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Exponential B-Spline Method for Second Order Singularly Perturbed Boundary Value Problem with Negative Shift

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Abstract:

In this paper, we present a numerical scheme based on exponential B-spline method for the solution of singularly perturbed delay differential equation.The equation is transformed to singularly perturbed boundary value problem by applying Taylor's series expansion. A three term recurrence relation is obtained and invariant embedded algorithm is applied to get the approximate solution .The convergence of the proposed scheme is discussed. The efficiency of the scheme is illustrated by presenting different examples .The results are compared with available literature and comparatively the method yields better results.
Keywords: Exponential B-spline, Negative shift, Invariant embedded algorithm, Tridiagonal system, Boundary layer.

1. Introduction

The numerical treatment of a singularly perturbed delay differential equations generated lots of interest in the recent years due to applicability these families of equations in the process of transforming a real life situation into a mathematical form for numerous disciplines of science and technology. An ordinary differential equation having a delay term and multiplied the highest order derivative by a small positive parameter , known as a singularly perturbed delay differential equation. Like in [8] Lange, C.G. and Miura, R.M. had studied initial exit time problem with the modelling of neuronal variability's activation. In [1] Derstine, M.W studied problems related to variational and bistable devices problem in control theory. Kadalbajoo et al. [3, 4] proposed some numerical approximations for a singularly perturbed differential-difference equation containing a delay term. Various Numerical approaches were proposed for differential equations with negative shifts, Mixed shifts and some schemes with fitted parameters by D.Kumara Swamy et al [7,13, 14,15,19].An integration method based on numerical approach was proposed by Y.N Reddy et al.[12] for a singularly perturbed differential equation having a negative shift. McCartin introduced the concept of Exponential B spline[9].He also described convergent rates and extremal features of the exponential spline approximation and also developed cardinal bases and B-spline bases for the space of exponential splines. Reza Mohammadi[10] proposed a method which is based on exponential B-spline to approximate the soluton of partial differential equation of convection-diffusion kind having boundary conditions of Dirichlet’s type. Von Neumann method was used to prove the stability of the method. A numerical scheme for a family of reaction-diffusion equations was proposed by A. S. V. Ravi Kanth et al.[6] based on Exponential B-spline for space derivative. In recent years, studies like wavelets theory, medical imaging, and image processing have all greatly benefited from the application of exponential splines. sinuk kang[5],the author came up with a technique for the cardinal exponential splines's space by using exponential B-spline as a Reisz basis.

W. K. Zahra et al. [18] developed a method that attempted to solve a singularly perturbed boundary value problem with a small unknown perturbation parameter using exponential splines and shishkin mesh discretization .Chandra Sekhara Rao , Mukesh Kumar[11] proposed a method for a self adjoint singularly perturbed boundary value problem based on the Exponential B-spline collocation approach.

We have implemented Exponential B-spline approach to approximate the solution of a differential equation of second order with negative shift having a boundary with a layer structure.We have discussed the problem in section 2 of this paper. The numerical method have been developed in Section 3 and the convergence analysis was performed in Section 4. The effectiveness of the proposed scheme have been covered in Section 5.

2. Objectives

The main objective of this paper is to implement Exponential B-spline method on a singularly perturbed delay differential equation to approximate its solution. We discuss the convergence analysis of the proposed method.

We consider the following linear differential equation of second order with negative shift to implement the proposed method,

$$\varepsilon u''(t) + m(t)u'(t-\delta) + n(t)u(t) = 0\,, \quad 0 \le t \le 1, \tag{2.1}$$

with imposed boundary conditions,

$$u(t) = \varphi\ ,\ -\delta \le t \le 0 \ \text{ and } \ u(1) = \psi \tag{2.2}$$

where ε and δ are perturbation parameter and delay argument respectively such that,

$$0 < \varepsilon \ll 1\,,\, 0 < \delta < 1 \ \text{ and } \delta = o(\varepsilon).$$

Furthermore, $m(t)$ and $n(t)$ are functions such that they are c^{∞} in the open interval (0,1) and φ , ψ are constants.

Let us assume $m(t) \ge \mathrm{n} > 0$ throughout the interval [0, 1],where, n is a positive constant .The boundary layer will be in the neighborhood of t = 0.Again assuming $m(t) \le \mathrm{n} < 0$ throughout the interval [0, 1], where, n is a negative constant.The boundary layer will be in the neighborhood of t = 1.

