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The M-polynomial of Schreier Graphs of the Basilica and Grigorchuk Groups: Comparative Evaluation

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Abstract:

A focus for comprehending the complex structures present in self-similar groups has been the study of Schreier graphs related to the Basilica and Grigorchuk group in recent years. Basic to group theory, Schreier graphs give a geometric picture of these groups' actions on sets and shed light on the connectivity and symmetry characteristics of these groups. This paper examines the M-polynomial of Schreier graphs of the Basilica and Grigorchuk groups, examining its consequences for topological indices and comparing the determined values.

Keywords: M-polynomial, Topological indices ,Schreier Graph, Basilica Group Grigorchuk Group

1. Introduction

Schreier graphs emerge as a fundamental representation of the actions performed by finitely generated groups on sets, offering a graphical depiction of these group actions. Particularly within the monarchy of automaton groups, Schreier graphs stand out as a natural focus of investigation. Finite invertible automata, functioning as finite input/output devices, intricately capture the sequential operations executed by a set of states on a finite alphabet. These automata, in turn, correspond to automorphisms of rooted regular trees, inducing bijections that collectively form what is known as an automaton group (for more details about automaton group please see [1-2].

The dynamic behaviour of an automaton group when acting upon a rooted tree manifests in the preservation of the hierarchical structure inherent in the tree's levels. By restricting this action to a specific level of the tree, one can construct a unique Schreier graph corresponding to that level. As a result, each automata group generates an unlimited number of finite Schreier graphs, each of which replicates the group's actions on the finite levels of the corresponding rooted regular tree. Within the empire of automaton groups laid captivating instances characterized by their unconventional and sometimes enigmatic properties. Notably, among these is the Basilica group, a pioneering example showcasing intriguing traits. The Basilica group is a unique entity that is produced by a three-state automata and is derived from the work of R. Grigorchuk and A. Zuk [3]. This group serves as a noteworthy illustration within the broader spectrum of automaton groups, highlighting the richness and diversity inherent in their structures.

A finitely produced torsion group and the first known example of a finitely generated group of intermediate growth are obtained from the Grigorchuk group, which also provides the simplest solution to the Burnside question. For more details and additional references, see [4] and [5].

The M- polynomial is defined by S. Klavzar or E. Deutsch in 2015. One important component in the monarchy of degree-oriented topological indices is the M-polynomial. It is the most widely used general progressive topological polynomial. Additionally, it has many potential applications is various field, which they can be found in [6-10].

The M-polynomial of the Schreier graphs associated with the actions of the two well-known automorphism groups of the binary rooted tree, Basilica group and Grigorchuk group is investigated in this paper. These groups can be associated with a compact limit space that is homeomorphic to the Basilica fractal since it can be described as an iterated monodromy group of the complex polynomial $z^2 – 1$. This is the first example of an amenable group that does not belong to a subexponentially pliable group [11]. This group has been demonstrated to have strong connections with complex dynamics and prophinite group theory [12]. The self-similarity value of some limit objects related to these groups reflects a fractal nature [8].

The primary conclusion of section 2 is the M-polynomial of the Basilica group's and Grigorchuk group's Schreier graphs, from which various degree-based topological indices are constructed. In Section 3, the selected topological are calculated and we make a comparison between the topological indices obtained from the M-polynomial and the estimated values of a few selected topological indices. Finally, the conclusion of this work is done in section 4.

2. Main Results

In this section the M-polynomial of Schreier graphs of the Basilica group is derived. Using that some degree based topological indices is derived.

2.1 The M-polynomial of Schreier graphs (without loops) of the Basilica group

The Basilica group is an automorphism collection that is self-similar and generated by the elements $a = e(b, id)$ and $b = \in (a, id)$ of the rooted binary tree. You may find the substitution rules in [13], which are used to generate the corresponding Schreier graphs iteratively. For n>1, let B_n be the notation for the Schreier graphs of the Basilica group, which are thought to be loop-free because the calculations for graphs with loops are trivial. Figure 1 display these graphs, as shown in [14].

Definition 2.1 Let $B_n = B_n$ (ɣ, Ȅ) be a connected graph where ɣ is the set of vertices and Ȅ is the set of edges of B_n. The M – Polynomial of graph B_n is defined as $f(w, z) = M_{poly}(B_n, \mathrm{w}, \mathrm{z}) = \sum_{\mathbb{I} \le i \le j \le \mathbb{J}} |Ȅ_{ij}(B_n)| w^i z^j$, where $\mathbb{I} = Min\{d(\mathcal{v}) / \mathcal{v} \epsilon ɣ(B_n)\}$; $\mathbb{J} = Max\{d(\mathcal{v}) / \mathcal{v} \epsilon ɣ(B_n)\}$, and $Ȅ_{ij}(B_n)$ is the edge $\mathcal{vu}\ \epsilon$ Ȅ for which $\{d(\mathcal{v}), d(\mathcal{u})\} = \{i, j\}$. Where $d(\mathcal{v})$ and $d(\mathcal{u})$ are degree of a vertices $\mathcal{v}$ and $\mathcal{u}$ respectively.

On the bases of growth of the Schreier graphs (without loops) of Basilica groups having edges partition $Ȅ_{ij} = \{\mathcal{vu}\ \epsilon\ Ȅ(Ǵ) / d(\mathcal{v}) = i, d(\mathcal{u}) = j\}$ is the set of all edges with end degrees $d(\mathcal{v}) = i, d(\mathcal{u}) = j)$. On the bases of degrees, we divide the edges into two partitions for $B_n, n > 1$.

$$\grave{\grave{E}}_{24} = \{\mathcal{v}\mathcal{u} \in \grave{\grave{E}}(\acute{G})/d(\mathcal{v}) = 2, d(\mathcal{u}) = 4\}$$

$$\grave{\grave{E}}_{44} = \{\mathcal{v}\mathcal{u} \in \grave{\grave{E}}(\acute{G})/d(\mathcal{v}) = 4, d(\mathcal{u}) = 4\}.$$

