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Fixed Point Theorems on Modular Revised Fuzzy Metric Spaces

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Abstract:

In this paper, we present a new space which is a melange between a revised fuzzy metric space and a modular metric space. We state some properties and examples of our new space. Then, we formulate and prove the existence and uniqueness results of a fixed point for continuous mappings under this new space. To support our results, we introduce some examples and an application.

Keywords: Modular metric space, Revised fuzzy metric space, Modular revised fuzzy metric space, fixed point.

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1. Introduction

In 1965, Zadeh [24] presented the concept of a fuzzy set. Ten years later, Kramosil and Michalek [12] stated the definition of fuzzy metric spaces. In 1988, Grabiec [7] implemented the notion of fuzzy metric space to extend the Banach contraction theorem over this space. Posteriorly, George and Veeramani [6] employed the definition of t-norm to formulate and introduce some results on the notion of a fuzzy metric space. Then, several researchers presented different contraction conditions over fuzzy metric spaces.

In 2010, Chistyakov [3-5] introduced the notion of modular metric spaces. Then, numerous mathematicians discussed different results in their works over modular metric spaces, for example, look at the references [16, 18-19, 21,2-24].Muraliraj and Thangathamizh [17] used the revised fuzzy set technique to start a family of revised fuzzy mappings that are extensions of multivalued mappings and produced a result in revised fuzzy metric space for these mappings in 2021. Muraliraj and Thangathamizh [16] were the first to establish the idea of revised fuzzy contractive mappings and show a fixed point theorem in revised fuzzy metric spaces for these mappings. The rational type revised fuzzy-contraction condition in RFM spaces was recently established by Muraliraj et al. [14], who also proved other FP theorems with an application. The concept of revised fuzzy cone metric (RFCM) space was first presented in 2023 by Thangathamizh et al. [15]. Under the presumption that "the revised fuzzy cone contractive sequences are Cauchy," they demonstrated a few fundamental features of FP as well as a "revised fuzzy cone Banach contraction theorem." Afterwards, several FP theorems in RFCMS were proven by Muraliraj and Thangathamizh [20, 25-27] without requiring that the "Revised fuzzy cone contractive sequences are Cauchy.

In this paper, we introduce a new space named modular revised fuzzy metric space. We launch some fixed point results over a modular fuzzy metric spaces. To analyse our work, we state some examples, corollaries, and an application.

2. Preliminaries

In this section, we will recall some definitions which are crucial in this paper.

Definition 2.1. [24]

Let Y *be any set. A fuzzy set E in* Y *is a function with domain* Y *and values in* [0,1].

Definition 2.2. [23]

Given a binary operation $\circledast : [0,1]^2 \to [0,1]$. *An operator* $\circledast$ *is a continuous t-conorm if* $\forall \alpha, \beta, \gamma, \delta \in [0,1]$ *satisfy:*

(1) $\alpha \circledast \beta = \beta \circledast \alpha$.

(2) $(\alpha \circledast \beta) \circledast \gamma = \alpha \circledast (\beta \circledast \gamma)$.

(3) $\alpha \circledast 0 = \alpha$.

(4) *If* $\alpha \leq \gamma$ *and* $\beta \leq \delta$, *then* $\alpha \circledast \beta \leq \gamma \circledast \delta$.

A. Sostak [2] in 2018 introduced the concept of a revised fuzzy metric space using the defintion of t-conorm as follows:

Definition 2.3. [2]

The triplet $(Y, \Lambda, \circledast)$ *is called a revised fuzzy metric space if* Y *is an arbitrary set,* $\circledast$ *is a continuous t-conorm and* Λ *is a revised fuzzy metric on* $Y^2 \times (0,\infty) \to [0,1]$ *for all* ι, η, ϑ *in* Y, *and for all* $s, t > 0$ *satisfying the following conditions:*

(1) $\Lambda(\iota,\eta,0) = 0, \Lambda(\iota,\eta,t) < 1, \forall t > 0$.

(2) $\Lambda(\iota,\eta,t) = 0$ *if and only if* $\iota = \eta$, *for all* $t > 0$.

(3) $\Lambda(\iota,\eta,t) = \Lambda(\eta,\iota,t)$.

(4) $\Lambda(\iota,\eta,t) \circledast \Lambda(\eta,\vartheta,s) \geq \Lambda(\iota,\vartheta,t+s)$.

(5) $\Lambda(\iota,\eta,.) : (0,\infty) \to [0,1]$ *is right continuous. Here,* Λ *called a revised fuzzy metric on* Y.

Example 2.1. [2]

Let (Y, d) *be a metric space. Define* $\alpha \circledast \beta = \alpha + \beta - \alpha\beta$ *for all* $\alpha, \beta \in [0,1]$, *and* $\Lambda : Y^2 \times (0,\infty) \to [0,1]$ *as*

$$\Lambda(\iota,\eta,t) = \frac{d(\iota,\eta)}{t+ d(\iota,\eta)}$$

$\forall \iota,\eta \in \Xi$ *and* $t > 0$. *Then* $(\Xi,\Lambda,\circledast)$ *is a revised fuzzy metric space; called revised fuzzy metric induced by the metric* d.

The notions of convergence, completeness and compactness on revised fuzzy metric spaces were presented in [17] as follows:

Definition 2.4. [17]

Let $(\Xi,\Lambda,\circledast)$ *be a revised fuzzy metric space.*

(1) *A sequence* $\{\iota_\kappa\}_{\kappa\in N}$ *in* Y *is convergent to an element* $\iota \in Y$ *if* $\lim_{\kappa\to\infty} \Lambda(\iota_\kappa,\iota,t) = 0$, *for all* $t > 0$.