Taylor's series expansion yields,

$$u'(t-\delta) \approx u'(t) - \delta u''(t) \tag{2.3}$$

Using equation (2.3) in equation (2.1),

$$-\varepsilon u''(t) + f(t)u'(t) + g(t)u(t) = 0 \tag{2.4}$$

Where,

$$f(t) = \frac{m(t)}{\sigma m(t)-1}\ , g(t) = \frac{n(t)}{\sigma m(t)-1}\ ,\ \sigma = \frac{\delta}{\varepsilon}$$

With reference to Elsgolt's and Norkin [2] equation (2.4) thus obtained from (2.1) is valid since,

$0 < \delta < 1$

3. Methods

Let us consider the equation (2.4),

$$Lu \equiv -\varepsilon u''(t) + f(t)u'(t) + g(t)u(t) = 0 \ , \ t \in [0,1] \quad (3.1)$$

Let $\Omega: 0 = t_0 < t_1 < \cdots\cdots < t_n = 1$ be partition on [0,1] with uniform step size $h = \frac{1}{n}$

Define the Exponential B-spline function $\mathcal{B}_j(t)$ **,** defined as follows [9]n

$$\mathcal{B}_j(t) = \begin{cases} k_5\left[(t_{j-2} - t) - \frac{1}{p}\sinh\{p(t_{j-2} - t)\}\right] \ , & t \in [t_{j-2}, t_{j-1}] \ , \\ k_1 + k_2(t_j - t) + k_3 e^{p(t_j - t)} + k_4 e^{-p(t_j - t)} \ , & t \in [t_{j-1}, t_j] \\ k_1 + k_2(t - t_j) + k_3 e^{p(t - t_j)} + k_4 e^{-p(t - t_j)} \ , & t \in [t_j, t_{j+1}], \\ k_5\left[(t - t_{j+2}) - \frac{1}{p}\sinh\{p(t - t_{j+2})\}\right] \ , & t \in [t_{j-2}, t_{j-1}], \\ 0 & \text{otherwise} \end{cases}$$

Where ,

$$k_1 = \frac{phc}{phc - s},$$

$$k_2 = \frac{p}{2}\left[\frac{c(c-1)+s^2}{(phc-s)(1-c)}\right],$$

$$k_3 = \frac{1}{4}\left[\frac{e^{-ph}(1-c)+s(e^{-ph}-1)}{(phc-s)(1-c)}\right],$$

$$k_4 = \frac{1}{4}\left[\frac{e^{ph}(c-1)+s(e^{ph}-1)}{(phc-s)(1-c)}\right],$$

$$k_5 = \frac{p}{2(phc-s)}$$

where $s = \sinh(ph)$, $c = \cosh(ph)$ and p is a non-negative parameter.

Let $\mathcal{U}(t)$ approximate the exact solution $u(t)$ of the equation (3.1),then we have,

$$\mathcal{U}(t) = \sum_{j=-1}^{n+1} \gamma_j \mathcal{B}_j(t) \qquad \text{[Mac-Cartin 1991]} \qquad (3.2)$$

Now by using the conditions (3.2) the values of the unknowns γ_j can be found.

The values of the $\mathcal{U}(t)$ and its 1st order and 2nd order derivatives can be determined at the mesh points t_j as follows:

$$\mathcal{U}(t_j) = \gamma_{j-1}\mathcal{B}_{j-1}(t_j) + \gamma_j\mathcal{B}_j(t_j) + \gamma_{j+1}\mathcal{B}_{j+1}(t_j)$$

$$\mathcal{U}(t_j) = \frac{s-ph}{2(phc-s)}\gamma_{j-1} + \gamma_j + \frac{s-ph}{2(phc-s)}\gamma_{j+1} \qquad (3.3)$$

$$\mathcal{U}'(t_j) = \frac{p(1-c)}{2(phc-s)}\gamma_{j-1} - \frac{p(1-c)}{2(phc-s)}\gamma_{j+1} \qquad (3.4)$$

$$\mathcal{U}''(\mathfrak{t}_j) = \frac{\mathfrak{p}^2 s}{2(\mathfrak{p}\hbar c - s)}[\Upsilon_{j-1} - 2\Upsilon_j + \Upsilon_{j+1}], \quad 0 \le j \le n \tag{3.5}$$

Now, the approximation of the equation (3.1) at the mesh points can be described as

$$-\varepsilon\,\mathcal{U}''(\mathfrak{t}_j) + f(\mathfrak{t}_j)\,\mathcal{U}'(\mathfrak{t}_j) + g(\mathfrak{t}_j)\mathcal{U}(\mathfrak{t}_j) = 0\,,\ \mathfrak{t}_j \in [0,1]$$

$$-\varepsilon\,\mathcal{U}''(\mathfrak{t}_j) + f_j\,\mathcal{U}'(\mathfrak{t}_j) + g_j\mathcal{U}(\mathfrak{t}_j) = 0\ , \quad \text{where } f(\mathfrak{t}_j) = f_j,\ g(\mathfrak{t}_j) = g_j \tag{3.6}$$

