mathbb{Z}_n, \mathcal{P})$ under given relations becomes a commutative association scheme. For arbitrary relations $S_i, S_m, S_k$ in $\mathcal{P}$, we find cardinality $p^k_{lm} = |aS_i \cap bS_m|$ whenever $(a, b) \in S_k$. Let $(a, b)$ be an arbitrary element of $Y$ in $S_k$ and let $aS_i = a'$ and $bS_m = b'$. This implies, $a = a' + l, b' = b + m$ and $a = b + k$ and we get $p^k_{lm} = 1$ if $a' = b'$ which implies $p^k_{lm} = 1$ if $k = l + m$. Further, $p^k_{lm} = 0$ if $a' \neq b'$, equivalently $p^k_{lm} = 0$, whenever $k \neq l + m$. $\square$

In the next section of this paper, we work on non symmetric association scheme of the cyclic groups $\mathbb{Z}_2^r, \mathbb{Z}_n \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, and general linear group of order 2 over $\mathbb{Z}_2$.

2. Association schemes for some finite Groups

**Theorem 1.** Let $Y = \mathbb{Z}_2^r = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $r \geq 2$. We define relations $S_k$ on $\mathcal{P}$ by $S_k = \{(a, b)\mid b_1 \equiv (t_s + a_s) \mod 2 \mid t_s, a_s \in \mathbb{Z}_2 \forall 1 \leq s \leq r, \ a = (a_1, a_2, \ldots, a_r), b = (b_1, b_2, \ldots, b_r) \in Y\}$

where $k = 2^{r-1}t_1 + 2^{r-2}t_2 + \cdots + 2t_{r-1} + t_r$.

Then $\mathcal{A} = (Y, \mathcal{P})$ is a symmetric and commutative association scheme with parameters

$$p^k_{lm} = \begin{cases} 1 & \text{if } t_s^{(k)} \equiv t_s^{(l)} + t_s^{(m)} \mod 2 \forall 1 \leq s \leq r, \\ 0 & \text{otherwise} \end{cases}$$

where $k = \sum_{s=1}^r 2^{r-s}t_s^{(k)}; \ l = \sum_{s=1}^r 2^{r-s}t_s^{(l)}; \ m = \sum_{s=1}^r 2^{r-s}t_s^{(m)}$ for some $t_s^{(k)}, t_s^{(l)}, t_s^{(m)} \in \mathbb{Z}_2$.
Proof. Observe that \(|Y| = 2^r = |S_k|\) for all \(0 \leq k \leq 2^r - 1\). The relations \(S_k\) are disjoint and \(\cup S_k: 0 \leq k \leq 2^r - 1 = \mathcal{P}\). For arbitrary relations \(S_l, S_m, S_k\) in \(\mathcal{P}\), we prove that the parameters \(p^k_{lm} = |aS_l \cap bS_m^*|\) is constant for \((a, b) \in S_k\). Let \((a, b)\) be an arbitrary element of \(Y\) in \(S_k\) where \(a = (a_1, a_2, ..., a_r), b = (b_1, b_2, ..., b_r)\). Let \(aS_l = a' = (a'_1, a'_2, ..., a'_r)\) and \(bS_m^* = b' = (b'_1, b'_2, ..., b'_r)\). Now \((a, b) \in S_k\) implies \(b_s \equiv (t_s^{(k)} + a_s) \mod 2\) where \(k = \sum_{s=1}^{r} 2^{r-s} t_s^{(k)}\) for some \(t_s^{(k)} \in \mathbb{Z}_2 \forall 1 \leq s \leq r\). Similarly, \((a, a') \in S_l\) and \((b', b) \in S_m\), implies \(a'_s \equiv (t_s^{(l)} + a_s) \mod 2\) where \(l = \sum_{s=1}^{r} 2^{r-s} t_s^{(l)}\) for some \(t_s^{(l)} \in \mathbb{Z}_2 \forall 1 \leq s \leq r\), and \(b_s \equiv (t_s^{(l)} + b'_s) \mod 2\) where \(m = \sum_{s=1}^{r} 2^{r-s} t_s^{(m)}\) for some \(t_s^{(m)} \in \mathbb{Z}_2 \forall 1 \leq s \leq r\). We observe that \(p^k_{lm} = 1\) if and only if \(a' = b'\) that is, if \(t_s^{(k)} \equiv (t_s^{(l)} + t_s^{(m)}) \mod 2\) for all \(1 \leq s \leq r\).