(2) *A sequence* $\{\iota_\kappa\}_{\kappa\in N}$ *in* Y *is Cauchy if for all* $0 < \varepsilon < 1$ *and for* $t > 0$, *there exists a number* $\kappa_0 \in N$ *such that* $\Lambda(\mathrm{L_k}, \mathrm{l_Y}, \mathrm{t}) < \varepsilon$ *for each* $\kappa, Y \geq \kappa_0$.

(3) *A revised fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

(4) *A revised fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.*

In 2010, Chistyakov [3-8] defined the notion of modular metric spaces as follows:

Definition 2.5. [3]

A modular metric on a nonempty set Y *is a function* $\Theta: (0,\infty) \times Y^2 \to [0,\infty)$ *that will be written as* $\Theta_{\%}(\iota,\eta) = \Theta(\%,\iota,\eta)$; *for all* $\iota,\eta,\vartheta \in Y$ *and for all* $\%,\sigma > 0$, *satisfy the following three conditions:*

(1) $\Theta_{\%}(\iota,\eta) = 0$ *if and only if* $\iota = \eta$, $\forall\ \% > 0$ *and* $\iota,\eta \in Y$.

(2) $\Theta_{\%}(\iota,\eta) = \Theta_{\%}(\eta,\iota)$, $\forall \% > 0$ *and* $\iota,\eta \in Y$.

(3) $\Theta_{\%+\sigma}(\iota,\eta) \leq \Theta_{\%}(\iota,\vartheta) + \Theta_\sigma(\vartheta,\eta)$; *for all* $\%,\sigma > 0$ *and* $\iota,\eta,\vartheta \in Y$.

Remark 2.1.

Let Θ *be modular on a set* Y. *Then for given* $\iota,\eta \in Y$, *the function* $0 < \% \to \Theta_{\%}(\iota,\eta) \in (0,\infty)$ *is non increasing on* $(0,\infty)$. *In fact if* $0 < \% < \sigma$, *then by above definition*

$$\Theta_\sigma(\iota,\eta) \leq \Theta_{\sigma-\%}(\iota,\iota) + \Theta_{\%}(\iota,\eta) = \Theta_{\%}(\iota,\eta) \text{ for all } \iota,\eta \in Y.$$

Definition 2.6. [5]

Given a modular Θ *on* Y, *a sequence* $\{\iota_\kappa\}_{\kappa\in N}$ *in* Y_Θ *is said to be modular convergent to an element* $\iota \in Y_\Theta$ *if there exists a number* $\% > 0$, *possibly depending on* $\{\iota_\kappa\}$ *and* ι, *such that*

$$\lim_{\kappa\to\infty} \Theta_{\%}(\iota_\kappa,\iota) = 0. \text{ i.e } \iota_\kappa \to \iota \text{ as } \kappa \to \infty.$$

Definition 2.7. [5]

Given a modular Θ *on* Y, *a sequence* $\{\iota_\kappa\}_{\kappa\in N}$ *in* Y_Θ *is said to be modular Cauchy if there exists a number* $\% = \%(\{\iota_\kappa\}) > 0$ *such that* $\lim_{\kappa,\xi\to\infty} \Theta_{\%}(\iota_\kappa,\iota_\xi) = 0$.

Definition 2.8. [5]

A modular space Y_{Θ} is said to be modular complete if each Cauchy sequence in Y_{Θ} is modular convergent. In fact, if $\{\iota_\kappa\} \subset Y_{\Theta}$ and there exists $\% = \%(\{\iota_\kappa\}) > 0$ such that $\lim_{\kappa,Y\to\infty} \Theta_{\%}(\iota_\kappa, \iota_Y) = 0$, then there exists $\iota \in Y_{\Theta}$, such that $\lim_{\kappa\to\infty} \Theta_{\%}(\iota_\kappa, \iota) = 0$.

Definition 2.9.

A modular Θ on Y is said to be satisfied the Δ_2-condition if $\lim_{n\to\infty} \Theta_{\%}(\iota_\kappa, \iota) = 0$, for some $\% > 0$ implies that $\lim_{n\to\infty} \Theta_{\%}(\iota_\kappa, \iota) = 0$, for all $\% > 0$.

3. Main results

In this section, we construct a new space called a modular revised fuzzy metric space. We present some examples of this space. Also, we formulate and prove some new fixed point results under this space. We start by presenting the following definitions.

Definition 3.1.

A modular revised fuzzy metric space is the triplet $(Y, \zeta_{\%}, \circledast)$ such that Y is an arbitrary set, $(\circledast)$ is a continuous t-conorm and $\zeta_{\%}$ is a revised fuzzy metric on $(0,\infty) \times Y^2 \times (0,\infty) \to [0,1]$; for all ι, η, ϑ in Y, and $s, t > 0$ satisfying the following conditions:

(1) $\zeta_{\%}(\iota, \eta, 0) = 0$, $\zeta_{\%}(\iota, \eta, t) < 1$, *for all* $t, \% > 0$.

(2) $\zeta_{\%}(\iota, \eta, t) = 0$ *if and only if* $\iota = \eta$, *for all* $t, \% > 0$.

(3) $\zeta_{\%}(\iota, \eta, t) = \zeta_{\%}(\eta, \iota, t)$, *for all* $t, \% > 0$.

(4) $\zeta_{\sigma}(\iota, \eta, t) \circledast \zeta_{\%}(\eta, \vartheta, s) \leq \zeta_{\sigma+\%}(\iota, \vartheta, t+s)$, *for all* $t, s, \sigma, \% > 0$.

(5) $\zeta_{\%}(\iota, \eta, .) : (0,\infty) \to [0,1]$ *is right continuous. Here,* $\zeta_{\%}$ *is called a modular revised fuzzy metric.*

Definition 3.2.

Let $(Y, \zeta_{\%}, \circledast)$ be a modular revised fuzzy metric space.