Plugging (3.3), (3.4) and (3.5) into (3.6) we get

$$\frac{-\varepsilon\mathfrak{p}^2 s}{2(\mathfrak{p}\hbar c - s)}\left[\Upsilon_{j-1} - 2\Upsilon_j + \Upsilon_{j+1}\right] + f_j\left[\frac{\mathfrak{p}(1-c)}{2(\mathfrak{p}\hbar c - s)}\Upsilon_{j-1} - \frac{\mathfrak{p}(1-c)}{2(\mathfrak{p}\hbar c - s)}\Upsilon_{j+1}\right] + g_j\left[\frac{s-\mathfrak{p}\hbar}{2(\mathfrak{p}\hbar c - s)}\Upsilon_{j-1} + \Upsilon_j + \frac{s-\mathfrak{p}\hbar}{2(\mathfrak{p}\hbar c - s)}\Upsilon_{j+1}\right] = 0$$

By performing mathematical maneuver we obtain a three term recurrence relation as follows:

$$\mathcal{K}_j\Upsilon_{j-1} + \mathfrak{L}_j\Upsilon_j + \mathrm{M}_j\Upsilon_{j+1} = R_j \tag{3.7}$$

Where,

$$\mathcal{K}_j = -\varepsilon\mathfrak{p}^2 s + f_j\mathfrak{p}(1-c) + g_j(s - \mathfrak{p}\hbar)$$

$$\mathfrak{L}_j = 2\varepsilon\mathfrak{p}^2 s + 2g_j(\mathfrak{p}\hbar c - s)$$

$$\mathrm{M}_j = -\varepsilon\mathfrak{p}^2 s - f_j\mathfrak{p}(1-c) + g_j(s - \mathfrak{p}\hbar)$$

$$R_j = 0$$

System (3.7) consists of $(n+1)$ equations with $(n+3)$ unknowns, say

$\Upsilon_{-1}, \Upsilon_0, \Upsilon_1 \ldots\ldots\ldots..\Upsilon_{n+1}$. Plugging the boundary conditions (2.2) in order to solve the system (3.7), we get two more additional equations,

$$\begin{cases} \Upsilon_{-1} = \dfrac{2(\mathfrak{p}\hbar c - s)}{s-\mathfrak{p}\hbar}(\varphi - \Upsilon_0) - \Upsilon_1 \\ \Upsilon_{n+1} = \dfrac{2(\mathfrak{p}\hbar c - s)}{s-\mathfrak{p}\hbar}(\psi - \Upsilon_n) - \Upsilon_{n-1} \end{cases} \tag{3.8}$$

Now (3.8) can be used to eliminate Υ_{-1} and Υ_{n+1} from (3.7) which yields a tridiagonal system in the unknowns $\Upsilon_0, \Upsilon_1 \ldots\ldots\ldots..\Upsilon_n$ of the form,

$$\mathcal{A}\Upsilon = \mathfrak{D} \tag{3.9}$$

Where,

$$\Upsilon = (\Upsilon_0, \Upsilon_1 \ldots\ldots\ldots..\Upsilon_n)'$$

$$\mathfrak{D} = (-2(\mathfrak{p}\hbar c - s)\mathcal{K}_0\varphi\,, 0, 0, \ldots\ldots\ldots\ldots., -2(\mathfrak{p}\hbar c - s)\mathrm{M}_n\psi)'$$

$$\mathcal{A}=\begin{pmatrix} \mathfrak{L}_0(s-\wp\hbar)-2(\wp\hbar c-s)\mathcal{K}_0 & (M_0-\mathcal{K}_0)(s-\wp\hbar) & 0 \\ \mathcal{K}_1 & \mathfrak{L}_1 & M_1 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ & (\mathcal{K}_n-M_n)(s-\wp\hbar) & \mathfrak{L}_n(s-\wp\hbar)-2(\wp\hbar c-s)M_n \end{pmatrix}$$

The tridiagonal system (3.7) can be solved using invariant embedded algorithm.

4. Results

Convergence Analysis: The followings results are required to discuss the convergent analysis of the proposed scheme,

Lemma 1. If $\{\mathcal{B}_{-1},\mathcal{B}_0,\ldots\ldots\ldots\ldots\ldots\ldots.\mathcal{B}_{n+1}\}$ is exponential spline basis, then see.[11]
$\sum_{j=-1}^{n+1}|\mathcal{B}_j(t)|\le\frac{5}{2}$, $0\le t\le 1$

