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Fixed Point Theorems in Revised Fuzzy Metric Space via R_F −Contraction

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Abstract:

The aim of this paper is to introduce a new type of contraction called R_F −contraction. As compared to the F-contraction in the existing literature, our R_F −contraction is much simpler and more straightforward, since it contains only one condition that is, the function revised fuzzy is strictly non-increasing. Moreover, some fixed-point theorems for R_F −contraction are presented. Further, some examples are given to illustrate its validity and superiority. In addition, by applying a very significant lemma, we show that our proofs of most fixed-point theorems are shorter and more elegant than ones in the literature.

Keywords and phrases: T-conorm, revised fuzzy metric, R_F −contraction, fixed point.

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1. Introduction

George and Veeramani [3] proposed axioms to fuzzy metric spaces [for short, FMS] based on Zadeh's theory of fuzzy sets [25]. The triangular norm (for short, t-norm), initially proposed by Schweizer and Sklar [21], is one of the most significant axioms in binary functions. In various domains, such as fuzzy sets, fuzzy logic, and its applications, this is a critical process. The fixed-point [for short, FP] theory constructed in FMSs, which was pioneered by Grabiec [4], where a FM version of the Banach contraction principle was introduced, is one of the most fascinating motives. Following that, Gregori and his coauthors presented a number of fuzzy contractive mappings [for short, FCM] in FMS (see [5]). Mihet [11], on the other hand, established a fixed-point theorem for weak Banach contraction in W-complete FMS, and expanded prior results including additional types of contractions such as Edelstein fuzzy contractive mappings, fuzzy y-contractive mappings, and so on (for details, see [11]). Wardowski [23] recently developed a novel idea of fuzzy H-contractive mapping and deduced some interesting FP theorems. Wardowski [24] also developed a FP theorem in metric spaces and introduced the F-contraction contraction. [7,11,22] recently proposed further contractions in FMS.

Alexander Sostak [1] introduced the notion of Revised Fuzzy metric [for short, RFM] in the year 2018, which allows for the incremental assessment of the membership of components in a set. Muraliraj and Thangathamizh developed Revised Fuzzy contraction mappings [for short, RFCM] and established FP findings for them [11-14, 24-26]. Many generic topological ideas and conclusions were then applied to the revised fuzzy topological space.

We intraduce a new contraction called RF-contraction throughout this research, which differs from [7,17,23] in that it incorporates a simpler criterion, namely that the mapping is strictly non-increasing. Furthermore, in the context of RFMS, we deal with FP theorems for R_F −contraction. In particular, we prove a lemma in RFMS with regard to the Cauchy sequence. Second, we introduce the idea of R_F −contraction, which requires just that the function be strictly non-increasing. Third, we get certain FP theorems for R_F −contraction with shorter requirements and simple proofs using the preceding lemma. Fourth, we provide some instances to back up our findings. Our examples demonstrate that our findings are true generalizations in the literature.

2. Preliminaries

We'll go over a few fundamental definitions and ideas in the sections that follow.

Definition 2.1 ([21]).

A binary operation $T:[0,1]^2 \to [0,1]$ is called a triangular norm (for short, t-norm) if the following conditions hold:

(i) $T(g,0)=g$, for each $g \in [0,1]$;

(ii) $T(g,d) \le T(l,m)$, for any $g \le l, d \le m$ and $g,d,l,m \in [0,1]$;

(iii) T is associative and commutative.

Three basic examples of continuous t-conorms are as follows: $T_{max}(g,d)=max\{g,d\}$, $T_P(g,d)=g+d-gd$ and $T_L(g,d)=ming\{g+d,1\}$ (maximum, product, and Lukasiewicz t-conorm, respectively).

Definition 2.2 ([1]).

A triple (X,W,T) is called a RFMS if X is a nonempty set, T is a continuous t-conorm, and $W: X^2 \times (0,+\infty) \to [0,1]$ be a RF satisfying the following conditions:

(RF 1) $W(q,r,t)<1$ for all $q,r \in X$ and $t>0$;

(RF 2) $W(q,r,t)=0(t>0)$ if and only if $q=r$;

(RF 3) $W(q,r,t)=W(r,q,t)$ for all $q,r \in X$ and $t>0$;

(RF 4) $W(q,l,t+s) \le T(W(q,r,t),W(r,l,s))$ for all $q,r,l \in X$ and $t,s>0$;

(RF 5) $W(q,r,-):(0,+\infty)\to[0,1]$ is continuous for all $q,r \in X$.

If (RF 4) is replaced by the following condition:

(RF 4)' $W(q,l,t) \le T(W(q,r,t),W(r,l,t))$ for all $q,r,l \in X$ and $t>0$;

then (X,W,T) is called a strong RFMS.

Moreover, if (X,W,T) is a RFMS, then W is a continuous function on $X^2 \times (0,+\infty)$ and $W(q,r,-)$ is non-increasing for all $q,r \in X$.