(1) *A sequence $\{\iota_\kappa\}_{\kappa\in N}$ in Y is convergent to an element $\iota \in Y$ if $\lim_{\kappa\to\infty} \zeta_{\%}(\iota_\kappa, \iota, t) = 0$ for all $t > 0$ and some $\% > 0$.*

(2) *A sequence $\{\iota_\kappa\}_{\kappa\in N}$ in Y is Cauchy if for all $0 < \varepsilon < 1$, there exists a number $\kappa_0 \in N$ such that $\zeta_{\varrho}(\iota_\kappa, \iota_\xi, t) < \varepsilon$, for each $\kappa, Y \geq \kappa_0$ and some $\% > 0$.*

(3) *A modular revised fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

(4) *A modular revised fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.*

Definition 3.3.

A revised fuzzy modular metric $\zeta_{\%}$ *on* Y *is said to be satisfied the* Δ_{2t}*-condition if* $\lim_{\kappa\to\infty} \Lambda(\iota_\kappa, \iota, t) = 0$, *for some* $\% > 0$ *and for some* $t > 0$ *imply that* $\lim_{\kappa\to\infty} \zeta_{\%}(\iota_\kappa, \iota, t) = 0$, *for all* $\% > 0$, *and for all* $t > 0$.

Example 3.1.

Let $(Y, \Theta_{\%})$ *be a modular metric space. Define* $\alpha * \beta = \alpha + \beta - \alpha\beta$ *for all* $\alpha, \beta \in [0,1]$, *and* $\zeta_{\%}: (0, \infty) \times Y^2 \times (0, \infty) \to [0,1]$ *by*

$$\zeta_{\varrho}(\iota, \eta, t) = \frac{\Theta_{\varrho}(\iota, \eta)}{t + \Theta_{\varrho}(\iota, \eta)}$$

Then $(Y, \zeta_{\%}, \circledast)$ *is a modular revised fuzzy metric space.*

Remark 3.1.

(1) *For* $\Theta_{\varrho}(\iota, \eta) = \frac{|\iota-\eta|}{\varrho}$, *we have* $\zeta_{\varrho}(\iota, \eta, t) = \frac{\frac{|\iota-\eta|}{\varrho}}{t + \frac{|\iota-\eta|}{\varrho}}$ *is a modular revised fuzzy metric.*

(2) *For* $\Theta_{\varrho}(\iota, \eta) = \frac{|\iota-\eta|}{\varrho+1}$, *we have* $\zeta_{\varrho}(\iota, \eta, t) = \frac{\frac{|\iota-\eta|}{\varrho+1}}{t + \frac{|\iota-\eta|}{\varrho+1}}$ *is a modular revised fuzzy metric.*

Example 3.2.

Let $(Y, \Theta_{\%})$ *be a modular metric space. Define* $\alpha * \beta = \alpha + \beta - \alpha\beta$ *for all* $\alpha, \beta \in [0,1]$, *and* $\zeta_{\%}: (0, \infty) \times Y^2 \times (0, \infty) \to [0,1]$ *by*

$$\zeta_{\varrho}(\iota, \eta, t) = exp^{-\{\Theta_{\varrho}(\iota,\eta)\}}\left(1 - exp^{\{\Theta_{\varrho}(\iota,\eta)\}}\right)$$

Then $(Y, \zeta_{\%}, \circledast)$ *is a modular revised fuzzy metric space.*

Remark 3.2.

(1) *For* $\Theta_{\varrho}(\iota, \eta) = \frac{|\iota-\eta|}{\varrho}$, *we have* $\zeta_{\varrho}(\iota, \eta, t) = exp^{-\left\{\frac{|\iota-\eta|}{\varrho}\right\}}\left(1 - exp^{\left\{\frac{|\iota-\eta|}{\varrho}\right\}}\right)$ *is a modular revised fuzzy metric.*

(2) *For* $\Theta_{\varrho}(\iota, \eta) = \frac{|\iota-\eta|}{\varrho+1}$, *we have* $\zeta_{\varrho}(\iota, \eta, t) = exp^{-\left\{\frac{|\iota-\eta|}{\varrho+1}\right\}}\left(1 - exp^{\left\{\frac{|\iota-\eta|}{\varrho+1}\right\}}\right)$ *is a modular revised fuzzy metric.*

Theorem 3.1.

On a complete modular revised fuzzy metric space $(Y, \zeta_{\%}, \circledast)$, *consider a continuous mapping* $\Gamma: Y \to Y$. *Suppose there exist a strictly non-decreasing, continuous function* $\Upsilon: (0,1] \to [0, \infty)$ *with* $\Upsilon(1) = 0$ *and a real number* H *with* $0 < H < 1$ *such that*

$$\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_\iota, \Gamma_\eta, t)} - 1\right) \leq H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota, \eta, t)} - 1\right), \text{for all } \iota, \eta \in Y, \iota \neq \eta. \quad (\mathbf{3}.1)$$

Then Γ *has a unique fixed point in* Y.

Proof.

Let ι_0 be an arbitrary point in Y. Choose $\iota_1 \in Y$ such that $\iota_1 = \Gamma\iota_0$. Continuing this process, we construct a sequence (ι_κ) such that $\iota_{\kappa+1} = \Gamma\iota_\kappa$, for $\kappa = 0,1,2..$ Let $\iota = \iota_{\kappa-1}$ and $\eta = \iota_\kappa$. Replacing this in (**3**.1), we get

$$\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_\iota,\Gamma_\eta,t)} - 1\right) = \Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_{\iota_{\kappa-1}},\Gamma_{\iota_\kappa},t)} - 1\right) = \Upsilon\left(\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)} - 1\right)$$

$$\leq H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_\kappa,t)} - 1\right) < \Upsilon\left(\frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_\kappa,t)} - 1\right). \quad (\mathbf{3}.2)$$

Since Y is a strictly non-decreasing function, we obtain

$$\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)} - 1 < \frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_\kappa,t)} - 1. \quad (\mathbf{3}.3)$$

We use the same method for $\iota = \iota_{\kappa-2}$ and $\eta = \iota_{\kappa-1}$, we get

$$\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa-1},t)} - 1 < \frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_{\kappa-2},t)} - 1. \quad (\mathbf{3}.4)$$

Therefore, (**3**.3) and (**3**.4) imply that $\zeta_{\%}(\iota_\kappa,\iota_{\kappa-1},t)$ is a strictly non-increasing sequence of positive real numbers in [0,1].