Theorem 1 . Let exact solution $u(t)$ of the problem (2.1) can be interpolated using exponential B-spline to a unique $\mathcal{U}(t)$. And if $u(t)\in c^4[0,n]$ and $f,g\in c^2[0,n]$, then $\exists$ a constant $\varkappa_j$, independent of $\hbar$ such that see.[11]

$$\|D^j(u(t)-\mathcal{U}(t))\|_\infty\le\varkappa_j\hbar^{j-1} \quad , \quad j=0,1,2\ldots.$$

Theorem 2. Let exact solution $u(t)$ of the problem (3.1) can be approximated using exponential B-spline by $\mathcal{U}(t)$. If $u(t)\in c^4[0,n]$ and $f,g\in c^2[0,n]$, then $\exists$ a constant $\mathfrak{H}$, such that

$$\|u(t)-\mathcal{U}(t)\|_\infty\le\mathfrak{H}\hbar^2 \ ,$$

For sufficiently small $\hbar$ and $\mathfrak{H}$ is a positive constant.

Proof . Let us consider the following for the problem (2.1).

$u(t)$ be the exact solution, $\tilde{\mathcal{U}}(t)=\sum_{j=-1}^{n+1}\tilde{\gamma}_j\,\mathcal{B}_j(t)$ be an unique exponential B-spline interpolating the solution $u(t)$ and $\mathcal{U}(t)=\sum_{j=-1}^{n+1}\gamma_j\,\mathcal{B}_j(t)$ be the approximate solution of the equation.

Now, from lemma 1 we have

$$\sum_{j=-1}^{n+1}|\mathcal{B}_j(t)|\le\frac{5}{2} \quad , \quad 0\le t\le 1$$

Again, using Theorem 1

$$|Lu(t_j)-L\tilde{\mathcal{U}}(t_j)|\le|-\varepsilon(u''(t_j)-\tilde{\mathcal{U}}(t_j))+f(t)(u'(t_j)-\tilde{\mathcal{U}}(t_j))+g(t)(u(t_j)-\tilde{\mathcal{U}}(t_j))|$$

$$\le(\varepsilon\varkappa_2+\|f\|_\infty\varkappa_1\hbar+\|g\|_\infty\varkappa_0\hbar^2)\hbar^2=\varkappa\hbar^2$$

Where, $\varkappa=\varepsilon\varkappa_2+\|f\|_\infty\varkappa_1\hbar+\|g\|_\infty\varkappa_0\hbar^2$

Thus we obtain,

$$|L\mathcal{U}(t_j)-L\tilde{\mathcal{U}}(t_j)|=|0-L\tilde{\mathcal{U}}(t_j)|=|Lu(t_j)-L\tilde{\mathcal{U}}(t_j)|\le\varkappa\hbar^2 \tag{4.1}$$

As $LU(t_j) = 0$, $0 \le j \le n$ with the boundary conditions given in (2.2) gives a system of linear equations,

$$\mathcal{A}\Upsilon = \mathfrak{D}$$

Now, suppose $L\tilde{U}(t_j) = \omega(t_j), 0 \le j \le n$,

with boundary conditions $\tilde{U}(t_0) = \varphi$ and $\tilde{U}(t_n) = \psi$ leads to a linear system

$$\mathcal{A}\bar{\Upsilon} = \bar{\mathfrak{D}}$$

Where,

$$\bar{\Upsilon} = (\bar{\Upsilon}_0, \bar{\Upsilon}_1 \dots\dots\dots..\bar{\Upsilon}_n)'$$

$$\bar{\mathfrak{D}} = (\bar{\mathfrak{D}}_0(s - \beta h) - 2(\beta hc - s)\mathcal{K}_0\varphi , \bar{\mathfrak{D}}_1, \bar{\mathfrak{D}}_2, \dots\dots\dots\dots., \bar{\mathfrak{D}}_n(s - \beta h) - 2(\beta hc - s)M_n\psi)'$$

Where, $\bar{\mathfrak{D}}_j = 2(\beta hc - s)\omega(t_j)$, $0 \le j \le n$

Then it follows

$$\mathcal{A}(\Upsilon - \bar{\Upsilon}) = (\mathfrak{D} - \bar{\mathfrak{D}}) \tag{4.2}$$

Where,

$$(\Upsilon - \bar{\Upsilon}) = [\, \Upsilon_0 - \bar{\Upsilon}_0, \Upsilon_1 - \bar{\Upsilon}_1, \Upsilon_2 - \bar{\Upsilon}_2, \dots\dots\dots\dots \Upsilon_n - \bar{\Upsilon}_n]'$$

$$(\mathfrak{D} - \bar{\mathfrak{D}}) = [-2(\beta hc - s)(s - \beta h)\omega(t_0), -2(\beta hc - s)\omega(t_1), \dots, -2(\beta hc - s)(s - \beta h)\omega(t_n)]'$$

Using (4.1) we have,

$$\|\mathfrak{D} - \bar{\mathfrak{D}}\|_\infty = \max_{0\le j\le n}\left|\mathfrak{D}_j - \bar{\mathfrak{D}}_j\right| \le \varkappa h^2(\beta h)^3 \tag{4.3}$$

As $0 < \varepsilon \ll 1$ and with sufficiently small h it can be verified that the coefficient matrix $\mathcal{A}$ is Irreducible and monotone [21, 22]. Hence, $\mathcal{A}^{-1}$ exists.