In the sequel, unless there is a special explanation, we always denote by N, the set of all positive integers; N_0, the set of all nonnegative integers; R, the set of all real numbers and R_+, the set of all positive real numbers.

Definition 2.3 ([12]).

Let (X,W,T) be a RFMS and $\{q_n\}_{n\in\mathrm{N}}$ be a sequence in X. Then, we say the following:

(i) $\{q_n\}_{n\in\mathrm{N}}$ converges to $q \in X$ (say $\lim_{n\to\infty} q_n = q$), if $\lim_{n\to\infty} W(q_n,q,t)=0$ for any $t>0$;

(ii) $\{q_n\}_{n\in\mathrm{N}}$ is a Cauchy sequence if, for any $\varepsilon \in (0,1)$ and $t>0$, there exists $n_0 \in \mathrm{N}$ such that $W(q_m,q_n,t)<\varepsilon$ for any $m,n \in n_0$;

(iii) (X,W,T) is complete if every Cauchy sequence is convergent.

Definition 2.4

Let (X,W,T) be a RFMS and $f:X\to X$ a mapping. Then f is called a revised fuzzy contraction if there exists $k\in(0,1)$ such that

$$W(f(q),f(r),t) \le k(W(q,r,t)) \tag{1}$$

for all $q,r \in X$ and $t>0$. In this case, k is called the contractive constant of f.

We say that the mapping $T:X\to X$ is called a Tirado contraction if there exists $k\in(0,1)$ such that

$$W(Tq,Tr,t) \le k(W(q,r,t)) \text{ for all } q,r \in X \text{ and } t>0. \tag{2}$$

3. Methods

Lemma 3.1.

Let (X,W,T) be a RFMS and $\{q_n\}$ be a sequence in X such that for each $n\in N$,

$\lim_{t\to 0^+} W(q_n, q_{n+1}, t) < 1$, and for any $t>0$, (3)

$\lim_{t\to\infty} W(q_n, q_{n+1}, t) = 0$. (4)

If $\{q_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon \in (0,1), t_0 > 0$, and two sequences of positive integers $\{n_k\}$, $\{m_k\}$, $n_k > m_k > k, k \in N$, such that the following sequences

$\{W(q_{mk}, q_{nk}, t_0)\}, \{W(q_{mk}, q_{nk+1}, t_0)\}, \{W(q_{mk-1}, q_{nk}, t_0)\}, \{W(q_{mk-1}, q_{nk+1}, t_0)\}, \{W(q_{mk+1}, q_{nk+1}, t_0)\}$

tend to ε as $k \to \infty$.

Proof.

Let $\{q_n\}$ be a sequence in X, which is not a Cauchy sequence. Then, by Definition 2.3, there exists $\varepsilon \in (0,1), t_0 > 0$ and sequence $\{n_k\}$, $\{m_k\}$, $n_k > m_k > k, k \in N$, such that for any $k \in N$, we have

$W(q_{mk}, q_{nk}, t_0) > \varepsilon$. (5)

and

$W(q_{mk}, q_{nk}, t_0) \le \varepsilon$. (6)

clearly, by (5), one has

$\lim_{k\to\infty} \inf W(q_{mk}, q_{nk}, t_0) > \varepsilon$. (7)

Using Condition (RF 4), for any $k \in N$ and $p \in (0, t_0)$, it is not hard to verify that

$W(q_{mk}, q_{nk}, t_0) \le T\left(W(q_{mk}, q_{mk-1}, p), W(q_{mk-1}, q_{nk}, t_0 - p)\right)$. (8)

Note that, by (3) and (4), it follows that

$\lim_{n\to\infty}\left(\lim_{p\to 0^+} W(q_n, q_{n+1}, p)\right) = 0$.. (9)

If we take $p \to 0^+$ in (8), then by (9), (6) and the continuity of T, we obtain

$$\lim_{k\to\infty} W(q_{mk}, q_{nk}, t_0) \le \lim_{k\to\infty}\left(\lim_{p\to 0^+} T\left(W(q_{mk}, q_{mk-1}, p), W(q_{mk-1}, q_{nk}, t_0 - p)\right)\right).$$

$$= T\left(\lim_{k\to\infty}\left(\lim_{p\to 0^+} W(q_{mk}, q_{mk-1}, p)\right), \lim_{k\to\infty}\left(\lim_{p\to 0^+} W(q_{mk-1}, q_{nk}, t_0 - p)\right)\right)$$

$$= T\left(0, \lim_{k\to\infty} W(q_{mk-1}, q_{nk}, t_0)\right) = \lim_{k\to\infty} W(q_{mk-1}, q_{nk}, t_0) < \varepsilon.$$

This inequality and (7) imply

$$\lim_{k\to\infty} W\left(q_{mk}, q_{nk}, t_0\right) = \varepsilon\ . \tag{10}$$