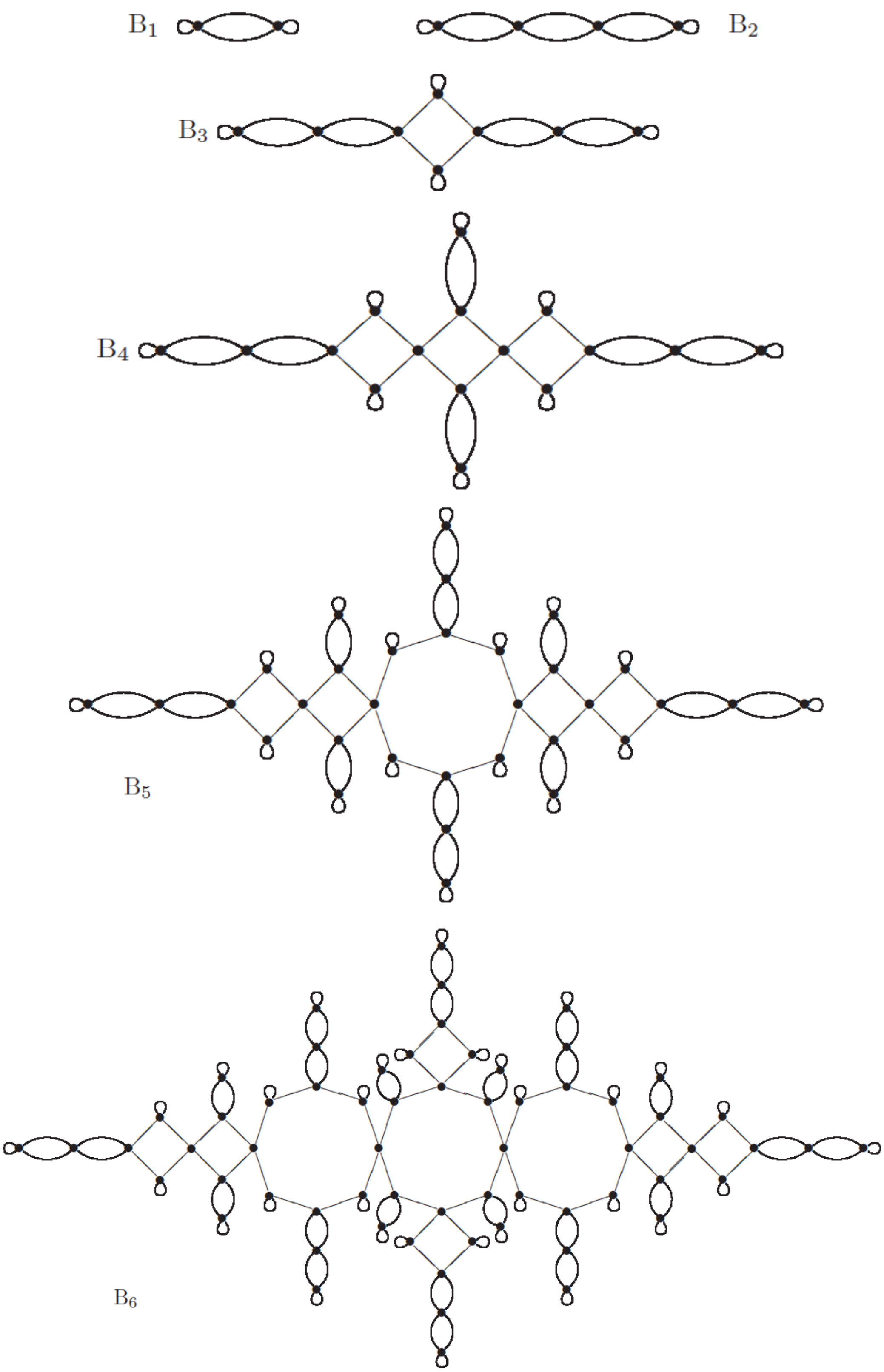

Fig. 1 Structure of B_n for some n

The following Table 1 depicts the selected topological indices for this study and its derivation formulas [6] through M-polynomial.

Table 1. Degree based M – Polynomial

Topological index	Degree based	Derivation
First Zagreb index (M_1)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} d(\upsilon)+d(u)$	$(D_w+D_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=z=1}$
Second Zagreb index (M_2)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} d(\upsilon)d(u)$	$(D_wD_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=z=1}$
Modified second Zagreb index (M_2^m)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} \frac{1}{d(\upsilon)d(u)}$	$(S_wS_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=z=1}$
Hyper Zagreb index (*HM*)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} [d(\upsilon)+d(u)]^2$	$(D_w+D_z)^2\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=z=1}$
Forgotten index (*F*)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} d(\upsilon)^2+d(u)^2$	$\left({D_w}^2+{D_z}^2\right)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=z=1}$
Inverse sum index (*I*)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} \frac{d(\upsilon)d(u)}{d(\upsilon)+d(u)}$	$(S_wJ\ D_wD_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=1}$
Augmented Zagreb index (A)	$\sum_{e=\upsilon u\in \grave{\grave{E}}} \left(\frac{d(\upsilon)d(u)}{d(\upsilon)+d(u)-2}\right)^3$	$\left(S_wQ_{-2}J\ {D_w}^3{D_z}^3\right)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big\vert_{w=1=11}$

Note that $D_w=w\frac{\partial}{\partial w}f(w,z)$, $D_z=z\frac{\partial}{\partial z}f(w,z)$, $S_w=\int_0^w\frac{f(t,z)}{t}dt$, $S_z=\int_0^z\frac{f(w,t)}{t}dt$,

$J\big(f(w,z)\big)=f(w,w)$,$Q_\propto\big(f(w,z)\big)=w^\propto f(w,z)$.

Theorem 2.1. Let B_n be Schreier graph (without loops) of Basilica group. Then the M – Polynomial of B_n is $f(w,z)=M_{poly}(B_n,\mathrm{w},\mathrm{z})=(2^n)w^2z^4+(2^{n-1})w^4z^4$.

Proof. From the following Table 2 we can understand the edge growth pattern and edge partition of Schreier graph of Basilica group.

Table 2. Edge Growth pattern and edge partition of Schreier graph of Basilica group

Basilica Group (B_n)	Number of vertices	Number of edges	Edge growth pattern	Edge Partitions	
				(2,4)	(4,4)
B2	4	6	$(2^2)+(2^1)$	4	2
B3	8	12	$(2^3)+(2^2)$	8	4
B4	16	24	$(2^4)+(2^3)$	16	8
B5	32	48	$(2^5)+(2^4)$	32	16
B6	64	96	$(2^6)+(2^5)$	64	32
.	.	.	.	.	.
.	.	.	.	.	.
Bn	2^n		$(2^n)+(2^{n-1})$	2^n	2^{n-1}

From Table 1, it is clear that the number of edges in the above two categories are $\left\vert\grave{\grave{E}}_{24}\right\vert=2^n$ and $\left\vert\grave{\grave{E}}_{44}\right\vert=2^{n-1}$.