Put $\Sigma_\kappa(\%,t) = \zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)$. Then $\{\Sigma_\kappa(\%,t)\}$ is a strictly non-increasing sequence. So $\exists\ \Sigma(\%,t)$ such that $\lim_{\kappa\to\infty}\Sigma_\kappa(\%,t) = \Sigma(\%,t)$. Assume that $0 < \Sigma(\%,t) < 1$. By (**3**.2), we have

$$\Upsilon(\Sigma_\kappa(\%,t)) \leq H\Upsilon(\Sigma_{\kappa-1}(\%,t)).$$

So,

$$\lim_{\kappa\to\infty}\Upsilon(\Sigma_\kappa(\%,t)) \leq \lim_{\kappa\to\infty}H\Upsilon(\Sigma_{\kappa-1}(\%,t)).$$

The continuity of Υ implies that

$\Upsilon(\Sigma(\%,t)) \leq H\Upsilon(\Sigma(\%,t))$, a contradiction. Then $\Sigma(\%,t) = 0$.

Now, we will prove that $\{\iota_\kappa\}$ is a Cauchy sequence. Assume not, then for $0 < \varepsilon < 1$,

there exist two sub-sequences $\{\iota_{Y(i)}\}$ and $\{\iota_{\kappa(i)}\}$ such that for each $i \in N$.

let $\kappa(i), Y(i) \in N$ satisfying $\kappa(i), Y(i) \geq \kappa$ and $\kappa(i) < Y(i) < i$, such that

$$\frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1 \geq \varepsilon, \frac{1}{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)-1},t)} - 1 < \varepsilon, \frac{1}{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)},t)} - 1 < \varepsilon. \quad (\mathbf{3}.5)$$

Consider

$$\varepsilon \leq \frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1 \leq \frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\xi(i)-1},\frac{t}{2}\right)} - 1 \circledast \frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)-1},\iota_{\xi(i)},\frac{t}{2}\right)} - 1.$$

By definition of Δ_{2t}-condition on Y and (**3**.5), we have

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$$\frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)-1},\iota_{\xi(i)},\frac{t}{2}\right)}-1<\varepsilon.$$

Hence,

$$\varepsilon \le \frac{1}{\zeta_{\varrho}\left(\iota_{\kappa(i)},\iota_{\xi(i)},t\right)}-1 \le \frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\kappa(i)-1},\frac{t}{2}\right)}-1 \circledast \varepsilon.$$

If $i \to \infty$, we have $\Sigma_{\kappa(i)}\left(\frac{\varrho}{2},\frac{t}{2}\right)=\frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\kappa(i)-1},\frac{t}{2}\right)}-1 \to 0.$

So, $\frac{1}{\zeta_{\varrho}\left(\iota_{\kappa(i)},\iota_{\xi(i)},t\right)}-1 \to \varepsilon.$

Then by (**3**.1), we have

$$\Upsilon\left(\frac{1}{\zeta_{\%}\left(\iota_{\kappa(i)},\iota_{Y(i)},t\right)}-1\right) \le H\Upsilon\left(\frac{1}{\zeta_{\%}\left(\iota_{\kappa(i)-1},\iota_{Y(i)-1},t\right)}-1\right) < \Upsilon\left(\frac{1}{\zeta_{\%}\left(\iota_{\kappa(i)-1},\iota_{Y(i)-1},t\right)}-1\right).$$ Thus

$$\varepsilon \le \frac{1}{\zeta_{\varrho}\left(\iota_{\kappa(i)},\iota_{\xi(i)},t\right)}-1 < \frac{1}{\zeta_{\varrho}\left(\iota_{\kappa(i)},\iota_{\xi(i)-1},\frac{t}{2}\right)}-1<\varepsilon,$$

which is impossible.

Hence $\{\iota_\kappa\}$ is a Cauchy sequence in a complete modular revised fuzzy metric space.

So $\exists \$ \in Y$ such that $lim\, \iota_\kappa = \$$, that means $\lim_{\kappa\to\infty} \zeta_{\%}(\iota_\kappa, \$, t)=0.$

To show \$ is a fixed point of Γ, we have :

Γ is continuous: $\iota_\kappa \to \$ \Rightarrow \Gamma\iota_\kappa \to \Gamma\$$. By (**3**.1), we have

$$\Upsilon\left(\frac{1}{\zeta_{\%}(\iota_\kappa,\Gamma\iota_\kappa,t)}-1\right) \le H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_\kappa,t)}-1\right).$$

Since $\Upsilon(1)=0$ and for $\kappa\to\infty$, we get

$\Upsilon\left(\frac{1}{\zeta_{\%}(\$,\Gamma\$,t)}-1\right) \le H\Upsilon\left(\frac{1}{\zeta_{\%}(\$,\$,t)}-1\right)=H\Upsilon(1)=0.$ So $\zeta_{\%}(\$,\Gamma\$,t)=0.$

Hence, $\zeta_{\%}(\$,\Gamma\$,t)=0 \Rightarrow \Gamma\$=\$$. Thus \$ is a fixed point of Γ.

Now, we will prove that \$ is unique.