To show \((Y, \mathcal{P})\) is a symmetric association scheme, we prove that \(S_k^* = S_k\) for all \(0 \leq k \leq 2^r\). Let \(S_k^* = S_k\) where \(K = \sum_{s=1}^{r} 2^{r-s} T_s\) and \(k = \sum_{s=1}^{r} 2^{r-s} t_s\) for some \(t_s, T_s \in \mathbb{Z}_2\). If \((b, a) \in S_k\), then further \(a_s \equiv T_s + b_s \mod 2\) and \(b_s \equiv t_s + a_s \mod 2\) which implies that \(T_s = t_s \forall 1 \leq s \leq r\). Thus \(K = k\), and hence, \(S_k^* = S_k\). □

Theorem 2. Let \(Y = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}\), where \(n_1, n_2, ..., n_r\) are pairwise co-prime. The relations \(S_k\) on \(\mathcal{P}\) defined by

\[
S_k = \{(a, b) | b_s \equiv (k + a_s) \mod n_s \forall 1 \leq s \leq r \mid a = (a_1, a_2, ..., a_r), b = (b_1, b_2, ..., b_r) \in Y\}
\]

where \(0 \leq k < n_1 n_2 \cdots n_r\),

is a non-symmetric and commutative association scheme with parameters

\[
p^k_{lm} = \begin{cases} 1 & \text{if } k \equiv (l + m) \mod n_s \forall 1 \leq s \leq r \\ 0 & \text{otherwise} \end{cases}
\]

where \(0 \leq k, l, m \leq n_1 n_2 \cdots n_r - 1\).

Proof. Observe that \(|Y| = n_1 n_2 \cdots n_r = |S_k|\) for all \(0 \leq k < n_1 n_2 \cdots n_r\). The relations \(S_k\) being disjoint, form partition of \(\mathcal{P}\). For arbitrary relations \(S_l, S_m, S_k\) in \(\mathcal{P}\), we prove that for each pair \(a, b\) with \((a, b) \in S_k\), the number of elements in the set \(\{c \in Y \mid (a, c) \in S_l, (c, b) \in S_m\}\) is invariant. Let \((a, b) \in S_k\) where \(a = (a_1, a_2, ..., a_r), b = (b_1, b_2, ..., b_r) \in Y\). Further, suppose that \(aS_l = a' = (a'_1, a'_2, ..., a'_r), bS_m^* = b' = (b'_1, b'_2, ..., b'_r)\). Now \((a, b) \in S_k\) implies \(b_s \equiv (k + a_s) \mod n_s \forall 1 \leq s \leq r\). Similarly, \((a, a') \in S_l\) and \((b', b) \in S_m\). We get \(a'_s \equiv (l + a_s) \mod n_s\) and \(b_s \equiv (m + b'_s) \mod n_s \forall 1 \leq s \leq r\).

The above equations give that, \(p^k_{lm} = 1\) if and only if \(a' = b'\). That is, \(p^k_{lm} = 1\) whenever \(k \equiv (l + m) \mod n_s \forall 1 \leq s \leq r\). Hence, \((Y, \mathcal{P})\) is a non-symmetric and commutative association scheme. □

Theorem 3. Let \(Y = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}\), where \(2 < n_1 \leq n_2 \leq \cdots \leq n_r\) and \(\gcd(n_1, n_2, ..., n_r) \neq 1\). The relations \(S_k\) on \(\mathcal{P}\) defined by

\[
g_{gcd}(n_1, n_2, ..., n_r) \neq 1\]
\[ S_k = \{(a,b)|b_r \equiv (k + a_r) \text{mod} n_r, b_s \equiv (k + t_s + a_s) \text{mod} n_s \forall 1 \leq s < r|\]
\[ t_s \in Z_{n_s}, a = (a_1, a_2, \ldots, a_r), b = (b_1, b_2, \ldots, b_r) \in Y \]

where \( k = n_2 n_3 \ldots n_r t_1 + n_3 n_4 \ldots n_r t_2 + \cdots + n_r t_{r-1} + t_r \).

is a non-symmetric and commutative association scheme with parameters

\[ p_{lm}^k = \begin{cases} 1 & \text{if } k \equiv (l + m) \text{mod} n_r, \text{and } k + t_{s(l)} + t_{s(m)} \equiv (l + m + t_{s(l)} + t_{s(m)}) \text{mod} n_s \forall 1 \leq s < r \\ 0 & \text{otherwise} \end{cases} \]

where \( k = n_2 n_3 \ldots n_r t_1(k) + n_3 n_4 \ldots n_r t_2(k) + \cdots + n_r d_{r-1}(k) + t_r(k); l = n_2 n_3 \ldots n_r t_1(l) + n_3 n_4 \ldots n_r t_2(l) + \cdots + n_r t_{r-1}(l) + t_r(l); m = n_2 n_3 \ldots n_r t_1(m) + n_3 n_4 \ldots n_r t_2(m) + \cdots + n_r t_{r-1}(m) + t_r(m) \)

for some \( t_{s(l)}, t_{s(l)}', t_{s(m)} \in Z_{n_s} \) where \( 1 \leq s \leq r \).

For each \( 0 \leq k < n_1 n_2 \ldots n_r, S_k^* = S_k \) can be calculated by solving the following equations:

\[ K + k \equiv 0 \text{ mod } n_r \]
\[ K + k + T_s + t_s \equiv 0 \text{ mod } n_s \forall 1 \leq s \leq r - 1 \]

where \( k = n_2 n_3 \ldots n_r t_1 + n_3 n_4 \ldots n_r t_2 + \cdots + n_r t_{r-1} + t_r \) and \( K = n_2 n_3 \ldots n_r T_1 + n_3 n_4 \ldots n_r T_2 + \cdots + n_r T_{r-1} + T_r \) for some \( t_s, T_s \in Z_{n_s} \).