Now, by the definition of M-polynomial, we have

$$f(w,z) = M_{poly}(B_n,\mathrm{w},\mathrm{z}) = \sum_{\mathbb{I}\le i\le j\le \mathbb{J}} |\text{Ȅ}_{ij}(B_n)| w^i z^j$$

$$= |\text{Ȅ}_{24}| w^2 z^4 + |\text{Ȅ}_{44}| w^4 z^4$$

$$\therefore f(w,z) = (2^n) w^2 z^4 + (2^{n-1}) w^4 z^4. \,\square$$

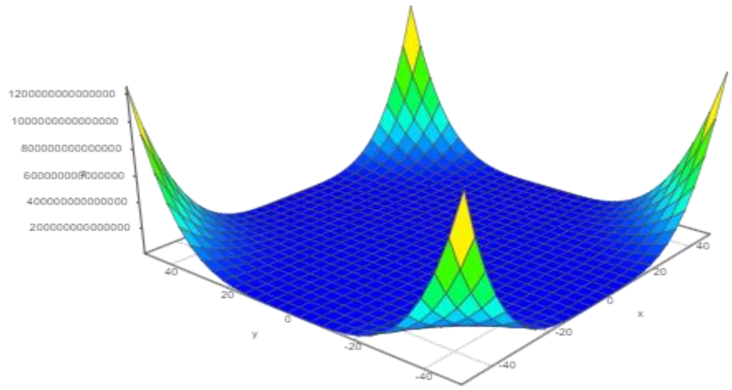

Fig. 2 Surface plot of M-polynomial of B_n.

Theorem 2.2 Let B_n be the Schreier graph of Basilica group. Then the M-polynomials for the topological indices of B_n for $n > 1$ are

a) $M_1(B_n) = 5 \times 2^{n+1}$

b) $M_2(B_n) = 2^{n+4}$

c) $M_2^m(B_n) = 5 \times 2^{n-5}$

d) $AZ(B_n) = 3 \times 2^{n+1}$

e) $HM(B_n) = 17 \times 2^{n+2}$

f) $F(B_n) = 3 \times 2^{n+3}$

g) $I(B_n) = \frac{7}{3} 2^n$.

Proof. For (a), $M_1(B_n) = (D_w + D_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big|_{w=z=1}$

$= |(2^n).2w^2z^4 + (2^n).4w^2z^4 + (2^{n-1})4w^4z^4 + (2^{n-1})4w^4z^4|_{w=z=1} = 5 \times 2^{n+1}$.

For (b), $M_2(B_n) = (D_w D_z)\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right)\Big|_{w=z=1}$

First

$D_z\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right) = (2^n)4w^2z^4 + (2^{n-1})4w^4z^4$

Now, $D_w D_z\left(M_{poly}(B_n,\mathrm{w},\mathrm{z})\right) = D_w[(2^n) \times 4 \times w^2z^4 + (2^{n-1}) \times 4 \times w^4z^4]$

$= (2^n) \times 4 \times 2 \times w^2z^4 + (2^{n-1}) \times 4 \times 4 \times w^4z^4 = 2^{n+4}$.

For (c), $M_2^m(B_n) = (S_w S_z)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=z=1}$, $S_w = \int_0^w \frac{f(t,z)}{t}dt = \int_0^w \frac{(2^n)t^2z^4+(2^{n-1})t^4z^4}{t}dt$

$\therefore S_w = (2^n)z^4\frac{w^2}{2} + (2^{n-1})z^4\frac{w^4}{4}$, $S_w S_z = \int_0^z \frac{f(w,t)}{t}dt = \int_0^z \frac{\left[(2^n)t^4\frac{w^2}{2}+(2^{n-1})t^4\frac{w^4}{4}\right]}{t}dt = (2^n)\frac{z^4}{4}\frac{w^2}{2} + \left(2^{n-1}\right)\frac{z^4}{4}\frac{w^4}{4}$

$$\therefore M_2^m(B_n) = (S_w S_z)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=z=1} = \frac{2^n}{8} + \frac{2^{n-1}}{16} = 2^{n-3} + 2^{n-5} = 5 \times 2^{n-5}$$

For (d), to find $\left({S_w}^3 Q_{-2} J\, {D_w}^3 {D_z}^3\right)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=1}$

$D_Z\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = (2^n)4w^2z^4 + (2^{n-1})4w^4z^4$, ${D_Z}^2\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 16 \times w^2z^4] + \left[\left(2^{n-1}\right) \times 16 \times w^4z^4\right]$, ${D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 64 \times w^2z^4] + \left[\left(2^{n-1}\right) \times 64 \times w^4z^4\right]$

$D_w {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 128 \times w^2z^4] + \left[\left(2^{n-1}\right) \times 256 \times w^4z^4\right]$

${D_w}^2 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 256 \times w^2z^4] + \left[\left(2^{n-1}\right) \times 1024 \times w^4z^4\right]$

${D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 512 \times w^2z^4] + \left[\left(2^{n-1}\right) \times 4096 \times w^4z^4\right]$

$J{D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [(2^n) \times 512 \times w^6] + \left[\left(2^{n-1}\right) \times 4096 \times w^8\right]$

$Q_{-2}J{D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = w^{-2}\left\{[(2^n) \times 512 \times w^6] + \left[\left(2^{n-1}\right) \times 4096 \times w^8\right]\right\}$

$S_w Q_{-2}J{D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = \int_0^w \left(\frac{[(2^n)\times 512\times t^6]+\left[\left(2^{n-1}\right)\times 4096\times t^8\right]}{t}\right)dt = (2^n) \times 512 \times \frac{w^6}{6} + \left(2^{n-1}\right) \times 4096 \times \frac{w^8}{8}$.

${S_w}^2 Q_{-2}J{D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = \int_0^w \frac{(2^n)\times 512\times\frac{t^6}{6}+\left(2^{n-1}\right)\times 4096\times\frac{t^8}{8}}{t}dt = (2^n) \times 512 \times \frac{w^6}{36} + \left(2^{n-1}\right) \times 4096 \times \frac{w^8}{64}$.