Assume not, $\exists\omega\in Y$, such that $\Gamma\omega=\omega$ where $\omega\neq\$$ and $\lim_{\kappa\to\infty}\iota_\kappa=\omega$. Then

$$\Upsilon\left(\frac{1}{\zeta_{\varrho}(\omega,\varpi,t)}-1\right)=\Upsilon\left(\frac{1}{\zeta_{\varrho}(\Gamma\omega,\Gamma\varpi,t)}-1\right) \le H\Upsilon\left(\frac{1}{\zeta_{\varrho}(\omega,\varpi,t)}-1\right)$$
$$\le H\Upsilon\left(\zeta_{\frac{\varrho}{2}}\left(\omega,\iota_\kappa,\frac{t}{2}\right)\circledast\zeta_{\frac{\varrho}{2}}\left(\iota_\kappa,\varpi,\frac{t}{2}\right)\right).$$

Since $\Upsilon(1)=0$ and for $\kappa\to\infty$ on both sides, we have

$$\Upsilon\left(\frac{1}{\zeta_{\varrho}(\omega,\varpi,t)}-1\right) \le H\Upsilon\left(\zeta_{\frac{\varrho}{2}}\left(\omega,\iota_\kappa,\frac{t}{2}\right)\circledast\zeta_{\frac{\varrho}{2}}\left(\iota_\kappa,\varpi,\frac{t}{2}\right)\right)=H\Upsilon(1)=0.$$

So, $\Upsilon\left(\frac{1}{\zeta_{\%}(\omega,\$,t)}-1\right)=0$.

Hence, $\zeta_{\%}(\omega,\$,t)=0 \Rightarrow w=\$$. Thus Γ has a unique fixed point *\$*.

Theorem 3.2.

On a complete modular revised fuzzy metric space $(Y,\zeta_{\%},\circledast)$, *consider a continuous mapping* $\Gamma: Y \to Y$. *Suppose there exist a strictly non-decreasing, continuous function* $\Upsilon: (0,1] \to [0,\infty)$ *with* $\Upsilon(1)=0$ *and a real number* H *with* $0<H<1$ *such that*

$$\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma\iota,\Gamma\eta,t)}-1\right) \leq H\left(\frac{1}{\frac{\zeta_{\varrho}(\iota,\eta,t)+\zeta_{\varrho}(\Gamma\iota,\iota,t)}{4}}-1+\frac{1}{\frac{\zeta_{\varrho}(\eta,\Gamma\eta,t)}{2}}-1\right) \quad (\mathbf{3}.6)$$

for all $\iota,\eta \in Y, \iota \neq \eta$. *Then* Γ *has a unique fixed point in* Y.

Proof.

Let ι_0 be an arbitrary point in Y. Choose $\iota_1 \in Y$ such that $\iota_1 = \Gamma\iota_0$. Continuing this process, we construct a sequence (ι_κ) such that $\iota_{\kappa+1}=\Gamma\iota_\kappa$, for κ = 0,1,2..

Let $\iota=\iota_{\kappa-1}$ and $\eta=\iota_\kappa$. Replacing this in (**3**.6), we get

$$\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma\iota,\Gamma\eta,t)}-1\right)=\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma\iota_{\kappa-1},\Gamma\iota_\kappa,t)}-1\right)=\Upsilon\left(\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)}-1\right)$$

$$\leq \Upsilon H\left(\frac{1}{\frac{\zeta_{\varrho}(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_{\varrho}(\Gamma\iota_{\kappa-1},\iota_{\kappa-1},t)}{4}}-1+\frac{1}{\frac{\zeta_{\varrho}(\iota_\kappa,\Gamma\iota_\kappa,t)}{2}}-1\right)$$

$$< \Upsilon\left(\frac{1}{\frac{\zeta_{\varrho}(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_{\varrho}(\Gamma\iota_{\kappa-1},\iota_{\kappa-1},t)}{4}}-1+\frac{1}{\frac{\zeta_{\varrho}(\iota_\kappa,\Gamma\iota_\kappa,t)}{2}}-1\right)$$

$$= \Upsilon\left(\frac{1}{\frac{\zeta_{\varrho}(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_{\varrho}(\iota_\kappa,\iota_{\kappa-1},t)}{4}}-1+\frac{1}{\frac{\zeta_{\varrho}(\iota_\kappa,\iota_{\kappa+1},t)}{2}}-1\right) \quad (\mathbf{3}.7)$$

Since Υ is a strictly non-decreasing function, we obtain

$$\frac{1}{\zeta_{\varrho}(\iota_\kappa,\iota_{\kappa+1},t)}-1 < \Upsilon\left(\frac{1}{\frac{\zeta_{\varrho}(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_{\varrho}(\iota_\kappa,\iota_{\kappa-1},t)}{4}}-1+\frac{1}{\frac{\zeta_{\varrho}(\iota_\kappa,\iota_{\kappa+1},t)}{2}}-1\right).$$

Hence

$$\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)}-1 < \frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa-1},t)}-1. \quad (\mathbf{3}.8)$$

We use the same method for $\iota=\iota_{\kappa-2}$ and $\eta=\iota_{\kappa-1}$, we get

$$\frac{1}{\zeta_{\%}(\iota_\kappa,\iota_{\kappa-1},t)}-1 < \frac{1}{\zeta_{\%}(\iota_{\kappa-1},\iota_{\kappa-2},t)}-1. \quad (\mathbf{3}.9)$$

Therefore, (**3**.8) and (**3**.9) imply that $\{\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)\}$ is a strictly non-increasing sequence of positive real numbers in $[0,1]$.

Put $\Sigma_\kappa(\%,t)=\zeta_{\%}(\iota_\kappa,\iota_{\kappa+1},t)$. Then $\{\Sigma_\kappa(\%,t)\}$ is a strictly non-increasing sequence.

So $\exists\, \Sigma(t,\%)$ such that $\lim \Sigma_\kappa(\%,t) = \Sigma(t,\%)$.