Proof. Again recall, \( |Y| = n_1 n_2 \ldots n_r = |S_k| \) for all \( 0 \leq k < n_1 n_2 \ldots n_r \). Above defined relations \( S_k \) being disjoint, form a partition of \( \mathcal{P} \). For arbitrary relations \( S_l, S_m, S_k \) in \( \mathcal{P} \), we show that for each \( (a,b) \in S_k \), the number of elements in the set \{ \( c \in Y | (a,c) \in S_l, (c,b) \in S_m \) \} is invariant. Let \( (a,b) \in S_k \) where \( a = (a_1, a_2, \ldots, a_r), b = (b_1, b_2, \ldots, b_r) \in Y \). Suppose \( aS_l = a' = (a'_1, a'_2, \ldots, a'_r), bS_m = b' = (b'_1, b'_2, \ldots, b'_r) \). Then \( (a,b) \in S_k \) implies \( b_r \equiv (k + a_r) \text{ mod } n_r, b_s \equiv (k + t_s(k) + a_s) \text{ mod } n_s \forall 1 \leq s \leq r - 1 \) where \( k = n_2 n_3 \ldots n_r t_1(k) + n_3 n_4 \ldots n_r t_2(k) + \cdots + n_r t_{r-1}(k) + t_r(k) \) for some \( t_s \in Z_{n_s} \forall 1 \leq s \leq r \).

Since \( (a,a') \in S_l \) and \( (b',b) \in S_m \), we have \( a_r' \equiv (l + a_r) \text{ mod } n_r, a_s' \equiv (l + t_s(l) + a_s) \text{ mod } n_s \forall 1 \leq s \leq r - 1 \) where \( l = n_2 n_3 \ldots n_r t_1(l) + n_3 n_4 \ldots n_r t_2(l) + \cdots + n_r t_{r-1}(l) + t_r(l) \) for some \( t_s(l) \in Z_{n_s} \forall 1 \leq s \leq r \), and

\[ b_r \equiv (m + b_r') \text{ mod } n_r, b_s \equiv (m + t_s(m) + b_s') \text{ mod } n_s \forall 1 \leq s \leq r - 1 \] where \( m = n_2 n_3 \ldots n_r t_1(m) + n_3 n_4 \ldots n_r t_2(m) + \cdots + n_r t_{r-1}(m) + t_r(m) \) for some \( t_s(m) \in Z_{n_s} \forall 1 \leq s \leq r \).

We observe that \( p_{lm}^k = 1 \) if and only if \( a' = b' \). That is, \( p_{lm}^k = 1 \) if \( k \equiv (l + m) \text{ mod } n_r \) and \( k + t_{s(k)} \equiv (l + m + t_{s(l)} + t_{s(m)}) \text{ mod } n_s \forall 1 \leq s \leq r - 1 \). Hence, \( (Y, \mathcal{P}) \) is a non-symmetric and commutative association scheme.

Next, we find \( S_k^* \). Let \( S_k^* = S_k \) where \( K = n_2 n_3 \ldots n_r T_1 + n_3 n_4 \ldots n_r T_2 + \cdots + n_r T_{r-1} + T_r \) for some \( T_s \in Z_{n_s} \). If \( (b,a) \in S_k \), then \( a_r \equiv K + b_r \text{ mod } n_r; a_s \equiv K + T_s + b_s \text{ mod } n_s \forall 1 \leq s \leq r - 1 \). Also, since \( (a,b) \in S_k \), we get \( b_r \equiv k + a_r \text{ mod } n_r; b_s \equiv k + t_s + a_s \text{ mod } n_s \forall 1 \leq s \leq r - 1 \).
This implies that $K + k \equiv 0 \mod n_r$ and $K + k + T_s + t_s \equiv 0 \mod n_s \forall 1 \leq s \leq r - 1$. Solving these equations, we get the value of $K$ as claimed. □

**Corollary 1.** Let $Y = M_n(\mathbb{Z}_m)$ be the set of all $n \times n$ matrices over $\mathbb{Z}_m$ where $m \geq 2$ and $n \geq 2$. Any element of $M_n(\mathbb{Z}_m)$ can be written as $n^2$-tuple in $\mathbb{Z}_m^n$. Therefore, relations defined on $\mathbb{Z}_m^n$ will form an association scheme over $M_n(\mathbb{Z}_m)$.