${S_w}^3 Q_{-2}J{D_w}^3 {D_Z}^3\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = \int_0^w \frac{(2^n)\times 512\times\frac{t^6}{36}+\left(2^{n-1}\right)\times 4096\times\frac{t^8}{64}}{t}dt = (2^n) \times 512 \times \frac{w^6}{216} + \left(2^{n-1}\right) \times 4096 \times \frac{w^8}{512}$.

$$\therefore AZ(B_n) = \left({S_w}^3 Q_{-2} J\, {D_w}^3 {D_z}^3\right)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=1} = [(2^n) \times 2] + \left[\left(2^{n-1}\right) \times 8\right] = 3 \times 2^{n+1}$$

For (e), to find $HM(B_n) = (D_w + D_z)^2\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=z=1}$, $(D_w + D_z)^2 = (D_w + D_z)(D_w + D_z)M_{poly}(B_n, \mathrm{w}, \mathrm{z}) = (D_w + D_z)\left\{(2^n).2w^2z^4 + (2^n).4w^2z^4 + (2^{n-1})4w^4z^4 + (2^{n-1})4w^4z^4\right\}$

$(D_w + D_z)^2 = \left|9w^2z^4 2^{n+2} + w^4z^4 2^{n+5}\right|_{w=1}$, $\therefore\ HM(B_n) = 17 \times 2^{n+2}$.

For (f)

To find $F(G) = \left({D_w}^2 + {D_z}^2\right)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=z=1}$

$D_w\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = D_w[(2^n)w^2z^4 + (2^{n-1})w^4z^4] = [2^{n+1} \times w^2z^4] + [2^{n+1} \times w^4z^4]$

${D_w}^2\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [2^{n+2} \times w^2z^4] + [2^{n+3} \times w^4z^4]$, $D_Z\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [2^{n+2} \times w^2z^4] + [2^{n+1} \times w^4z^4]$

$$D_Z{}^2\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [2^{n+4} \times w^2z^4] + [2^{n+3} \times w^4z^4]$$

$$\therefore \left({D_w}^2 + {D_z}^2\right)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = 3 \times 2^{n+3}$$

$$\therefore F(B_n) = 3 \times 2^{n+3}.$$

For (g)

To find $I(B_n) = (S_wJ\, D_wD_z)\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right)\Big|_{w=1}$

From (ii) $D_wD_z\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = (2^n) \times 4 \times 2 \times w^2z^4 + (2^{n-1}) \times 4 \times 4 \times w^4z^4$

$JD_wD_z\left(M_{poly}(B_n, \mathrm{w}, \mathrm{z})\right) = [8 \times 2^n w^6] + [16 \times 2^{n-1}w^8]$, $(S_wJ\, D_wD_z)\left(M_{poly}(\mathrm{Ɠ}, \mathrm{w}, \mathrm{z})\right) = \int_0^w \left[\frac{[8\times 2^n t^6]+[16\times 2^{n-1}t^8]}{t}\right] dt = \frac{8\times 2^n w^6}{6} + \frac{16\times 2^{n-1}w^8}{8}$

$$\therefore I(\mathrm{Ɠ}) = (S_wJ\, D_wD_z)\left(M_{poly}(\mathrm{Ɠ}, \mathrm{w}, \mathrm{z})\right)\Big|_{w=1} = \frac{7}{3}2^n. \square$$

2.2 The M-polynomial of Schreier graphs (with loops) of the Basilica group

In this section the Schreier graph of the Basilica group with loops considered and the M-polynomial for the graph is derived in the following Theorem 2.3. The surface plot of this M-polynomial can be seen in Figure 4. Moreover, by the use of this M-polynomial the chosen topological indices are derived in Theorem 2.4.

Theorem 2.3. Let ${B_n}^*$ be Schreier graph (with loops) of Basilica group. Then the M – Polynomial of ${B_n}^*$ is $f(w, z) = M_{poly}({B_n}^*, \mathrm{w}, \mathrm{z}) = (2^{n+1})w^4z^4$.

Proof. From Figure 1, degree of each vertex in ${B_n}^*$ is four if we consider the loops. Hence the Schreier graph of Basilica group is a 4- regular graph. So, the number vertices in the graph are 2^n and the number of edges is 2^{n+1}.

That is, $\left|V({B_n}^*)\right| = 2^n$, $|E({B_n}^*)| = 2^{n+1}$.

Since ${B_n}^*$ is a 4-regular graph, there will be only one edge partition which is (4, 4) and therefore $\left|\grave{\ddot{E}}_{44}\right| = 2^{n+1}$

Therefore, we have by the definition 2.1,

$f({B_n}^*, w, z) = M_{poly}({B_n}^*, \mathrm{w}, \mathrm{z}) = \sum_{\mathbb{I}\le i\le j\le \mathbb{J}} \left|\grave{\ddot{E}}_{ij}({B_n}^*)\right| w^iz^j = \left|\grave{\ddot{E}}_{44}\right| w^4z^4$

$$\therefore f(B_n{}^*, w, z) = (2^{n+1})w^4z^4.$$

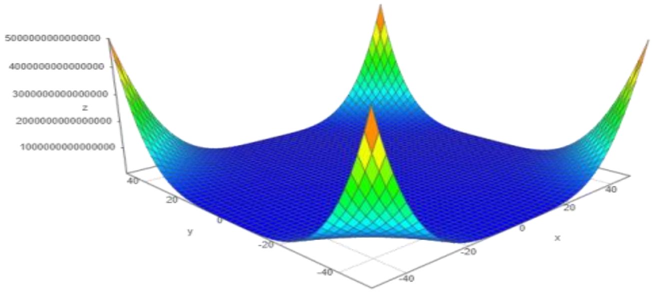

Fig. 3 Surface plot of M-polynomial of $B_n{}^*$

Theorem 2.4 Let $B_n{}^*$ be the Schreier graph (including loops) of Basilica group. Then the topological indices of $B_n{}^*$, $n > 1$ using the M-polynomial are

a) $M_1(B_n{}^*) = 2^{n+4}$

b) $M_2(B_n{}^*) = 2^{n+5}$

c) $M_2^m(B_n{}^*) = 2^{n-3}$

d) $AZ(B_n{}^*) = 2^{n+4}$

e) $HM(B_n{}^*) = 2^{n+7}$

f) $F(B_n{}^*) = 2^{n+6}$

g) $I(B_n{}^*) = 2^{n+2}$.

Proof. These results can be proved using Theorem 2.2. □

2.3 The M-polynomial of Schreier graphs (without loops) of the Grigorchuk group

Let Γ_n be the Schreier graph of the Grigorchuk group. In finding the M-polynomial, this graph is considered as unlabelled and without loops. The graph Γ_n for $n > 1$ can be seen in the following Figure 4.