Assume that

$0 < \Sigma(t,\%) \ < \ 1.$

By (**3**.7), we have

$\Upsilon\big(\Sigma_\kappa(\varrho,t)\big) \le \Upsilon H\left(\frac{\Sigma_{\kappa-1}(\varrho,t)}{2} + \frac{\Sigma_\kappa(\varrho,t)}{2}\right).$

So,

$\lim_{\kappa\to\infty} \Upsilon\big(\Sigma_\kappa(\varrho,t)\big) \le \lim_{\kappa\to\infty} \Upsilon H\left(\frac{\Sigma_{\kappa-1}(\varrho,t)}{2} + \frac{\Sigma_\kappa(\varrho,t)}{2}\right)$

By the continuity of Υ, we have

$\Upsilon\big(\Sigma(\varrho,t)\big) \le \Upsilon H\left(\frac{\Sigma(\varrho,t)}{2} + \frac{\Sigma(\varrho,t)}{2}\right)$, a contradiction. Then $\Sigma(\%,t) = 0$.

Now, we will prove that $\{\iota_\kappa\}$ is a Cauchy sequence.

Assume not, then for $0 < \varepsilon < 1$, there exists two sub-sequences $\{\iota_{\Upsilon(i)}\}$ and $\{\iota_{\kappa(i)}\}$ such that for each $i \in N$, let $\kappa(i), \Upsilon(i) \in N$ satisfying $\kappa(i), \Upsilon(i) \ge \kappa$ and $\kappa(i) > \Upsilon(i) > i$, such that

$\frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1 \ge \varepsilon, \frac{1}{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)-1},t)} - 1 < \varepsilon, \frac{1}{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)},t)} - 1 < \varepsilon.$
(**3**.10)

Consider

$\varepsilon \le \left(\frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1\right) \le \left(\frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\kappa(i)-1},\frac{t}{2}\right)} - 1\right) \circledast \left(\frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)-1},\iota_{\xi(i)},\frac{t}{2}\right)} - 1\right).$

By definition of Δ_{2t}-condition on Y and (**3**.10), we have

$\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)-1},\iota_{\xi(i)},\frac{t}{2}\right) < \varepsilon.$

Thus

$\varepsilon \le \left(\frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1\right) \le \left(\frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\kappa(i)-1},\frac{t}{2}\right)} - 1\right) \circledast \varepsilon.$

If $i \to \infty$, we have $\Sigma_{\kappa(i)}\left(\frac{\varrho}{2},\frac{t}{2}\right) \le \frac{1}{\zeta_{\frac{\varrho}{2}}\left(\iota_{\kappa(i)},\iota_{\kappa(i)-1},\frac{t}{2}\right)} - 1 \to 0.$.

So

$\zeta_\varrho\big(\iota_{\kappa(i)},\iota_{\xi(i)},t\big) \to \varepsilon.$

Then by (**3**.6), we have

$\Upsilon\left(\frac{1}{\zeta_\varrho(\iota_{\kappa(i)},\iota_{\xi(i)},t)} - 1\right) \le \Upsilon H\left(\frac{1}{\frac{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)-1},t)+\zeta_\varrho(\Gamma\iota_{\kappa(i)-1},\iota_{\kappa(i)-1},t)}{4}} - 1 + \frac{1}{\frac{\zeta_\varrho(\iota_{\xi(i)-1},\Gamma\iota_{\xi(i)-1},t)}{2}} - 1\right)$

$$< \Upsilon\left(\frac{1}{\frac{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)-1},t)+\zeta_\varrho(\Gamma\iota_{\kappa(i)-1},\iota_{\kappa(i)-1},t)}{4}} - 1 + \frac{1}{\frac{\zeta_\varrho(\iota_{\xi(i)-1},\Gamma\iota_{\xi(i)-1},t)}{2}} - 1\right)$$

$$= \Upsilon\left(\frac{1}{\frac{\zeta_\varrho(\iota_{\kappa(i)-1},\iota_{\xi(i)-1},t)+\zeta_\varrho(\iota_{\kappa(i)},\iota_{\kappa(i)-1},t)}{4}} - 1 + \frac{1}{\frac{\zeta_\varrho(\iota_{\xi(i)-1},\iota_{\xi(i)},t)}{2}} - 1\right)$$

Since Υ is a strictly decreasing function, (**3**.10) implies that

$\varepsilon < \frac{\varepsilon+0}{4} + \frac{1}{2} = \frac{\varepsilon}{4} + \frac{1}{2} < \varepsilon,$

which is impossible.

Hence $\{\iota_\kappa\}$ is a Cauchy sequence in a complete modular revised fuzzy metric space.

So $\exists \$ \in Y$ such that $\lim_{\kappa\to\infty} \iota_\kappa = \$$, that means $\lim_{\kappa\to\infty} \zeta_{\%}(\iota_\kappa, \$, t) = 0$.

To show \$ is a fixed point of Γ,

we have :

Γ is continuous: $\iota_\kappa \to \$ \Rightarrow \Gamma\iota_\kappa \to \Gamma\$$.

By (**3**.6), we have

$$\Upsilon\left(\frac{1}{\zeta_\varrho(\iota_\kappa,\Gamma\iota_\kappa,t)} - 1\right) \leq \Upsilon H\left(\frac{1}{\frac{\zeta_\varrho(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_\varrho(\Gamma\iota_{\kappa-1},\iota_{\kappa-1},t)}{4}} - 1 + \frac{1}{\frac{\zeta_\varrho(\iota_\kappa,\Gamma\iota_\kappa,t)}{2}} - 1\right)$$

$$= H\Upsilon\left(\frac{1}{\frac{\zeta_\varrho(\iota_{\kappa-1},\iota_\kappa,t)+\zeta_\varrho(\iota_\kappa,\iota_{\kappa-1},t)}{4}} - 1 + \frac{1}{\frac{\zeta_\varrho(\iota_\kappa,\Gamma\iota_\kappa,t)}{2}} - 1\right)$$

Since Y(1) = 0 and for $\kappa \to \infty$, we get

$$\Upsilon\left(\frac{1}{\zeta_\varrho(\omega,\Gamma\varpi,t)} - 1\right) = H\Upsilon\left(\left(\frac{1}{\frac{\zeta_\varrho(\omega,\varpi,t)+\zeta_\varrho(\omega,\varpi,t)}{4}} - 1\right) + \left(\frac{1}{\frac{\zeta_\varrho(\omega,\varpi,t)}{2}} - 1\right)\right)$$

$$= H\Upsilon\left(\frac{1}{\zeta_\varrho(\omega,\Gamma\varpi,t)} - 1\right) = H\Upsilon(1) = 0.$$

Hence, $\zeta_{\%}(\$,\Gamma\$,t) = 0 \Rightarrow \Gamma\$ = \$$. Thus \$ is a fixed point of Γ.