Let us prove that

$$\lim_{k\to\infty} W\left(q_{mk}, q_{nk+1}, t_0\right) = \varepsilon\ . \tag{11}$$

$$\lim_{k\to\infty} W\left(q_{mk}, q_{nk+1}, t_0\right) \le \lim_{k\to\infty}\left(\lim_{p\to 0^+} T\left(W\left(q_{mk}, q_{nk}, t_0 - p\right), W\left(q_{nk}, q_{nk+1}, p\right)\right)\right).$$

$$= T\left(\lim_{k\to\infty} W\left(q_{mk}, q_{nk}, t_0\right), 0\right) = \lim_{k\to\infty} W\left(q_{mk}, q_{nk}, t_0\right) = \varepsilon\,. \tag{12}$$

On the other hand, by (10) and (4), we have

$$\varepsilon = \lim_{k\to\infty} W\left(q_{mk}, q_{nk}, t_0\right) \le \lim_{k\to\infty}\left(\lim_{p\to 0^+} T\left(W\left(q_{mk}, q_{nk+1}, t_0 - p\right), W\left(q_{nk+1}, q_{nk}, p\right)\right)\right)$$

$$\le T\left(\lim_{k\to\infty} W\left(q_{mk}, q_{nk+1}, t_0\right), 0\right) = \lim_{k\to\infty} W\left(q_{mk}, q_{nk+1}, t_0\right). \tag{13}$$

Then, by (12) and (13), we obtain (11).

The left proofs are similar to the above argument, and therefore we omit them.

Remark 3.2.

Condition (3) in Lemma 3.1 can be omitted if $\left(X, W, T\right)$ is a strong RFMS.

In this case, instead of (8), we have

$$W\left(q_{mk}, q_{nk}, t_0\right) \le T\left(W\left(q_{mk}, q_{mk-1}, p\right), W\left(q_{mk-1}, q_{nk}, t_0 - p\right)\right).$$

In the following, denote by F the class of all mappings $F : [0,1] \to (0, +\infty)$ satisfying the following condition: for all $q, r \in X$, $q < r$ implies $F(q) < F(r)$. That is to say, F is strictly non-increasing on $[0,1]$.

Definition 3.3.

Let $\left(X, d\right)$ be a metric space and $F : R_+ \to R$ be a mapping, satisfying the following:

(RF 1) F is strictly non increasing on R_+;

(RF 2) For each sequence $\{\alpha_n\}_{n\in \mathrm{N}}$ of positive numbers, $\lim_{k\to\infty} F\left(\alpha_n\right) = -\infty$ if, and only if

$\lim_{k\to\infty} \alpha_n = 0$;

(RF 3) There exists $k \in (0,1)$ such that $\lim_{k\to 0^+} \alpha^k F\left(a\right) = 0$.

The mapping $J : X \to X$ is said to be an $F-$contraction if there exists $t > 0$ such that

$$\tau + F\left(d\left(J(q), J(r)\right)\right) \le F\left(d(q,r)\right) \text{ for all } q, r \in X \text{ with } d\left(j(q), j(r)\right) > 0 .$$

Definition 3.4.

Let (X,W,T) be a RFMS and $F \in \mathrm{F}$. The mapping $J : X \to X$ is said to be a R_F −contraction if there exists $\tau \in (0,1)$ such that

$$F\left(d\left(f(q), f(r)\right)\right) \le \frac{1}{\tau}\left(F\left(d(q,r)\right)\right) \text{ for all } q,r \in X, q \ne r, \text{ and } t > 0. \tag{14}$$

Theorem 3.5.

Let (X,W,T) be a complete RFMS such that

$$\lim_{t \to 0^+} W(q,r,t) < 1, \text{ for all } q,r \in X .$$

If $J : X \to X$ is a continuous RF F-contraction, then J has a unique fixed-point in X .

Proof.

Choose $q_0 \in X$ and $q_{n+1} = J(q_n)$ for all $n \in N_0$. Suppose that $J : X \to X$ is a RF −contractive mapping. If $q_n = q_{n+1} = j(q_n)$ holds for some $n \in N_0$, then q_n is a FP. Assume that $q_n \ne q_{n+1}$ for any $n \in N_0$. By (14), for every $n \in N$ and $t > 0$, one has

$$F\left(W(q_{n+1}, q_n, t)\right) < \frac{1}{\tau}\left(F\left(W(q_{n+1}, q_n, t)\right)\right) \le F\left(W(q_n, q_{n-1}, t)\right).$$