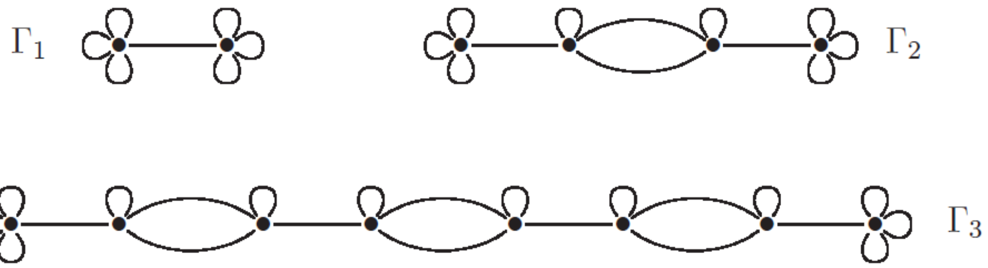

Figure 4. Structure of Γ_n for $n = 1,2,3$.

Theorem 2.5. Let Γ_n be Schreier graph (without loops) of Grigorchuk group. Then the M – polynomial of Γ_n is $f(\Gamma_n, w, z) = M_{poly}(\Gamma_n, w, z) = 2w^1z^3 + (2^n + 2^{n-1} - 4)w^3z^3$.

Proof. From the following Table 3, one can understand the edge growth pattern and edge partition of Schreier graph of Basilica group.

Table 3. Edge Growth pattern and edge partition of Schreier graph of Grigorchuk group

Grigorchuk group	No. of vertex (x)	No of Edges(y)	Edge growth pattern	Edge Partitions	
				(1,3)	(3,3)
Γ_2	4	4	$(2^1)+(2^1)$	2	2
Γ_3	8	10	(2^1)+ $(2)^3$	2	8
Γ_4	16	22	$(2^1)+(2^4+4)$	2	20
Γ_5	32	46	$(2^1)+(2^5+12)$	2	44
Γ_6	64	94	$(2^1)+(2^6+28)$	2	92
.	.	.	.	.	.
.	.	.	.	.	.
Γ_n	2^n	$3.2^{n-1}-2$	$(2^1)+2^n+(2^{n-1}-4)$	2^1	$2^n+2^{n-1}-4$

From Table 3, it is clear that the number of edges in the above two category are $|\grave{\ddot{E}}_{13}|=2$ and $|\grave{\ddot{E}}_{33}|=2^n+2^{n-1}-4$.

Now, by the definition 2.1, we have

$$f(\Gamma_n,w,z)=M_{poly}(\Gamma_n,\mathrm{w},\mathrm{z})=\sum_{\mathbb{I}\le i\le j\le \mathbb{J}}|\grave{\ddot{E}}_{ij}(\Gamma_n)|w^iz^j$$

$$=|\grave{\ddot{E}}_{13}|w^1z^3+|\grave{\ddot{E}}_{33}|w^3z^3$$

$$\therefore f(\Gamma_n,w,z)=2w^1z^3+(2^n+2^{n-1}-4)w^3z^3.\ \square$$

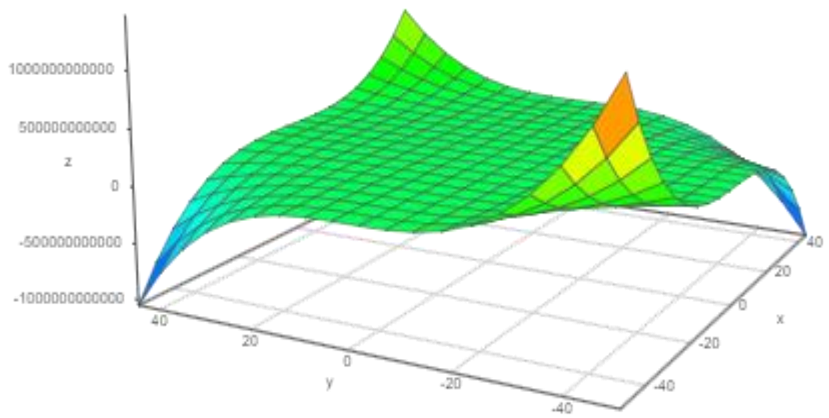

Figure 5. Surface plot of M-polynomial of Γ_n.

Theorem 2.6 Let Γ_n be the Schreier graph of Grigorchuk group. Then the M-polynomials for the topological indices of Γ_n , $n>1$ are

a) $M_1(\Gamma_n)$= $6\,(2^n+2^{n-1})-16$.

b) $M_2(\Gamma_n)=9(2^n+2^{n-1})-30$.

c) $M_2^m(\Gamma_n) = \frac{2^n+2^{n-1}+2}{9}$.

d) $AZ(\Gamma_n) = \frac{729(2^n+2^{n-1})-2484}{64}$.

e) $HM(\Gamma_n) = 36(2^n + 2^{n-1}) - 112$.

f) $F(\Gamma_n) = 18(2^n + 2^{n-1}) - 52$.

g) $I(\Gamma_n) = \frac{3(2^n+2^{n-1})-9}{2}$.

Proof. It can be verified using Theorem 2.2. □

2.4 The M-polynomial of Schreier graphs (with loops) of the Grigorchuk group

Let ${\Gamma_n}^*$ be the Schreier graph (including loops) of the Grigorchuk group. In finding the M-polynomial, this graph is considered as unlabelled and with loops. From Figure 4, it is clear that the number of vertices $|V({\Gamma_n}^*)| = 2^n$ and $|E({\Gamma_n}^*)| = 5.2^{n-1} + 2$ when we consider the loops.

Theorem 2.7. Let ${\Gamma_n}^*$ be Schreier graph (with loops) of Grigorchuk group. Then the M – polynomial of ${\Gamma_n}^*$ is $f({\Gamma_n}^*, w, z) = M_{poly}({\Gamma_n}^*, \mathrm{w}, \mathrm{z}) = 2w^5z^7 + 6w^7z^7 + (5.2^{n-1} - 6)w^5z^5$.

Proof. From the Figure 4, it is clear that there are three different categories of edge partitions of Schreier graph ${\Gamma_n}^*$ of Grigorchuk group and they are

$|\grave{\grave{E}}_{57}| = 2$, $|\grave{\grave{E}}_{77}| = 6$ and $|\grave{\grave{E}}_{55}| = 5.2^{n-1} - 6$.