Now, we will prove that \$ is unique.

Assume not, $\exists w \in Y$, such that $\Gamma w = w$ where $w \neq \$$ and $\lim_{\kappa\to\infty} \iota_\kappa = w$. By (**3**.6), we have

$$\Upsilon\left(\frac{1}{\zeta_\varrho(\omega,\Gamma\varpi,t)} - 1\right) = \Upsilon\left(\frac{1}{\zeta_\varrho(\Gamma\omega,\Gamma\varpi,t)} - 1\right)$$

$$\leq H\Upsilon\left(\left(\frac{1}{\frac{\zeta_\varrho(\omega,\varpi,t)+\zeta_\varrho(\Gamma\omega,\varpi,t)}{4}} - 1\right) + \left(\frac{1}{\frac{\zeta_\varrho(\omega,\Gamma\varpi,t)}{2}} - 1\right)\right) = H\Upsilon\left(\frac{1}{\frac{\zeta_\varrho(\omega,\Gamma\varpi,t)}{4}} - \frac{1}{2}\right)$$

Hence, $\zeta_{\%}(w,\$,t) \leq 0$.

Thus, $\zeta_{\%}(w,\$,t) = 0 \Rightarrow w = \$$. So Γ has a unique fixed point \$.

The two following examples satisfy Theorem (**3**.1).

Example 3.3.

Let $Y = [0,1]$ *and* $\zeta_{\varrho}(\iota,\eta,t) = \frac{\frac{|\iota-\eta|}{\varrho}}{t+\frac{|\iota-\eta|}{\varrho}}$. *Define* $\Gamma: [0,1] \to [0,1]$ *via* $\Gamma(\iota) = \frac{\iota}{3}$.

Also, define $\Upsilon: (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = \frac{1}{\iota} - 1$. *Note that* Υ *is a strictly non-decreasing, continuous function and* $\Upsilon(1) = 0$. *Now, we have:*

$\zeta_{\varrho}(\Gamma\iota,\Gamma\eta,t) = \frac{|\iota-\eta|}{3\varrho t+|\iota-\eta|}$, $\zeta_{\varrho}(\iota,\eta,t) = \frac{|\iota-\eta|}{3\varrho t+|\iota-\eta|}$

$\Upsilon\left(\frac{1}{\zeta_{\varrho}(\Gamma\iota,\Gamma\eta,t)} - 1\right) = \zeta_{\varrho}(\Gamma\iota,\Gamma\eta,t) = \frac{|\iota-\eta|}{3\varrho t}$; $\Upsilon\left(\frac{1}{\zeta_{\varrho}(\iota,\eta,t)} - 1\right) = \zeta_{\varrho}(\iota,\eta,t) = \frac{|\iota-\eta|}{\varrho t}$

So,

Hence, for $H = \frac{1}{3}$, *we get* $\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_{\iota},\Gamma_{\eta},t)} - 1\right) = H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota,\eta,t)} - 1\right)$.

*Thus, Theorem (***3***.1) implies that* Γ *has a unique fixed point* $0 \in Y$.

Example 3.4.

Let $Y = [0,1]$ *and* $\zeta_{\varrho}(\iota,\eta,t) = exp^{\left\{-\frac{|\iota-\eta|}{t\varrho}\right\}}\left(exp^{\left\{\frac{|\iota-\eta|}{t\varrho}\right\}} - 1\right)$.

Define $\Gamma: [0,1] \to [0,1]$ *via* $\Gamma(\iota) = \frac{\iota}{5}$. *Also, define* $\Upsilon: (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = -ln\iota$.

Note that Y *is a strictly non-decreasing, continuous function and* $\Upsilon(1) = 0$.

Now, we have:

$\zeta_{\varrho}(\Gamma\iota,\Gamma\eta,t) = exp^{\left\{-\frac{|\iota-\eta|}{5t\varrho}\right\}}\left(exp^{\left\{\frac{|\iota-\eta|}{5t\varrho}\right\}} - 1\right), \zeta_{\varrho}(\iota,\eta,t) = exp^{\left\{-\frac{|\iota-\eta|}{t\varrho}\right\}}\left(exp^{\left\{\frac{|\iota-\eta|}{t\varrho}\right\}} - 1\right)$.

So, $\Upsilon\left(\frac{1}{\zeta_{\varrho}(\Gamma\iota,\Gamma\eta,t)} - 1\right) = \frac{|\iota-\eta|}{5t\varrho}$; $\Upsilon\left(\frac{1}{\zeta_{\varrho}(\iota,\eta,t)} - 1\right) = \frac{|\iota-\eta|}{t\varrho}$.

Hence, for $H = \frac{1}{5}$, *we get* $\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_{\iota},\Gamma_{\eta},t)} - 1\right) = H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota,\eta,t)} - 1\right)$.

*Thus Theorem (***3***.1) implies that* Γ *has a unique fixed point* $0 \in Y$.

The following example satisfies Theorem (**3**.2).

Example 3.5.