**Proof.** Follows from Theorem 1 when $m = 2$, and from Theorem 3 when $m > 2$. □

Presentations of some general linear groups $GL(2, \mathbb{Z}_n)$ are given in [15] for $n = 4, 6, 8, 10$. With the help of these presentations, we can compute the canonical form for these groups (provided in Table 1) Canonical forms of some general linear groups

The canonical form of the presentation of $GL(2, \mathbb{Z}_2)$ is $A^aB^b$: $0 \leq a \leq 1, 0 \leq b \leq 2$. The group is isomorphic onto $S_3$, symmetric group on 3-symbols. Hence, another familiar presentation can be given as $GL(2, \mathbb{Z}_2) = \langle A, B | A^2 = B^3 = 1, AB = B^{-1}A \rangle$. Using this presentation, we get a non symmetric association scheme.

| Group     | Generators | Presentation          | Canonical form
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<tbody>
<tr>
<td>$GL(2, \mathbb{Z}_2)$</td>
<td>$A, B$</td>
<td>$τ^2 = σ^3 = (τσ)^3 = 1$</td>
<td>$A^aB^b$: $0 \leq a \leq 1, 0 \leq b \leq 2$</td>
</tr>
<tr>
<td>$GL(2, \mathbb{Z}_4)$</td>
<td>$A, B, C$</td>
<td>$A^2 = B^2 = C^4 = X^3 = Y^4$ = $(CBA)^4 = 1$, $C^2A = AC^2, C^2B = BC^2, BC = C^{-1}B$</td>
<td>$A^aC^bC^cX^d$: $0 \leq a \leq 1, 0 \leq b, c \leq 3, 0 \leq d \leq 2$</td>
</tr>
<tr>
<td>$GL(2, \mathbb{Z}_6)$</td>
<td>$A, B, C$</td>
<td>$A^2 = B^2 = C^4 = X^{12} = Y^6$, $(AB)^3 = (BC)^2 = 1$, $C^A = AC^2, C^2B = BC^2, (AC)^2 = C^2$, $(CA)^2B(CA)^2 = (AC)^2B(AC)^2$ where $X = B(CA)^2B$ and $Y = CAB$</td>
<td>$C^aX^bY^cB^d$: $0 \leq a, d \leq 1, 0 \leq b \leq 11, 0 \leq c \leq 5$</td>
</tr>
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</table>

**Theorem 4.** Let $Y = GL(2, \mathbb{Z}_2)$. Define relations $S_k$ on $\mathcal{P}$ by

$$S_k = \{(A^aB^b, A^{k+a}B^{k+b}) | 0 \leq a \leq 1, 0 \leq b \leq 2 \} \text{ for all } 0 \leq k \leq 5.$$ 

Then $(Y, \mathcal{P})$ is a non symmetric association scheme with parameters as follows:

$$p_{im}^k = \begin{cases} 1 & \text{if } k = l + m, \\ 0 & \text{if } k \neq l + m \end{cases}$$

where $0 \leq k, l, m \leq 5$.

**Proof.** Observe that $S_0 = \{(M, M) | M \in GL(2, \mathbb{Z}_2)\}$ is an identity relation. For arbitrary relations $S_i, S_m, S_k \in \mathcal{P}$, we find the cardinality $p_{im}^k$ such that $|MS_i \cap NS_m^k| = p_{im}^k$ for all $(M, N) \in S_k$. Let $(M, N) \in S_k$ and let $MS_i = M’$ and $NS_m^k = N’$. That is, $M = A^aB^b, N = A^{k+a}B^{k+b}, M = A^{a_1}B^{b_1}, M’ = A^{l+a_1}B^{l+b_1}, N’ = A^{a_2}B^{b_2}$, $N = A^{m+a_2}B^{m+b_2}$ where $0 \leq a, a_1, a_2 \leq 1$ and $0 \leq...
Since every pair \((M, N)\) in \(Y\) are \(k^{th}\) associates for exactly one \(k\), we find that \(p_{lm}^k\) can be either 0 or 1. Hence, with the help of the above equations, we obtain \(p_{lm}^k = 1\) if \(M' = N'\) equivalently, if \(k = l + m\). Moreover, \(p_{lm}^k = 0\) if \(M' \neq N'\) that is, whenever \(k \neq l + m\). Also it can be easily proved that the relations are not symmetric and hence \((Y, \mathcal{P})\) is a non symmetric association scheme. \(\square\)

**Theorem 5.** Let \(Y = GL(2, \mathbb{Z}_4)\) and \(\mathcal{P}\) be a partition of \(Y \times Y\). For all \(k\) written in form of \(48s + 16n + 4t + r\) \((s \in \mathbb{Z}_2; n \in \mathbb{Z}_3; t, r \in \mathbb{Z}_4)\), the relations \(S_k\) in \(\mathcal{P}\) defined by

\[
S_k = \{(A^a C^b Y^c X^d, A^{a+s} C^{b+t} Y^{c+r} X^{d+n}) | a \in \mathbb{Z}_2; b, c \in \mathbb{Z}_4; d \in \mathbb{Z}_3\}
\]

is a non-symmetric association scheme with parameters

\[
p_{lm}^k = \begin{cases} 
1 & \text{if } s(k) \equiv (s(l) + s(m)) \mod 2, \\
& n(k) \equiv (n(l) + n(m)) \mod 2, \\
& t(k) \equiv (t(l) + t(m)) \mod 4, \text{and} \\
& r(k) \equiv (r(l) + r(m)) \mod 4 \text{ otherwise} \\
0 & \text{otherwise}
\end{cases}
\]

*Proof.* Observe that \(|S_k| = |Y|\) for all \(0 \leq k < 96\). The relations \(S_k\) are disjoint and \(\cup S_k: 0 \leq k < 96 = \mathcal{P}\).