Then, we get

$$W(q_n, q_{n-1}, t) > W(q_{n+1}, q_n, t).$$

Thus, $\{W(q_{n+1}, q_n, t)\}$ $(t > 0)$ is a strictly non-increasing sequence bounded from above, so $\{W(q_{n+1}, q_n, t)\}$ $(t > 0)$ is convergent. In other words, there exists $a(t) \in [0,1]$ such that for any $t > 0$, one has

$$\lim_{n \to \infty} W(q_n, q_{n+1}, t) = a(t). \tag{15}$$

Clearly, for any $t > 0$ and $n \in N$, it follows that

$$W(q_n, q_{n+1}, t) < a(t). \tag{16}$$

Note that, by (15) and (16), for any $t > 0$, we have

$$\lim_{n \to \infty} F\left(W(q_n, q_{n+1}, t)\right) = F\left(a(t) - 0\right). \tag{17}$$

Assume that $a(t) < 1$ for some $t > 0$. By (14), it implies that

$$F\left(W\left(q_{n+1},q_n,t\right)\right)<\frac{1}{\tau}\left(F\left(W\left(q_{n+1},q_n,t\right)\right)\right)\leq F\left(W\left(q_n,q_{n-1},t\right)\right). \tag{18}$$

Taking the limit from both sides of (18) together with (17), we get

$$F\left(a\left(t\right)-0\right)\leq\frac{1}{\tau}\left(F\left(a\left(t\right)-0\right)\right)\leq F\left(a\left(t\right)-0\right),$$

which means that $F\left(a\left(t\right)-0\right)=0$. This is a contradiction with $F\left(a\left(t\right)-0\right)>0$.

Therefore, we have

$$\lim_{n\to\infty}W\left(q_n,q_{n+1},t\right)=0. \tag{19}$$

Further, we need to prove that $\{q_n\}$ is a Cauchy sequence. Suppose that this claim is not true. Using Lemma 3.1 and noting that (19) is in fact Condition (4), then there exist $\varepsilon\in\left(0,1\right),t_0>0$ and sequences $\left\{q_{m_k}\right\}$ and $\left\{q_{n_k}\right\}$ such that $\lim_{k\to\infty}W\left(q_{m_k},q_{n_k},t_0\right)=\varepsilon$.

By (14), we have

$$F\left(W\left(J\left(q_{m_k}\right),J\left(q_{n_k}\right),t_0\right)\right)<\frac{1}{\tau}\left(F\left(W\left(J\left(q_{m_k}\right),J\left(q_{n_k}\right),t_0\right)\right)\right)\leq F\left(W\left(q_{m_k},q_{n_k},t_0\right)\right).$$

Letting $k\to\infty$ from both sides of the above inequality, we have

$F\left(\left(\varepsilon\right)-0\right)\leq\frac{1}{\tau}\left(F\left(\left(\varepsilon\right)-0\right)\right)\leq F\left(\left(\varepsilon\right)-0\right)$, which establishes that $F\left(\left(\varepsilon\right)-0\right)=0$.

This is in contradiction with $F\left(\left(\varepsilon\right)-0\right)>0$.

Hence, $\{q_n\}$ is a Cauchy sequence. Since $\left(X,W,T\right)$ is complete, then there exists $q^{*}\in X$ such that

$$\lim_{n\to\infty}q_n=q^{*}. \tag{20}$$

Let us prove that q^{*} is a FP of J. As a matter of fact, it follows immediately from (20) and the continuity of J that

$$q^{*}=\lim_{n\to\infty}q_{n+1}=J\left(\lim_{n\to\infty}q_n\right)=f\left(q^{*}\right).$$

Finally, we prove the uniqueness of the FP. Suppose that q^{*} and r^{*} are distinct FPs of J. Again, by using (14), we easily obtain that

$$F\left(W\left((J\left(q^{*}\right),J\left(r^{*}\right),t\right)\right)<\frac{1}{\tau}\left(F\left(W\left((J\left(q^{*}\right),J\left(r^{*}\right),t\right)\right)\right)\leq F\left(W\left(q^{*},r^{*},t\right)\right).$$

As a consequence, we have

$W\left((J\left(q\right),J\left(r\right),t\right)<W\left(q,r,t\right)=W\left((J\left(q\right),J\left(r\right),t\right).$

This is a contradiction.

Remark 3.6.

Let $\left(X,W,T\right)$ be a RFMS.

(i) Define a strictly non- increasing function $F\left(t\right)=\frac{1}{1+t}$ for any $t\in\left(0,1\right)$ and let J be a RF-contraction. Then, the RF contraction (1) is obtained. Indeed, since J is RF-contractive, then there exists $t\in\left(0,1\right)$ such that

$\frac{1}{1+W\left(J\left(q\right),J\left(r\right),t\right)}\leq\frac{1}{\tau}\left(\frac{1}{1+W\left(q,r,t\right)}\right)$ that is,

$$\frac{1}{\tau}\left(1+W\left(J\left(q\right),J\left(r\right),t\right)\right)\geq1+W\left(q,r,t\right)$$

Therefore,

$W\left(J\left(q\right),J\left(r\right),t\right)\leq\tau\left(W\left(q,r,t\right)\right)$ holds for all $q,r\in X$, and $t>0$.