Now, by the definition of 2.1, we have

$$f({\Gamma_n}^*, w, z) = M_{poly}({\Gamma_n}^*, \mathrm{w}, \mathrm{z}) = \sum_{\mathbb{I} \le i \le j \le \mathbb{J}} |\grave{\grave{E}}_{ij}({\Gamma_n}^*)| w^i z^j$$

$$= |\grave{\grave{E}}_{57}| w^5 z^7 + |\grave{\grave{E}}_{55}| w^5 z^5 + |\grave{\grave{E}}_{77}| w^7 z^7$$

$$\therefore f({\Gamma_n}^*, w, z) = 2w^5z^7 + 6w^7z^7 + (5.2^{n-1} - 6)w^5z^5. \ \square$$

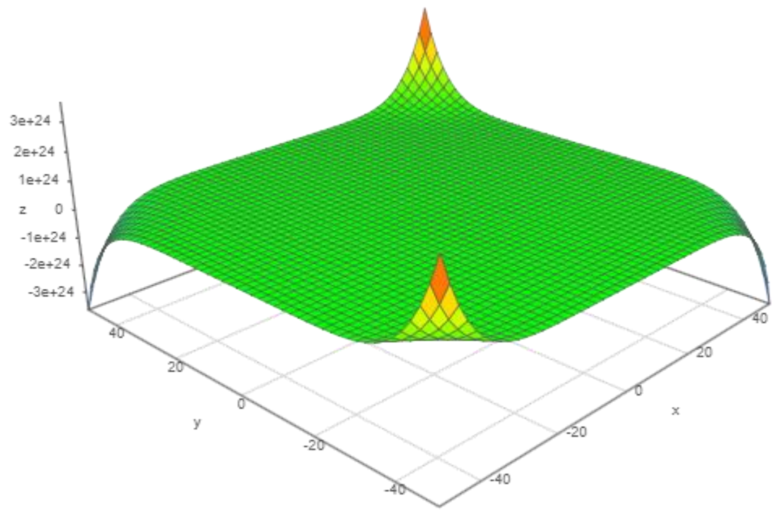

Figure 6. 3-D graph of M-polynomial of ${\Gamma_n}^*$.

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Theorem 2.8 Let ${\Gamma_n}^*$ be the Schreier graph (including loops) of Grigorchuk group. Then the topological indices of ${\Gamma_n}^*$, $n > 1$ are

a) $M_1({\Gamma_n}^*) = (25 \times 2^n) - 62$.

b) $M_2({\Gamma_n}^*) = (125 \times 2^{n-1}) + 214$.

c) $M_2^m({\Gamma_n}^*) = \frac{2^n}{10} - 0.0605$.

d) $AZ({\Gamma_n}^*) = (152.5875 \times 2^{n-1}) + 311.148$.

e) $HM({\Gamma_n}^*) = (250 \times 2^n) + 864$.

f) $F({\Gamma_n}^*) = (125 \times 2^n) + 436$.

g) $I({\Gamma_n}^*) = (12.5 \times 2^{n-1}) + 864$.

Proof. It can be verified using Theorem 2.2. □

3. Calculations and Comparative analysis

In this section the values of the selected topological indices calculated by using the results in the previous sections and comparative study done based on the determined results. Table 4 presents calculated values of various topological indices for B_n. These indices provide various measures of the structure and properties of B_n at different levels.

For example, $M_1(B_n)$ and $M_2(B_n)$ provide basic information about the size of the graph and its connectivity, while indices like $AZ(B_n)$, $HM(B_n)$, and $I(B_n)$ give understandings into more specific properties related to vertex independence, distances, and autonomy.

Table 4. Calculated values of topological indices of B_n.

Topological Index	**Basilica group Bn**					
		B_2	B_3	B_4	B_5	B_6
$M_1(B_n)$		40	80	160	320	640
$M_2(B_n)$		64	128	256	512	1024
$M_2^m(B_n)$		0.62	1.25	2.5	5	10
$AZ(B_n)$		24	48	96	192	384
$HM(B_n)$		272	544	1088	2176	4352
$F(B_n)$		96	192	384	768	1536
$I(B_n)$		9.33	18.67	37.33	74.67	149.33

When we display these numbers on a line graph (see Figure 7), we see that when the level of the Basilica group rises, all of the indices exhibit consistent patterns of exponential growth. The graph's rising complexity is reflected in the indices $M_1(B_n)$,$M_2(B_n)$ and $F(B_n)$, which grow quickly as the number of vertices and edges increases exponentially. Similarly, indices like $AZ(B_n)$ and $HM(B_n)$ also display significant growth, though at a slightly slower pace, indicating the increasing autonomy and harmonic complexity of the graph structure. Additionally, the indices $M_2^m(B_n)$ and $I(B_n)$ follow the general trend of exponential growth but with smaller numerical values. Overall, the line chart illustrates the dynamic growth and complexity of the Basilica group's Schreier graph across different topological indices, highlighting the convoluted interplay between vertices, edges, distances, and autonomy within the graph structure as the group level increases.

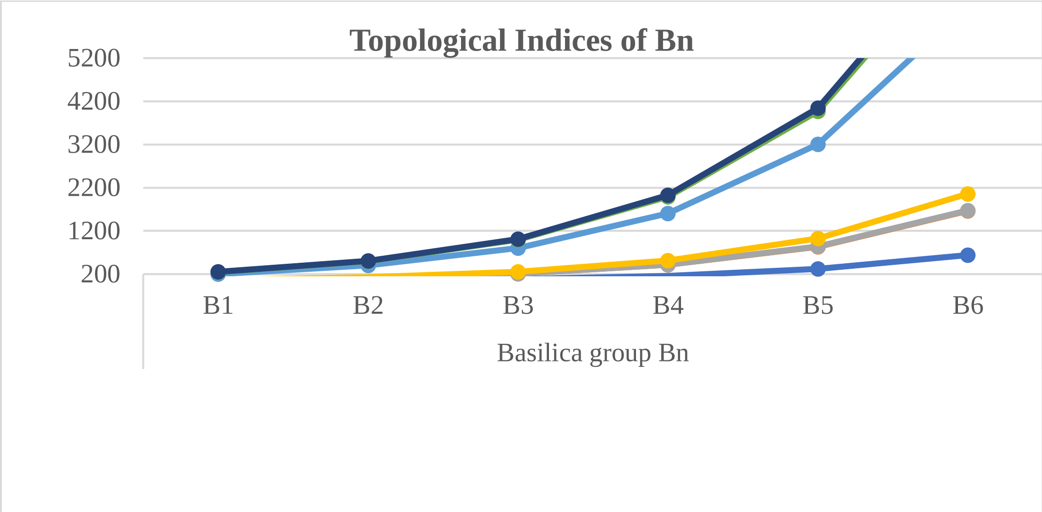

Fig. 7 Line chart for topological index value of Bn.