\(S_0 = \{(M, M): M \in GL(2, \mathbb{Z}_4)\}\) is an identity relation. Let \(S_t, S_m, S_k\) be arbitrary relations in \(\mathcal{P}\) and \((M, N)\) be any element of \(S_k\). Since \(M \in Y, M\) is of the form \(A^a C^b Y^c X^d\) for some \(a, b, c, d\) (from Table [table:canonical forms An,Sn]). Suppose \(MS_t = M'\) and \(NS_m = N'\) where \(M', N' \in Y\).

Now \((M, N) \in S_k, (M, M') \in S_t\) and \((N', N) \in S_m\), implies

\[
N = A^{(a+s(k))}_{mod 2} C^{(b+t(k))}_{mod 4} Y^{(c+r(k))}_{mod 4} X^{(d+n(k))}_{mod 3},
\]

\[
M' = A^{(a+s(l))}_{mod 2} C^{(b+t(l))}_{mod 4} Y^{(c+r(l))}_{mod 4} X^{(d+n(l))}_{mod 3}, \text{ and}
\]

\[
N' = A^{(a+s(k), s(m))}_{mod 2} C^{(b+t(k)-t(m))}_{mod 4} Y^{(c+r(k) - r(m))}_{mod 4} X^{(d+n(k) - n(m))}_{mod 3}
\]

respectively, where \(k = 48s(k) + 16n(k) + 4t(k) + r(k), l = 48s(l) + 16n(l) + 4t(l) + r(l)\) and \(m = 48s(m) + 16n(m) + 4t(m) + r(m)\), for some \(s(k), s(l), s(m) \in \mathbb{Z}_2; n(k), n(l), n(m) \in \mathbb{Z}_3\) and \(t(k), t(l), t(m), r(k), r(l), r(m) \in \mathbb{Z}_4\).

Since every pair \((M, N)\) in \(\mathcal{P}\) are \(k^{th}\) associates for exactly one \(k\), we find that \(p_{lm}^k\) can be either 0 or 1. Hence we obtain that \(p_{lm}^k = 1\) if and only if \(M' = N'\) that is, if \(s(k) \equiv (s(l) + s(m)) \mod 2, t(k) \equiv (t(l) + t(m)) \mod 4, r(k) \equiv (r(l) + r(m)) \mod 4\) and \(n(k) \equiv (n(l) + n(m)) \mod 3. \square\)

**Theorem 6.** Let \(Y = GL(2, \mathbb{Z}_6)\) and \(\mathcal{P}\) be a partition of \(Y \times Y\). For all \(k\) written in form of \(144s + 72n + 12t + r\) \((s, n \in \mathbb{Z}_2, t \in \mathbb{Z}_6, r \in \mathbb{Z}_{12})\), the relations \(S_k\) in \(\mathcal{P}\) defined by

\[
S_k = \{(C^a X^b Y^c B^d, C^{a+s} X^{b+t} Y^{c+r} B^{d+n}) | a, d \in \mathbb{Z}_2; b \in \mathbb{Z}_{12}; c \in \mathbb{Z}_6\}
\]

is a non-symmetric association scheme with parameters
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\[ p_{lm}^k = \begin{cases} 1 & \text{if } s^{(k)} \equiv (s^{(l)} + s^{(m)}) \mod 2, \\ n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 2, \\ t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 6, \text{and} \\ r^{(k)} \equiv (r^{(l)} + r^{(m)}) \mod 12 \\ 0 & \text{otherwise} \end{cases} \]

**Proof.** Here, \(|c_k| = |Y|\) for all \(0 \leq k < 288\). The relations \(S_k\) are disjoint and \(\bigcup S_k : 0 \leq k < 288 = \mathcal{P}\). Proceeding as in proof of theorem 5, we can find cardinality \(p_{lm}^k\) such that for all \((x, y) \in S_k\), \(|xS_i \cap yS_m| = p_{lm}^k\) is a constant. □

**Note:** Let \(SL(2, \mathbb{Z}_p)\) be the special linear group over \(\mathbb{Z}_p\) and \(GL(2, \mathbb{Z}_p)\) be general linear group over \(\mathbb{Z}_p\). The structure of these conjugacy classes are worked out in detail in [16]. Using these classes one can compute group association scheme for \(SL(2, \mathbb{Z}_p)\) and \(GL(2, \mathbb{Z}_p)\) using Definition 2.

Let us represent \(S_n\) as the symmetric group of degree \(n\). Group association scheme for symmetric groups have been described by Tomiyama and Yamazaki in [17]. In [13], Sabharwal et al. identified the association schemes for the symmetric groups \(S_3\) and \(S_4\) without using the conjugacy classes. In next theorem, we have determined non symmetric commutative association scheme for the symmetric groups \(S_4\) and \(S_5\) and alternating groups \(A_3, A_4\) and \(A_5\) without using conjugacy classes.