(ii) let $\left(t\right)=\frac{1}{1+t}$, where $t\in\left(0,1\right)$, and suppose that J is a RF-contraction. Then we easily obtain the Tirado contraction.

Corollary 3.7.

Let $\left(X,d\right)$ be a complete RFMS, and $J:X\rightarrow X$ be a function such that there exists $K_i\in\left(0,1\right)\left(i=1,2,3,4\right)$ and for all $q,r\in X,q\neq r$, and one of the followings:

(1) $W\left(J\left(q\right),J\left(r\right),t\right)\leq\frac{1}{K_1}\left(W\left(q,r,t\right)\right);$

(2) $\frac{exp^{\{-W\left(J\left(q\right),J\left(r\right),t\right)\}}}{1+e^{\{-W\left(J\left(q\right),J\left(r\right),t\right)\}}}\leq\frac{1}{K_2}\left(\frac{e^{\{-W\left(q,r,t\right)\}}}{1+e^{\{-W\left(q,r,t\right)\}}}\right);$

(3) $\frac{e^{\{-W\left(J\left(q\right),J\left(r\right),t\right)\}}-e^{\{W\left(J\left(q\right),J\left(r\right),t\right)\}}}{3+e^{\{-W\left(J\left(q\right),J\left(r\right),t\right)\}}-e^{\{W\left(J\left(q\right),J\left(r\right),t\right)\}}}\leq\frac{e^{\{-W\left(q,r,t\right)\}}-e^{\{W\left(q,r,t\right)\}}}{1+e^{\{-W\left(q,r,t\right)\}}-e^{\{W\left(q,r,t\right)\}}};$

(4) $exp\left\{W\left(J\left(q\right),J\left(r\right),t\right)\right\}$. Then, J has a unique FP in X .

Proof.

For Cases (1)–(4), put

$$F_1(t)=t, F_2(t)=\frac{exp^{\{-t\}}}{1+exp^{\{-t\}}}, F_3(t)=\frac{exp^{\{-t\}}-exp^{\{t\}}}{3+exp^{\{-t\}}-exp^{\{t\}}}, F_4(t)=exp^{\{t\}},$$

respectively. Using Theorem 3.5, we claim that J has a unique FP.

Example 3.8.

Let $X=R$ and define the usual metric $d(q,r)=|q-r|$ for all $q,r\in X$. Let T be a product t-conorm. Define a *RFM* as follows:

$$W(q,r,t)=exp^{\left\{-\frac{d(q,r)}{t+1}\right\}}\left(exp^{\left\{\frac{d(q,r)}{t+1}\right\}}-1\right),$$

Where $q,r\in X$, and $t>0$. Clearly, $W(q,r,t)$ satisfies the conditions of (RF 1)–(RF 3) and (RF 5). Moreover, for all $q,r,l\in X$ and $t,s>0$, it is clear that

$$W(q,l,t+s)=e^{\left\{-\frac{d(q,l)}{t+s+1}\right\}}\left(e^{\left\{\frac{d(q,l)}{t+s+1}\right\}}-1\right)=W(q,r,t).W(r,l,s),$$

that is, Condition (RF 4) holds.

Let $J(x)=\frac{1}{4}x(x\in X), F(y)=-\frac{1}{ln\, y}$ $(0<y<1) and\ \tau=\frac{1}{2}$. Since

$$F\left(W\left(f(q),f(r),t\right)\right)=F\left(W(q,r,t)\right)$$

holds for all $q,r\in X$, $q\neq r$ and $t>0$, then Condition (14) is fulfilled. Hence, by Theorem 3.5, it follows that J has a unique FP. It is worth mentioning that this example is true for arbitrary function $J(q)=kq$, where $0<k<1$ is a constant with $t>k$.

Example 3.9.

Let (X,d) be a metric space and T a t-conorm. Then for all $q,r\in X$ and $t>0$,

$$W(q,r,t)=\frac{d(q,r)}{t+1+d(q,r)}$$

defines a RFM.

Define a function $F(x)=x$ on $[0,1]$ and let $\tau\in(0,1)$ be a constant. If Condition (14) is fulfilled, then

$$W\left(f(q),f(r),t\right)=F\left(W\left(f(q),f(r),t\right)\right)\le\tau\left(F\left(W(q,r,t)\right)\right)=\tau\left(\frac{d(q,r)}{t+1+d(q,r)}\right)=W\left(q,r,\frac{t}{\tau}\right)$$

holds for all $q, r \in X$ and $t > 0$. That is to say, we obtain the contractive condition (1) from [22].

Theorem 3.10.

Let (X, W, T) be a complete *RFMS* and $F \in \mathrm{F}$ be a continuous mapping.

If $J : X \to X$ is a *RFM*, $F-$contraction, then J has a unique fixed-point in X.

Proof.