Therefore, from (4.2) we must have

$$\|\Upsilon - \bar{\Upsilon}\|_\infty \le \|\mathcal{A}^{-1}\|_\infty \|\mathfrak{D} - \bar{\mathfrak{D}}\|_\infty \tag{4.4}$$

Let ρ_j , $0 \le j \le n$ be the row sum of the matrix $\mathcal{A}$ such that

$$\rho_0 = \mathfrak{L}_0(s - \beta h) - 2(\beta hc - s)\mathcal{K}_0 + (M_0 - \mathcal{K}_0)(s - \beta h)$$

$$\rho_j = \mathcal{K}_j + \mathfrak{L}_j + M_j \qquad\qquad j = 1,2, \dots\dots\dots\dots., n - 1$$

$$\rho_n = (\mathcal{K}_n - M_n)(s - \beta h) + \mathfrak{L}_n(s - \beta h) - 2(\beta hc - s)M_n$$

Now from theory of matrices we have,

$$\|\mathcal{A}^{-1}\|_\infty \le \frac{1}{\rho} \le \frac{1}{|\rho|} \le \frac{1}{(\beta h)^3} \tag{4.5}$$

Where, $\rho = \min\{\rho_0, \rho_1, \rho_3, \dots\dots\dots \rho_n\}$

Now, substituting the values of (4.3) and (4.5) in (4.4) gives

$$\|\gamma-\bar{\gamma}\|_\infty \le \varkappa\hbar^2(\beta\hbar)^3\frac{1}{|\rho|} \le \varkappa\hbar^2(\beta\hbar)^3\frac{1}{(\beta\hbar)^3} = \varkappa\hbar^2 \tag{4.6}$$

Using Lemma 1 and equation (4.6) ,we obtained

$$\left\|\mathcal{U}(t)-\tilde{\mathcal{U}}(t)\right\|_\infty \le \sum_{j=-1}^{n+1}(\gamma_j-\bar{\gamma}_j)\left|\mathcal{B}_j(t)\right|$$

$$\le \sum_{j=-1}^{n+1}\left|\mathcal{B}_j(t)\right|\|\gamma-\bar{\gamma}\|_\infty \le \frac{5\varkappa\hbar^2}{2} \tag{4.7}$$

Applying the Theorem 1 we have

$$\left\|u(t)-\tilde{\mathcal{U}}(t)\right\|_\infty \le \varkappa_0\hbar^4 \tag{4.8}$$

Now combining the results (4.7) and (4.8) ,

$$\|u(t)-\mathcal{U}(t)\|_\infty \le \mathfrak{K}\hbar^2$$

Where, $\mathfrak{K} = \varkappa_0\hbar^2 + \frac{5\varkappa}{2}$

Hence, the Theorem is proved.

Numerical Experiments: The efficiency of the proposed method is demonstrated by the four model problems. Our proposed solutions are compared with the exact solution those are available in the literature for various values of ε and δ . We are using double mesh principle given by

$E^n = \max_{0\le i\le n} |u_i{}^n - u_{2i}{}^{2n}|$ to calculate the absolute error wherein the exact solutions are not available.

Example 5.1: Let us take a differential equation containing a negative shift with left end layer

$\varepsilon u''(t) + u'(t-\delta) - u(t) = 0\,, 0 \le t \le 1$, with the boundary conditions

$u(0) = 1\ and\ u(1) = 1$

We have the exact solution as

$$u(t) = \frac{(1-e^{w_2})e^{w_1 t} + (e^{w_1}-1)e^{w_2 t})}{(e^{w_1}-e^{w_2})}$$

Where,

$$w_1 = \frac{-1-\sqrt{1+4\,(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \text{ and } w_1 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$$

The comparisons of the absolute error are presented in the Table 1 and table 2 with

left end layer for various values of ε and δ .And the impact of the parameter are shown in graph presented in the Figure 1 and Figure 2.

Example 5.2: Let us take a variable coefficient differential equation with negative shift

$\varepsilon u''(t) + e^{-0.5t}u'(t-\delta) - u(t) = 0$, with $u(0) = 1\ and\ u(1) = 1$

We are using double mesh principle to calculate the maximum absolute error and presented in the Table 3. for various values of ε and δ. The graph of the computed solution is presented in the Figure 3 .