**Lemma 2.** Let \(X = A_3 = \sigma^i : 0 \leq i \leq 2\) where \(\sigma^3 = 1\). Then the relations \(S_k\) on \(\mathcal{P}\) defined by

\[ S_k = \{(\sigma^i, \sigma^{k+i}) | 0 \leq i \leq 2\} \text{ for all } 0 \leq k \leq 2 \]

is a non-symmetric commutative association scheme and intersection numbers of this association scheme are as follows:

\[ p_{lm}^k = \begin{cases} 1 & \text{if } k = l + m, \\ 0 & \text{if } k \neq l + m \end{cases} \]

**Proof.** Since \(A_3\) is isomorphic to \(\mathbb{Z}_3\) by mapping \(\sigma^i \mapsto i\), and the set of relations \(\{(i, k + i) | i = 0,1,2\}\) forms AS for \(\mathbb{Z}_3\), we can conclude the result. □

Presentations of alternating and symmetric groups of degree less than 8 are given in [18]. With the help of these presentations, we can compute the canonical form for these groups (provided in Table 2 Canonical forms of alternating and symmetric groups of degree 4 and 5). In [13], canonical form of \(S_4\) is discussed.

**Table 2** Canonical forms of alternating and symmetric groups of degree 4 and 5

<table>
<thead>
<tr>
<th>Group</th>
<th>Generators</th>
<th>Presentation</th>
<th>Canonical form</th>
</tr>
</thead>
</table>
| \(A_4\) | \(\tau, \sigma\) | \(A^2 = B^3 = I, AB = B^{-1}A\) | \(\tau^a \sigma \tau^b \sigma^c;\)  
\(0 \leq a, b \leq 1,\)  
\(0 \leq c \leq 2\) |
| \(S_4\) | \(\tau, \sigma\) | \(\tau^2 = \sigma^3 = (\tau \sigma)^3 = 1\) | \(A^2 \tau^a \sigma^b \tau^c;\)  
\(0 \leq a \leq 1, 0 \leq b \leq 2,\)  
\(0 \leq c \leq 3\) |

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Theorem 7. Let \( Y = A_k \) and \( \mathcal{P} \) be a partition of \( Y \times Y \). For all \( k \) written in form of \( 3n + t \ (n \in \mathbb{Z}_4, t \in \mathbb{Z}_3) \), the relations \( S_k \) on \( \mathcal{P} \) defined by

\[
S_k = \left\{ (\alpha_{i,j}^{(0)}, \alpha_{i,j}^{(1)}, \alpha_{i,j}^{(2)}, \alpha_{i,j}^{(3)}) \mid 0 \leq i \leq 2, 0 \leq j \leq 3 \right\}
\]

where \( \alpha_{i,j}^{(0)} = \sigma^i \), \( \alpha_{i,j}^{(1)} = \sigma \tau \sigma^i \), \( \alpha_{i,j}^{(2)} = \tau \sigma^i \), \( \alpha_{i,j}^{(3)} = \tau \sigma \tau \sigma^i \)

is a non-symmetric association scheme with parameters

\[
p_{lm}^k = \begin{cases} 1 & \text{if } n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 4, \text{ and } t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 3 \\ 0 & \text{otherwise} \end{cases}
\]

where \( k = 3n^{(k)} + t^{(k)}; l = 3n^{(l)} + t^{(l)}; m = 3n^{(m)} + t^{(m)}; \) for some \( t^{(k)}, t^{(l)}, t^{(m)} \in \mathbb{Z}_3 \) and \( n^{(k)}, n^{(l)}, n^{(m)} \in \mathbb{Z}_4 \).

Proof. Observe that \( |S_k| = |Y| \) for all \( 0 \leq k < 12 \). The relations \( S_k \) are disjoint and \( \cup S_k : 0 \leq k < 12 = \mathcal{P} \).

\[
S_0 = \left\{ (\alpha_{i,j}^{(0)}, \alpha_{i,j}^{(1)}, \alpha_{i,j}^{(2)}, \alpha_{i,j}^{(3)}) \mid 0 \leq i \leq 2, 0 \leq j \leq 3 \right\}
\]

is an identity relation. For arbitrary relations \( S_l, S_m, S_k \) in \( \mathcal{P} \), we prove that for all \( (x, y) \in S_k, |xS_l \cap yS_m| \) is constant. Now let \( (x, y) \) be an arbitrary element of \( S_k \). Since \( x \in Y, x \) is of the form \( \tau^a \sigma^b \sigma^c \) for some \( a, b, c \) (from Table [table:canonical forms \( An, Sn \)]) and further \( x = \alpha_{i,j}^{(j)} \) for some \( i, j \). Suppose \( xS_l = x' \) and \( yS_m = y' \) where \( x', y' \in Y \).