Table 5. Calculated values of topological indices of Bn*

Topological Index	Basilica group Bn*					
		B2*	B3*	B4*	B5*	B6*
$M_1(B_n{}^*)$		64	128	256	512	1024
$M_2(B_n{}^*)$		128	256	512	1024	2048
$M_2^m(B_n{}^*)$		0.5	1	2	4	8
$AZ(B_n{}^*)$		64	128	256	512	1024
$HM(B_n{}^*)$		512	1024	2048	4096	8192
$F(B_n{}^*)$		256	512	1024	2048	4096
$I(B_n{}^*)$		16	32	64	128	256

Table 5 presents the computed values of various topological indices for the Schreier graph with loops in the Basilica group at different levels labelled as $B_1{}^*$ to $B_6{}^*$. A thorough understanding of the structural complexity of the graph at various group levels is offered by these indices. Analysis reveals that most indicators show a consistent exponential development trend as the group level increases. This can be seen in Figure 8. Interestingly, indices such as $M_1(B_n{}^*)$, $M_2(B_n{}^*)$, $AZ(B_n{}^*)$, and $F(B_n{}^*)$ show strong increases, reflecting the graph's growing vertex and edge network. In addition, the Hyper Zagreb index $HM(B_n{}^*)$ shows significant increase, indicating increased harmonic complexity in the graph structure. Interestingly, however, the indices $I(B_n{}^*)$ and $M_2^m(B_n{}^*)$ exhibit exponential development patterns with much smaller numerical values.

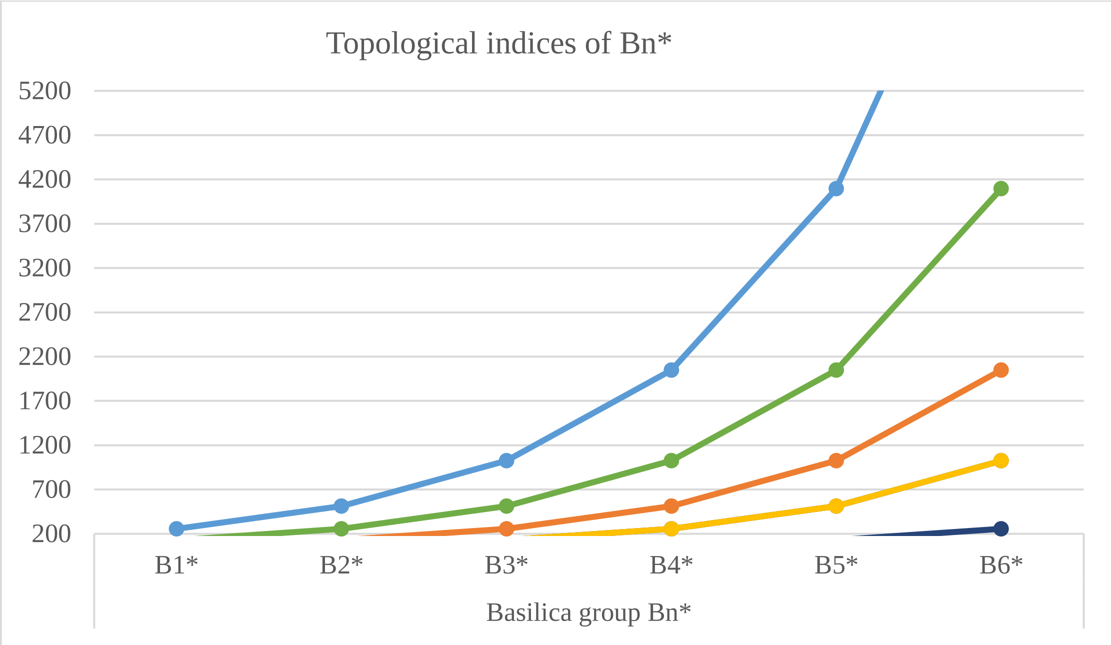

Fig. 8 Line chart for calculated values of topological indices of $B_n{}^*$

The following Table 6 provides calculated values of various topological indices for Schreier graph of the Grigorchuk group at different levels.

Table 6. Calculated values of topological indices of Γ_n

Topological Index	Grigorchuk group				
	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
$M_1(\Gamma_n)$	20	56	128	272	560
$M_2(\Gamma_n)$	24	78	186	402	834
$M_2^m(\Gamma_n)$	0.8889	1.6	2.88889	5.55556	10.8889
$AZ(\Gamma_n)$	29.531	98	234.563	507.938	1054.69
$HM(\Gamma_n)$	104	320	752	1616	3344
$F(\Gamma_n)$	-4	44	140	332	716
$I(\Gamma_n)$	4.5	14	31.5	67.5	139.5

A 3- D graph (see Figure 9) representing these patterns would give a clear picture of how each topological index varies with the Grigorchuk group level, revealing details about the structural characteristics and complexity of the related graph at various group sizes.

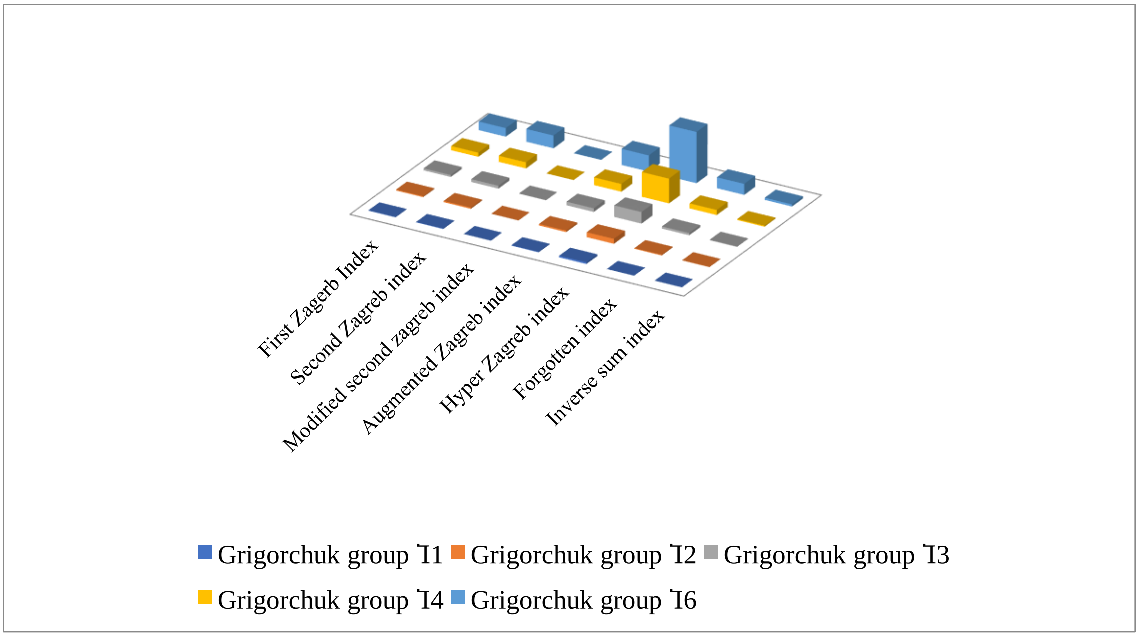

Fig. 9 3-D plot of topological indices of Γ_n.