Example 5.3: Let us take a differential equation containing a negative shift with right end layer

$\varepsilon u''(t) - u'(t - \delta) - u(t) = 0$, with the boundary conditions

$u(0) = 1 \; and \; u(1) = -1$

We have the exact solution as

$$u(t) = \frac{((1+e^{w_2})e^{w_1 t} - (e^{w_1}+1)e^{w_2 t})}{(e^{w_2} - e^{w_1})}$$

$$\text{Where } w_1 = \frac{1 - \sqrt{1 + 4\,(\varepsilon - \delta)}}{2(\varepsilon + \delta)} \quad \text{and } w_1 = \frac{1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon + \delta)}$$

The comparison of the absolute errors are presented in the Table 4 and table 5 with

left end layer for various values of ε and δ .And the impact of the parameter are shown in graph presented in the Figure 4. and Figure 5.

Example 5.4: Let us take a variable coefficient differential equation with a delay term

$$\varepsilon u''(t) - e^{t} u'(t - \delta) - t u(t) = 0 \text{ , with } \; u(0) = 1 \; and \; u(1) = 1$$

The maximum absolute errors are calculated by double mesh principle is presented in the Table 6. for various values of ε and δ. The graph of the computed solution is presented in the Figure 7 .

Table 1: Maximum Absolute Error with $\varepsilon = 0.1$ for various values of δ and n (Example 5.1)

n →	10^2		10^3		10^4	
δ ↓	Proposed method	Previous result[13]	Proposed method	Previous result[13]	Proposed method	Previous result[13]
0.01	2.5733e-05	1.3798e-04	2.5763e-07	1.3907e-05	2.7092e-09	1.3887e-06
0.03	2.9690e-05	1.0765e-04	2.9705e-07	1.0849e-05	2.9838e-09	1.0600e-06
0.06	6.1468e-05	6.1798e-05	6.2550e-07	6.2273e-06	6.2161e-09	6.3164e-07
0.08	1.4035e-05	3.0995e-05	1.4099e-06	3.1233e-06	1.400e-08	3.3537e-07

Table 2: Absolute Error with $\varepsilon = 0.02$, $\delta = 0.001$and $h = 0.01$ (Example 5.1)

t	Solution by Proposed Method	Exact Solution	Solution by Method [15]	Absolute Error by Proposed method	Absolute Error by method[15]
0.0	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.02	0.59650758	0.59611217	0.40030832	3.95409e-04	1.95804e-01
0.04	0.46326487	0.46292257	0.38470877	3.42308e-04	7.82138e-02
0.06	0.42273389	0.42247446	0.39155769	2.59429e-04	3.09168e-02
0.08	0.41407601	0.41386729	0.39941373	2.08724e-04	1.44536e-02
0.1	0.41644444	0.41626092	0.40746127	1.83512e-04	8.79965e-03
0.2	0.45613474	0.45597317	0.45020470	1.61571e-04	5.76847e-03
0.3	0.50314705	0.50299127	0.49743206	1.55780e-04	5.55921e-03

0.4	0.55502160	0.55487431	0.54961366	1.47294e-04	5.26065e-03
0.5	0.61224452	0.61210912	0.60726922	1.35403e-04	4.83990e-03
0.6	0.67536714	0.67524765	0.67097295	1.19493e-04	4.27470e-03
0.7	0.74499772	0.74489886	0.74135933	9.88619e-05	3.53953e-03
0.8	0.82180724	0.82173453	0.81912938	7.27047e-05	2.60515e-03
0.9	0.90653585	0.90649574	0.90505767	4.01012e-05	1.43807e-03
1.0	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Table 3: Maximum absolute Error with $\varepsilon = 0.1$ for various values of δ and ｎ using double mesh principle (for Example 5.2)

ｎ →	10^2	10^3	10^4
δ ↓	Proposed method	Proposed method	Proposed method
0.001	0.0265796	0.0032995	3.3605207e-04
0.003	0.0271238	0.0033683	3.4306667e-04
0.006	0.0279825	0.0034768	3.5414669e-04
0.008	0.0285853	0.0035531	3.6193372e-04

Table 4: Maximum absolute Error with $\varepsilon = 0.1$ for various values of δ and ｎ(for Example 5.3)

ｎ →	10^2		10^3		10^4	
δ ↓	Proposed Method	Results[13]	Proposed method	Results[13]	Proposed method	Results[13]
0.01	1.8402e-05	4.9650e-05	1.8410e-07	4.9729e-06	1.6235e-09	4.9586e-07
0.03	1.4093e-05	5.8439e-05	1.4097e-07	5.8534e-06	1.1188e-09	5.8693e-07
0.06	9.7089e-06	7.1489e-05	9.7114e-08	7.1607e-06	1.0594e-09	7.2219e-07
0.08	7.5492e-06	8.0100e-05	7.5512e-08	8.0235e06	6.7766e-10	8.0780e-07