Now \( (x, y) \in S_k \) implies \( y = \alpha_{(i+t^{(k)})mod3}^{((j+n^{(k)})mod4)} \) where \( k = 3n^{(k)} + t^{(k)} \) for some \( t^{(k)} \in \mathbb{Z}_3, n^{(k)} \in \mathbb{Z}_4 \).

Similarly, \( (x, x') \in S_l \) and \( (y', y) \in S_m \), implies \( x' = \alpha_{(i+t^{(l)})mod3}^{((j+n^{(l)})mod4)} \) where \( l = 3n^{(l)} + t^{(l)} \) for some \( t^{(l)} \in \mathbb{Z}_3, n^{(l)} \in \mathbb{Z}_4 \), and \( y' = \alpha_{(i+t^{(m)})mod3}^{((j+n^{(m)})-n^{(m)})mod4} \) where \( m = 3n^{(m)} + t^{(m)} \) for some \( t^{(m)} \in \mathbb{Z}_3, n^{(m)} \in \mathbb{Z}_4 \). Since every pair \( (x, y) \) in \( \mathcal{P} \) are \( k^{th} \) associates for exactly one \( k \), we find that \( p_{lm}^k \) can be either 0 or 1. Hence we obtain that \( p_{lm}^k = 1 \) if and only if \( x' = y' \) that is, if \( t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 3 \) and \( n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 4 \). □

Theorem 8. Let \( Y = A_5 \) and \( \mathcal{P} \) be a partition of \( Y \times Y \). For all \( k \) written in form of \( 12n + 4t + r \ (n \in \mathbb{Z}_5, t \in \mathbb{Z}_3, r \in \mathbb{Z}_4) \), the relations \( S_k \) on \( \mathcal{P} \) defined by

\[
S_k = \left\{ (\alpha_{i,j}^{(s)}, \alpha_{i+(j+r)mod5}^{((j+n+r)mod4)}) \mid i \in \mathbb{Z}_5, j \in \mathbb{Z}_3, s \in \mathbb{Z}_4 \right\}
\]
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where \( \alpha_{ij}^{(0)} = \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \), \( \alpha_{ij}^{(1)} = \sigma \tau \sigma \gamma^i \sigma \sigma^j \), \( \alpha_{ij}^{(2)} = \tau \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \), \( \alpha_{ij}^{(3)} = \tau \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \)

is a non-symmetric association scheme with parameters

\[
p_{lm}^k = \begin{cases} 
1 & \text{if } n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 5, \\
\quad t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 3, \text{and} \\
\quad r^{(k)} \equiv (r^{(l)} + r^{(m)}) \mod 4 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** Observe that \( |S_k| = |Y| \) for all \( 0 \leq k < 60 \). The relations \( S_k \) are disjoint and \( \cup \{S_k : 0 \leq k < 60 \} = \mathcal{P} \).

\( S_0 = \{ (\alpha_{ij}^{(s)}, \alpha_{ij}^{(s)}): 0 \leq i \leq 4, 0 \leq j \leq 2, 0 \leq s \leq 3 \} \) is a non-identity relation. Let \( S_l, S_m, S_k \) be arbitrary relations in \( \mathcal{P} \) and \( (x, y) \) be any element of \( S_k \). Since \( x \in Y \), \( x \) is of the form \( \tau^a \sigma \tau \sigma^2 \gamma^b \tau^c \sigma \sigma^d \) for some \( a, b, c, d \) (from Table [table:canonical forms An,Sn]) and further \( x = \alpha_{ij}^{(s)} \) for some \( i, j, s \). Suppose \( xS_l = x' \) and \( yS_m = y' \) where \( x', y' \in Y \).

Now \( (x, y) \in S_k \) implies \( y = \alpha_{ij}^{(s)} \) for some \( k \leq 12n^{(k)} + 4t^{(k)} + r^{(k)} \) for some \( t^{(k)} \in \mathbb{Z}_3, n^{(k)} \in \mathbb{Z}_5, r^{(k)} \in \mathbb{Z}_4 \). Similarly, \( (x, x') \in S_l \) and \( (y', y) \in S_m \) implies \( x' = \alpha_{ij}^{(s)} \) for some \( l = 12n^{(l)} + 4t^{(l)} + r^{(l)} \) for some \( t^{(l)} \in \mathbb{Z}_3, n^{(l)} \in \mathbb{Z}_5, r^{(l)} \in \mathbb{Z}_4 \); and \( y' = \alpha_{ij}^{(s)} \) for some \( m = 12n^{(m)} + 4t^{(m)} + r^{(m)} \) for some \( t^{(m)} \in \mathbb{Z}_3, n^{(m)} \in \mathbb{Z}_5, r^{(m)} \in \mathbb{Z}_4 \). Since every pair \( (x, y) \) in \( \mathcal{P} \) are \( k \)th associates for exactly one \( k \), we find that \( p_{lm}^k \) can be either 0 or 1. Hence we obtain that \( p_{lm}^k = 1 \) if and only if \( x' = y' \) that is, if \( t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 3, r^{(k)} \equiv (r^{(l)} + r^{(m)}) \mod 4 \) and \( n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 5 \).