Table 7 shows the determined values of several topological indices at different levels (represented by $\Gamma_2{}^*$ to $\Gamma_6{}^*$) for the Schreier graph of the Grigorchuk group, including loops. There are distinct trends among the indices: when the group level rises, the indices $M_1(\Gamma_n)$ and $M_2(\Gamma_n)$ show steady growth, signifying an increase in vertices and edges in the graph. Although its numerical values are less, the Modified Second Zagreb index displays a similar tendency. Especially, the Hyper and Augmented Zagreb indices show significant increases, indicating more intricacy and connection in the network structure. The Inverse Sum index gradually increases across several group levels, whereas the forgotten index consistently remains positive.

Table 7. Calculated values of topological indices of $\Gamma_n{}^*$

Topological Index	**Grigorchuk group**				
	$\Gamma_2{}^*$	$\Gamma_3{}^*$	$\Gamma_4{}^*$	$\Gamma_5{}^*$	$\Gamma_6{}^*$
$M_1(\Gamma_n{}^*)$	38	138	338	738	1538
$M_2(\Gamma_n{}^*)$	464	714	1214	2214	4214
$M_2^m(\Gamma_n{}^*)$	0.3395	0.7395	1.5395	3.1395	6.3395
$AZ(\Gamma_n{}^*)$	616.32	921.498	1531.848	2752.548	5193.948
$HM(\Gamma_n{}^*)$	1864	2864	4864	8864	16864
$F(\Gamma_n{}^*)$	936	1436	2436	4436	8436
$I(\Gamma_n{}^*)$	889	914	964	1064	1264

The graphical representation of the calculated values for the topological indices of $\Gamma_n{}^*$is presented in the Figure 10. In this plot, each axis represent a different aspect: one axis would represent the group level ($\Gamma_2{}^*$ to $\Gamma_6{}^*$), another axis would represent the specific topological index being measured, and the third axis would represent the corresponding calculated value for that index. The plot would consist of multiple lines or surfaces, each representing a different topological index, with points along each line or surface indicating the calculated values at different group levels.

This 3D visualization would provide a comprehensive depiction of how each topological index varies with both the group level and the specific measurement being considered, offering a clear and intuitive understanding of the evolving characteristics of $\Gamma_n{}^*$.

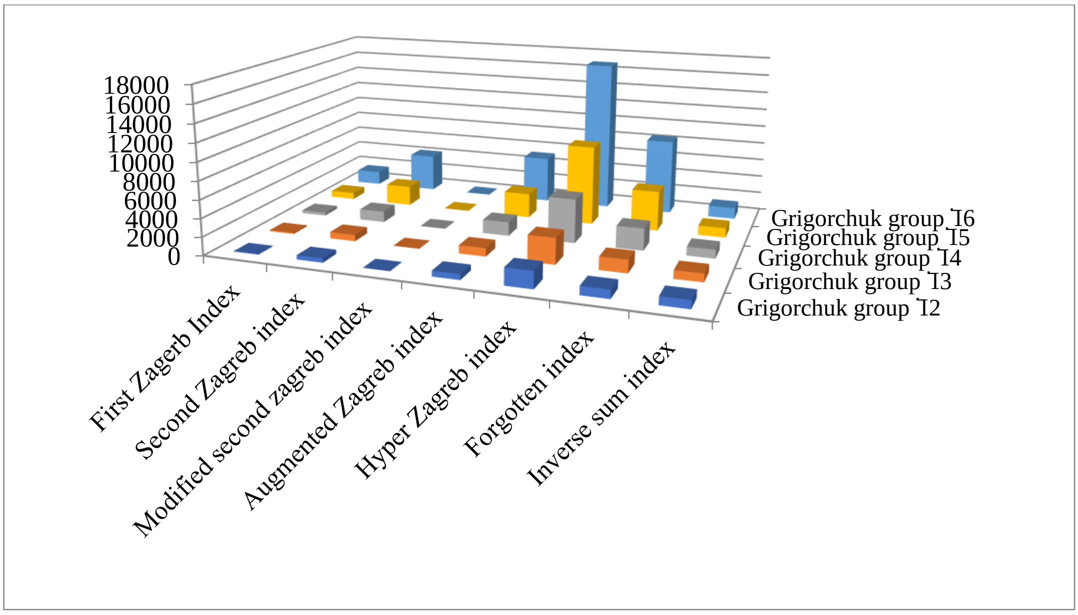

Fig. 10 Calculated values of topological indices of $\Gamma_n{}^*$.

4. Conclusion

The study explores into the characterization of self-similar groups through Schreier graphs, particularly focusing on the Basilica and Grigorchuk groups. By analyzing the M-polynomial of these graphs, the paper explores its implications on various topological indices, providing understandings into the connectivity and symmetry properties of these groups. Similarly, the calculated values of these indices for the Grigorchuk group Γn and its variant Γn*, offering a comparative analysis between the two groups.

From the calculated values, it is evident that there are discernible patterns in the values of the topological indices across different iterations (n) of both groups. For instance, the First Zagreb index generally exhibits exponential growth with increasing n for both Basilica and Grigorchuk groups, reflecting their complex structural characteristics.

Moreover, comparing the indices of the original groups with their variants (Bn vs. Bn* and Γn vs. Γn*), it is seeming that certain transformations or modifications lead to significant alterations in the topological properties, as evidenced by changes in index values. This analysis provides valuable perceptions into the topological characteristics of the Basilica and Grigorchuk groups, illuminating on their complex structures and behaviors. Such investigations contribute to a deeper understanding of self-similar groups and their geometric representations, with potential implications across various domains, including group theory, topology, and computational mathematics.

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