Choose $q_0 \in X$ and define a sequenced $\{q_n\}$ by $q_{n+1} = J(q_n)$ for all $n \in N_0$. If $q_n = q_{n+1}$ for some $n \in N_0$, then the proof is finished. Assume that $q_n \neq q_{n+1}$ for any $n \in N_0$. From the definition of the F-contractive, we have

$$F\left(W\left(q_{n+1}, q_{m+1}, t\right)\right) < \frac{1}{\tau}\left(F\left(W\left(q_{n+1}, q_{m+1}, t\right)\right)\right) \leq F\left(W\left(q_n, q_m, t\right)\right).$$

Then,

$W(q_{n+1}, q_{m+1}, t) > W(q_n, q_m, t)$. For any $n > m$. Let $a_m(t) = \inf_{n>m} W(q_n, q_m, t)$. Notice that

$$\inf_{n>m} W\left(q_{n+1}, q_{m+1}, t\right) \leq \inf_{n>m} W\left(q_n, q_m, t\right)$$

Then $a_{m+1}(t) \geq a_m(t)$, for any $m \in N$. Since $\{a_m(t)\}$ is bounded, then there exists $a(t) \in [0,1]$ such that $\lim_{n\to\infty} a_m(t) = a(t)$ for all $t > 0$. let us prove that $a(t) = 0$ for all $t > 0$. Suppose the contrary, and there exists $s > 0$ such that $0 \leq a(s) < 1$. Then by (14) $\tau \in [0,1]$ such that for any $t > 0$, one has

$$\lim_{m\to\infty} F\left(\inf_{n>m} W\left(q_{n+1}, q_{m+1}, t\right)\right) \leq \frac{1}{\tau} \lim_{m\to\infty} F\left(\inf_{n>m} W\left(q_{n+1}, q_{m+1}, t\right)\right) \leq \lim_{m\to\infty} F\left(\inf_{n>m} W\left(q_n, q_m, s\right)\right)$$

Using the assumption that F is continuous, we have

$$F\left(a(s)\right) \leq \frac{1}{\tau} F\left(a(s)\right) \leq F\left(a(s)\right),$$

which means that $F(a(s)) = 1$. This is in contraction with $F(a(s)) < 1$. Thus

$\lim_{m\to\infty}\left(\inf_{n>m} W\left(q_n, q_m, s\right)\right) = 0$. For any $t > 0$. Consequently,

$\lim_{m,n\to\infty} W\left(q_n, q_m, s\right) = 0$, for any $t > 0$.

Thus $\{q_n\}$ is a Cauchy sequence. Since (W, X, T) is complete, then there is $q \in X$ such that $\lim_{n\to\infty} q_n = q_n$.

Taking advantage of (14), we have

$$F\left(W\left(q_{n+1},J\left(q\right),t\right)\right)<\frac{1}{\tau}F\left(W\left(q_{n+1},J\left(q\right),t\right)\right)\leq W\left(q_{n},q,t\right).$$

For any $t>0$. Consequently, $\lim_{m,n\to\infty} W\left(q_{n},q_{m},s\right)=0$ for any $t>0$.

Thus, $\left\{q_{n}\right\}$

$$F\left(W\left(q_{n+1},q_{n},t\right)\right)<\frac{1}{\tau}\left(F\left(W\left(q_{n+1},q_{n},t\right)\right)\right)\leq F\left(W\left(q_{n},q_{n-1},t\right)\right) \text{ for all } n\in N.$$

Letting $n\to\infty$ and using the assumption that F is continuous, we have

$$F\left(W\left(q,J\left(q\right),t\right)\right)<\frac{1}{\tau}\ F\left(W\left(q,J\left(q\right),t\right)\right)\leq F\left(0\right).$$

Thus, it leads to $W\left(q,f\left(q\right),t\right)\leq 0$. Therefore, $q=J\left(q\right)$.

Suppose now that J has distinct FPs $q,r\in X$, then by (14), we obtain

$$F\left(W\left(q,r,t\right)\right)=F\left(W\left(J\left(q\right),J\left(r\right),t\right)\right)<\frac{1}{\tau}F\left(W\left(J\left(q\right),J\left(r\right),t\right)\right)\leq F\left(W\left(q,r,t\right)\right).$$

This is a contradiction. Hence, $q=r$.

Theorem 3.10.

Let $\left(X,W,T\right)$ be a complete RFMS such that $\lim_{t\to 0^{+}}W\left(q,r,t\right)<1$ for all $q,r\in X$. Let $f:X\to X$ be a mapping and $F\in\mathrm{F}$. Suppose that for all $q,r\in X$, $q\neq r$ and $t>0$, there exists $\tau\in\left(0,1\right)$ such that

$$F\left(W\left(f\left(q\right),f\left(r\right),t\right)\right)\leq\frac{1}{\tau}F\left(max\left\{W\left(q,r,t\right),W\left(q,J\left(q\right),t\right),W\left(r,J\left(r\right),t\right)\right\}\right). \tag{21}$$

Then, J has a unique FP, provided that J or F is continuous.

Proof.