**Theorem 9.** Let \( Y = S_4 \) and \( \mathcal{P} \) be a partition of \( Y \times Y \). For all \( k \) written in form of \( 4n + t \) \( (n \in \mathbb{Z}_6, t \in \mathbb{Z}_4) \), the relations \( S_k \) on \( \mathcal{P} \) defined by

\[
S_k = \left\{ (\alpha_{ij}^{(j)}, \alpha_{ij}^{(j+n) \mod 4}) \mid i \in \mathbb{Z}_4, j \in \mathbb{Z}_6 \right\}
\]

where \( \alpha_{i}^{(0)} = \tau \sigma^i \), \( \alpha_{i}^{(1)} = \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \), \( \alpha_{i}^{(2)} = \sigma^2 \tau \sigma^2 \gamma^i \sigma \sigma^j \), \( \alpha_{i}^{(3)} = \tau \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \), \( \alpha_{i}^{(4)} = \tau \sigma \tau \sigma^2 \gamma^i \sigma \sigma^j \)

is a non-symmetric association scheme with parameters

\[
p_{lm}^k = \begin{cases} 
1 & \text{if } n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 6, \text{and} \\
\quad t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 4, \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** \( S_0 = \{ (\alpha_{ij}^{(j)}, \alpha_{ij}^{(j)}): 0 \leq i \leq 2, 0 \leq j \leq 3 \} \) is an identity relation. It can be verified that \( (Y, \mathcal{P}) \) is an association scheme and the relations are non-symmetric.

Let \( S_l, S_m, S_k \in \mathcal{P} \). To find cardinality \( p_{lm}^k \) such that for all \( (x, y) \in S_k \), \( |xS_l \cap yS_m^*| = p_{lm}^k \).
Let \((x, y) \in S_k, xS_I = x'\) and \(yS_m = y'\) where \(x', y' \in Y\). Since \(x \in Y, x\) is of the form \(t'^a\sigma^b\tau\sigma^c\) for some \(a, b, c\) (from Table [Table:canonical forms An,Sn]) and further \(x = \alpha_i^{(j)}\) for some \(i, j\). Proceeding as in proof of previous theorem, we get \(y = \alpha_{i+(t(k)) \mod 4}^{(j+n(k)) \mod 6}\) where \(k = 4n^{(k)} + t^{(k)}\) for some \(t^{(k)} \in \mathbb{Z}_4, n^{(k)} \in \mathbb{Z}_6\); \(x' = \alpha_{i+(t(l)) \mod 4}^{(j+n(l)) \mod 6}\) where \(l = 4n^{(l)} + t^{(l)}\) for some \(t^{(l)} \in \mathbb{Z}_4, n^{(l)} \in \mathbb{Z}_6\), and \(y' = \alpha_{i+(t(m)) \mod 4}^{(j+n(m)) \mod 6}\) where \(m = 4n^{(m)} + t^{(m)}\) for some \(t^{(m)} \in \mathbb{Z}_4, n^{(m)} \in \mathbb{Z}_6\). Using these equations, we have \(p^k_{lm} = 1\) if and only if \(x' = y'\) that is, if \(t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 4\) and \(n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 6\).

**Theorem 10.** Let \(Y = S_5\) and \(P\) be a partition of \(X \times Y\). For all \(k\) written in form of \(24t + 4n + r\) \((t \in S_5, n \in S_6, r \in S_4)\), the relations \(S_k\) on \(P\) defined by

\[
S_k = \left\{ \left( \alpha_{i+j}^{(s)} \right)_{(i+t) \mod 4}^{(s+r) \mod 6} \right\} | i \in S_5, j \in S_6, s \in S_4
\]

where \(\alpha_{ij}^{(s)} = t^i\sigma^j\sigma^5\tau\sigma^j, \alpha_{ij}^{(1)} = t^i\sigma_4\tau^4\sigma^5\tau\sigma^j, \alpha_{ij}^{(2)} = t^i\sigma_4\tau^4\sigma_5\delta\tau\sigma^j, \alpha_{ij}^{(3)} = t^i\sigma_4\tau^4\sigma_5\delta\tau\sigma^j\)

is a non-symmetric association scheme with parameters

\[
p^k_{lm} = \begin{cases} 1 & \text{if } n^{(k)} \equiv (n^{(l)} + n^{(m)}) \mod 6, \\ t^{(k)} \equiv (t^{(l)} + t^{(m)}) \mod 5, \text{ and} \\ r^{(k)} \equiv (r^{(l)} + r^{(m)}) \mod 4 & \text{otherwise} \end{cases}
\]

**Proof.** Here, \(|S_k| = |Y|\) for all \(0 \leq k < 120\). The relations \(S_k\) are disjoint and \(\cup S_k : 0 \leq k < 120 = P\). Proceeding as in proof of theorem 8, we can find cardinality \(p^k_{lm}\) such that for all \((x, y) \in S_k, |xS_I \cap yS_m| = p^k_{lm}\) is a constant. 

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