Table 5: Absolute Error with $\varepsilon = 0.001$, $\delta = 0.003$ and $\hbar = 0.01$ (for Example 5.3)

ŧ	Solution by Proposed Method	Exact Solution	Solution by Method [15]	Absolute Error by Proposed method	Absolute Error by method[15]
0.0	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.1	0.90536618	0.90519656	0.90493550	1.69625e-04	4.30683e-04
0.2	0.81968792	0.81938081	0.81890826	3.07117e-04	7.79665e-04
0.3	0.74211773	0.74170069	0.74105916	4.17042e-04	1.05857e-03
0.4	0.67188829	0.67138491	0.67061074	5.03387e-04	1.27755e-03
0.5	0.60830494	0.60773531	0.60685947	5.69633e-04	1.44547e-03
0.6	0.55073872	0.55011991	0.54916868	6.18814e-04	1.57004e-03
0.7	0.49862021	0.49796665	0.49696223	6.53568e-04	1.65798e-03
0.8	0.45143388	0.45075769	0.44971877	6.76186e-04	1.71511e-03
0.9	0.40871297	0.40802431	0.40696648	6.88656e-04	1.74649e-03
0.92	0.40066672	0.39997663	0.39891664	6.90087e-04	1.75008e-03
0.94	0.39277848	0.39208729	0.39102493	6.91195e-04	1.75355e-03
0.96	0.38498654	0.38429459	0.38317158	6.91948e-04	1.81496e-03
0.98	0.36841582	0.36772893	0.36289935	6.86889e-04	5.51647e-03
1.0	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Table 6: Maximum absolute Error with $\varepsilon = 0.1$ for various values of δ and n using double mesh principle (Example 5.4)

n →	10^2	10^3	10^4
δ ↓	Proposed method	Proposed method	Proposed method
0.000	0.02230547	0.00305640	3.11355660e-04
0.003	0.02081696	0.00282670	2.87617565e-04
0.007	0.01910969	0.00256855	2.61006577e-04
0.015	0.01639981	0.00217005	2.20066455e-04

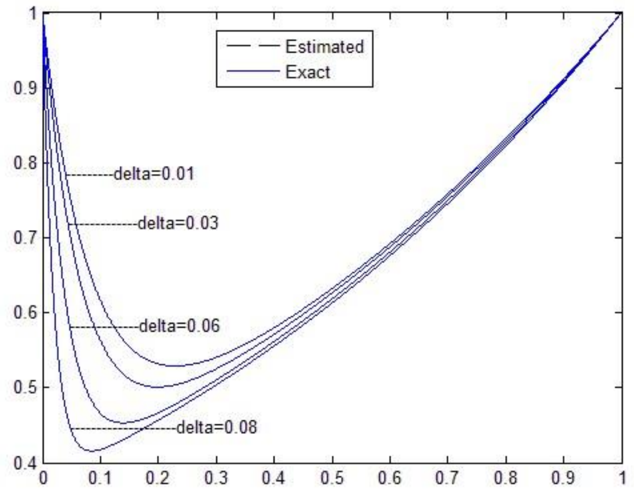

Figure 1. Graphical representation with $\varepsilon = 0.1$ for various values of δ (example5.1)

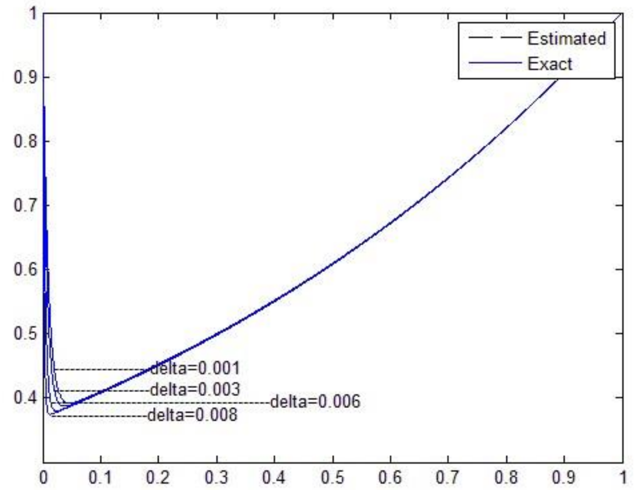

Figure 2. Graphical representation with $\varepsilon = 0.01$ for various values of δ (example5.1)