Choose $q_{0}\in X$ and define a sequence $\left\{q_{n}\right\}$ as follows: $q_{n+1}=J\left(q_{n}\right)\left(n\in N_{0}\right)$.

By (21), we have

$$\begin{aligned}F\left(W\left(q,q_{n+1},t\right)\right)&=F\left(W\left(J\left(q_{n-1}\right),J\left(q_{n}\right),t\right)\right)<\frac{1}{\tau}F\left(W\left(J\left(q_{n-1}\right),J\left(q_{n}\right),t\right)\right)\\&\leq F\left(max\left\{W\left(q_{n-1},q_{n},t\right),W\left(q_{n-1},q_{n},t\right),W\left(q_{n},q_{n+1},t\right)\right\}\right)\\&=F\left(max\left\{W\left(q_{n-1},q_{n},t\right),W\left(q_{n},q_{n+1},t\right)\right\}\right), \text{ for all } n\in N \text{ and } t>0.\end{aligned} \tag{22}$$

If $max\left\{W\left(q_{n-1},q_n,t\right),W\left(q_n,q_{n+1},t\right)\right\}=W\left(q_n,q_{n+1},t\right)$, then by (22), we get

$W\left(q_n,q_{n+1},t\right)<W\left(q_n,q_{n+1},t\right)$, which is a contradiction. If
$max\left\{W\left(q_{n-1},q_n,t\right),W\left(q_n,q_{n+1},t\right)\right\}=W\left(q_{n-1},q_n,t\right)$, then by (22), we have

$W\left(q_n,q_{n+1},t\right)<W\left(q_{n-1},q_n,t\right)$.

Following the proof of Theorem 3.5, we find $q^{*}\in X$ such that $\lim_{n\to\infty} q_n=q^{*}$.

Suppose first that J is continuous. Then, by the construction of sequence $\{q_n\}$ it follows that J has a FP q^{*}. Suppose that F is continuous. Then, by (21), we have

$$F\left(W\left(q_{n+1},J\left(q^{*}\right),t\right)\right)<\frac{1}{\tau}F\left(W\left(q_{n+1},J\left(q^{*}\right),t\right)\right)$$

$$\leq F\left(max\left\{W\left(q_n,q^{*},t\right),W\left(q_n,q_{n+1},t\right),W\left(q^{*},J\left(q^{*}\right),t\right)\right\}\right), \tag{23}$$

for all $n\in N$ and $t>0$. If $J\left(q^{*}\right)=q^{*}$, then taking $n\to\infty$ from both sides of (23), we have

$$F\left(W\left(q^{*},J\left(q^{*}\right),t\right)\right)\leq\frac{1}{\tau}F\left(W\left(q^{*},J\left(q^{*}\right),t\right)\right)\leq F(max\left\{0,0,W\left(q^{*},J\left(q^{*}\right),t\right)\right\}=F\left(W(q^{*},J\left(q^{*}\right),t\right),$$

which means that $F\left(W(q^{*},J\left(q^{*}\right),t\right)=1$. This is in contradiction with $F\left(W(q^{*},J\left(q^{*}\right),t\right)<1$.

Finally, we prove the uniqueness of the FP. Assume that J has two distinct FPs, p^{*},q^{*}. Then, by (21), we have

$$F\left(W\left(p^{*},q^{*},t\right)\right)=F\left(W\left((J\left(p^{*}\right),J\left(q^{*}\right),t\right)\right)<\frac{1}{\tau}F\left(W\left((J\left(p^{*}\right),J\left(q^{*}\right),t\right)\right)$$

$$.\leq F\left(max\left\{W\left(p^{*},q^{*},t\right),W\left(p^{*},J\left(p^{*}\right),t\right),W\left(q^{*},J\left(q^{*}\right),t\right)\right\}\right),$$

$$=F\left(max\left\{W\left(p^{*},q^{*},t\right),W\left(p^{*},p^{*},t\right),W\left(q^{*},q^{*},t\right)\right\}\right)$$

$$=F\left(max\left\{W\left(p^{*},q^{*},t\right),0,0\right\}\right)=F\left(W\left(p^{*},q^{*},t\right)\right),$$

This is a contradiction. Therefore, $p^{*}=q^{*}$.

Refrences

[1] Alexander Sostak *"George-Veeramani Fuzzy Metrics Revised"* Axioms 2018,7,60.

[2] Chandok, S.; Huang, H.; Radenovi´c, S. "*Some fixed-point results for the generalized F-suzuki type contractions in b-metric spaces"*.Sahand Commun. Math. Anal. 2018, 11, 81–89.

[3] George.A, Veeramani. P. "*On some results in fuzzy metric spaces",* Fuzzy Sets and Syst. 64(1994), 395-399.

[4] Grabiec. M, *Fixed Points in fuzzy metric spaces*. Fuzzy Sets and Syst. 27(1988), 385-389.