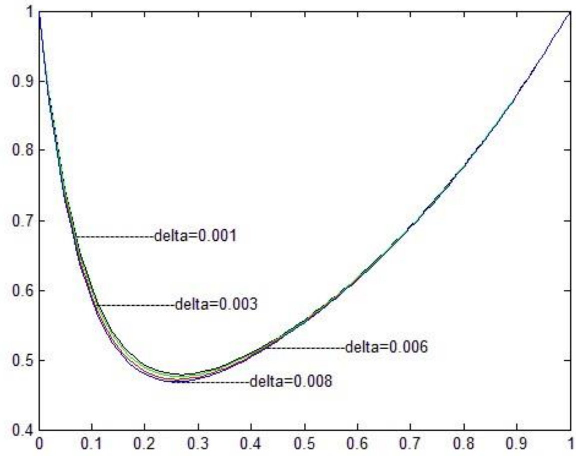

Figure 3. Graphical representation with $\varepsilon = 0.1$ for various values of δ (example5.2)

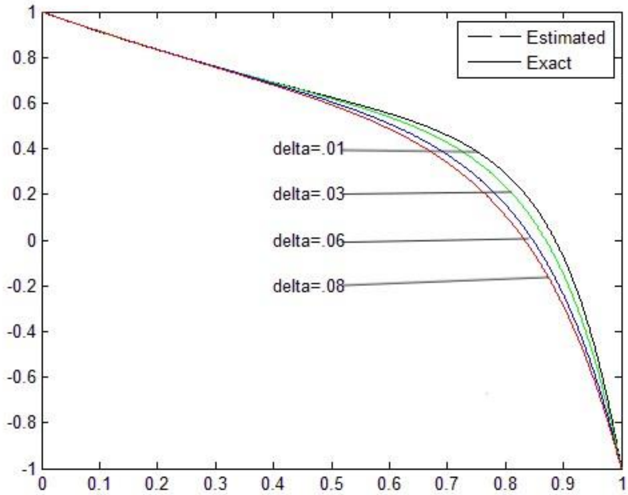

Figure 4. Graphical representation with $\varepsilon = 0.1$ for different values of δ (example5.3)

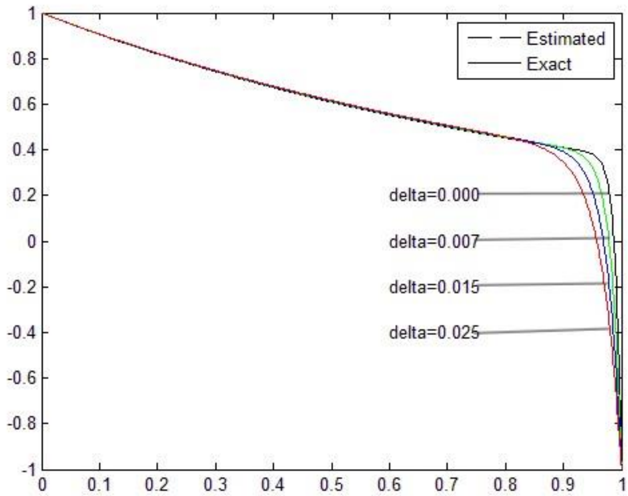

Figure 5. Graphical representation with $\varepsilon = 0.01$ for various values of δ (example5.3)

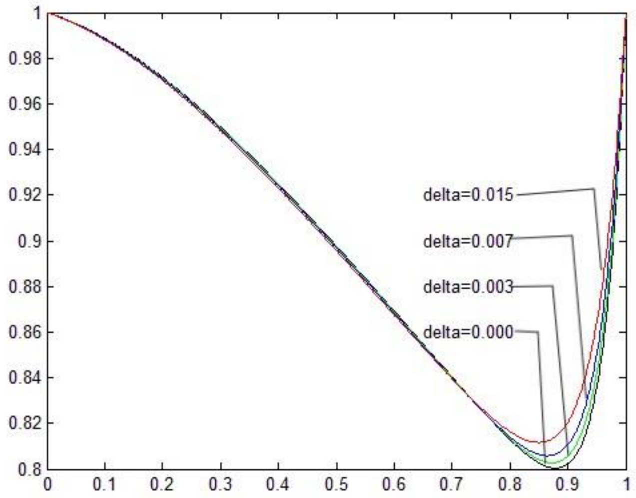

Figure 6. Graphical representation with $\varepsilon = 0.1$ for various values of δ (example5.4)

5. Discussion

In an effort to solve delay differential equations with boundary layers, we employed the exponential B-spline method in this study. With this approach, we first applied Taylor's series to transform a second order singularly perturbed differential equation with a delay term problem into a neutral type singularly perturbed differential equation. The suggested approach is applied to four model problems, and the results are contrasted to the precise solutions that were found. In cases where exact solutions were not found, the absolute error was calculated using the double mesh concept. We have finally drawn the graphs showing the answers for various values of ε and δ.

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