[5] Gregori, V., Miñana, J.-J., Miravet, D. *Contractive sequences in fuzzy metric spaces*. Fuzzy Sets Syst. 2020, 379, 125–133.

[6] Gupta. V, Mani. N, A. Saini, *Fixed Points theorems and its applications in fuzzy metric spaces*, Conference Paper, 2013 (2013), 11 pages. 1, 1.6.

[7] Huaping Huang, Biljana Cari´c , Tatjana Došenovi´c , Dušan Raki´c and Mirjana Brdar, "*Fixed-Point Theorems in Fuzzy Metric Spaces via Fuzzy F-Contraction*", Mathematics 2021, 9, 641.

[8] Jehad R, madhan Kider *"Some Properties of Algebra Fuzzy Metric Space"*, Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.12(2) 2020 , pp Math 43–56 43.

[9] Jehad R, madhan Kider *"Application of Fixed Points in Algebra Fuzzy Normed Spaces"*, Journal of Physics: Conference Series 1879 (2021) 022099, IOP Publishing.

[10] J. Kramosil, J. Michalek: *Fuzzy metric and statistical metric spaces*. Kibernetika.11(1975), 336–344.

[11] Mihet D. Erratum to " *Fuzzy y-contractive mappings in non-Archimedean fuzzy metric spaces*, Fuzzy Sets and Systems. 159(2008), 739-744]". Fuzzy Sets Syst. 2010, 161, 1150–1151

[12] Muraliraj A, Thangathamizh R, Popovic N, Savic A, Radenovic S. The First Rational Type Revisd Fuzzy-Contractions in Revisd Fuzzy Metric Spaces with an Applications. Mathematics. 2023;11(10):2244.

[13] Muraliraj. A and Thangathamizh. R, "*Some topological properties of revised fuzzy cone metric space*", Ratio Mathematica, Volume 26, Number 2, (2023)

[14] Muraliraj A and Thangathamizh, "*New Relation-Theoretic Fixed Point Theorems in Revised Fuzzy Metric Spaces with an Application to Fractional Differential Equations*", Communications in Mathematics and Applications,Vol.12, No 22. (2023)

[15] Muraliraj. A and Thangathamizh. R, *"Fixed Point theorems in Revised Fuzzy Metric Space"*, Advances in Fuzzy Sets and Systems, Volume 26, Number 2, 2021.

[16] A. Muraliraj. and R. Thangathamizh, *"Introduction on Revised Fuzzy modular spaces"*, ISSN 0973-1768 Volume 17, Number 2 (2021), pp. 303-317.

[17] Thangathamizh R, Muraliraj A and Shanmugavel P *"New Approach of Lebesgue Integral in Revised Fuzzy Cone Metric Spaces vie Unique Coupled fixed point theorems",* Military Technical Courier, 3, 2024. Doi: 10.5937/vojtechg72-48816.

[18] Rhoades. E.B, *A Comparison of Various Definitions of Contractive Mappings*, *Trans. Amer. Math. Soc.* 226 (1977), 257-290.

[19] Secelean, N.A. *Iterated function system consisting of F-contractions*. Fixed Point Theory Appl. 2013, 2013, 277.

[20] B. Schweizer, A. Sklar: *Statistical metric spaces*. Pacific J. Math. 10(1960), 314–334.

[21] Shukla, S.; Gopal, D.; Sintunavarat, W. *A new class of fuzzy contractive mappings and fixed-point theorems*. Fuzzy Sets Syst.2018, 350, 85–94.

[22] Wardowski, D. *Fuzzy contractive mappings and fixed-points in fuzzy metric space*. Fuzzy Sets Syst. 2013, 222, 108–114.

[23] Wardowski, D. *Fixed Points of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. 2012, 2012,94.

[24] L. A. Zadeh: *Fuzzy sets*. Inf. Control., 8(1965), 338–353.

[25] Muraliraj A and Shanmugavel P, Thangathamizh R *"Existence of fixed point theorems in Revised fuzzy modular spaces",* Advances in nonlinear variational inequalities, 2024. **(Accepted for Publication)**

[26] Thangathamizh R, Balamurugan K, Karnan C, Shanmugavel P, and Balraj D, *"Revised fuzzy differential equations using weakly compatible self-mappings in revised fuzzy metric spaces",* Advances in nonlinear variational inequalities, 2024. **(Accepted for Publication)**

[27] Thangathamizh R, Abdelhamid Moussaoui, Tatjana Dosenovic, Stojan Radenovic *"Fixed Point Results in Controlled Revised Fuzzy Metric spaces with an Application to the Transformation of Solar Energy to Electric Power",* Military Technical Courier, 4,2024. Doi.10.5937/vojtehg72-49064.

[28] Muraliraj A and Shanmugavel P, Thangathamizh R *"Fixed point theorems in Molular Revised fuzzy modular spaces",* Advances in nonlinear variational inequalities, 2024. **(Accepted for Publication)**