Let $Y = [0,1]$ *and* $\zeta_\varrho(\iota,\eta,t) = \frac{\frac{|\iota-\eta|}{\varrho}}{t+\frac{|\iota-\eta|}{\varrho}}$. *Define* $\Gamma : [0,1] \to [0,1]$ *via* $\Gamma(\iota) = c, c \in [0,1]$. *Also, define* $\Upsilon: (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = \frac{1}{\iota} - 1$. *note that* Υ *is a strictly non- decreasing, continuous function and* $\Upsilon(1) = 0$. *Now, we have:*

$$\zeta_\varrho(\Gamma\iota,\Gamma\eta,t) = \frac{|c-c|}{t+|c-c|} = 0; \quad \frac{\zeta_\varrho(\iota,\eta,t)}{4} = \frac{1}{4} \times \frac{|\iota-\eta|}{\varrho t+|\iota-\eta|}$$

$$\zeta_\varrho(\Gamma\iota,\iota,t) = \frac{1}{4} \times \frac{|c-\iota|}{\varrho t+|c-\iota|} \text{ and } \zeta_\varrho(\eta,\Gamma\eta,t) = \frac{1}{2} \times \frac{|c-\eta|}{\varrho t+|c-\eta|}.$$

So,

$$\left(\frac{1}{\frac{\zeta_\varrho(\iota,\eta,t)+\zeta_\varrho(\Gamma\iota,\iota,t)}{4}} - 1\right) + \left(\frac{1}{\frac{\zeta_\varrho(\eta,\Gamma\eta,t)}{2}} - 1\right) = 0 = \zeta_\varrho(\Gamma\iota,\Gamma\eta,t).$$

So, for H *such that* $0 < H < 1$, *we have*

$$\left(\frac{1}{\frac{\zeta_\varrho(\iota,\eta,t)+\zeta_\varrho(\Gamma\iota,\iota,t)}{4}} - 1\right) + \left(\frac{1}{\frac{\zeta_\varrho(\eta,\Gamma\eta,t)}{2}} - 1\right) \leq 0 = \zeta_\varrho(\Gamma\iota,\Gamma\eta,t).$$

*Thus Theorem (***3***.2) implies that* Γ *has a unique fixed point* $c \in Y$.

4. Application

In this section, we use our obtained results to show that the following integral equation has a solution:

$$\iota(\kappa) = h(\kappa) + \int_0^1 \Omega(k,s)\varsigma\left(s,\iota(s)\right)ds, \kappa \in [0,1]. \quad (\mathbf{4}.1)$$

Let $Y = C([0,1])$ be the space of all continuous functions defined on [0,1].

Define a modular revised fuzzy metric:

$\zeta_{\%}(\iota,\eta,t) : (0,\infty)^2 \times C([0,1])^2 \to [0,1]$,

By

$$\zeta_\varrho(\iota,\eta,t) = \frac{\sup_{\kappa\in[0,1]} \frac{|\iota(\kappa)-\eta(\kappa)|}{\varrho}}{t + \sup_{\kappa\in[0,1]} \frac{|\iota(\kappa)-\eta(\kappa)|}{\varrho}}$$

Then $(Y, \zeta_{\%}, \circledast)$ is a complete modular revised fuzzy metric space.

Theorem 4.1.

Suppose we have the following hypotheses:

(1) $\exists$*a continuous function* $g: [0,1] \to [0,1]$ *such that*

$$|(s,\iota) - (s,\eta)| \leq g(s)|\iota - \eta|. \quad (\mathbf{4}.2)$$

And

$$\int_0^1 g(k)dk \leq \frac{1}{3} \tag{4.3}$$

(2) $\Omega(k,s) \geq 0, \forall\, k,s \in [0,1]$ **(4.4)**

*Then the integral equation (**4**.1) has a solution* $\iota^{\circledast} \in C^2([0,1])$.

Proof.

Take the operator:

$$\Gamma\iota(\kappa) = h(\kappa) + \int_0^1 \Omega(k,s)\varsigma\left(s,\iota(s)\right)ds, \kappa \in [0,1]$$

For all $\iota, \eta \in C([0,1])$, we have

$$\zeta_\varrho(\iota,\eta,t) = \frac{\sup_{\kappa\in[0,1]} \frac{|\iota(\kappa)-\eta(\kappa)|}{\varrho}}{t+\sup_{\kappa\in[0,1]} \frac{|\iota(\kappa)-\eta(\kappa)|}{\varrho}} = \frac{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|h(\kappa)+\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))ds-h(\kappa)-\int_0^1 \Omega(k,s)\varsigma(s,\eta(s))ds\right|}{t+\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|h(\kappa)+\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))ds-h(\kappa)-\int_0^1 \Omega(k,s)\varsigma(s,\eta(s))ds\right|}$$

$$= \frac{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))ds-\varsigma(s,\eta(s))ds\right|}{t+\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))-\varsigma(s,\eta(s))ds\right|}$$

By (**4**.2) and (**4**.4), we have

$$\left|\int_0^1 \Omega(k,s)\varsigma\left(s,\iota(s)\right)-\varsigma\left(s,\eta(s)\right)ds\right| \leq \int_0^1 \Omega(k,s)\left|\varsigma\left(s,\iota(s)\right)-\varsigma\left(s,\eta(s)\right)ds\right|$$

$$\leq \int_0^1 g(s)\,|\iota(s)-\eta(s)|ds.$$

Hence by (**4**.3), we get

$$\frac{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))-\varsigma(s,\eta(s))ds\right|}{t+\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))-\varsigma(s,\eta(s))ds\right|} \leq \frac{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\int_0^1 g(s)|\iota(s)-\eta(s)|ds}{t+\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\int_0^1 g(s)|\iota(s)-\eta(s)|ds} = \frac{\sup_{\kappa\in[0,1]} \frac{1}{3\varrho}|\iota(\kappa)-\eta(\kappa)|}{t+\sup_{\kappa\in[0,1]} \frac{1}{3\varrho}|\iota(\kappa)-\eta(\kappa)|}$$

Thus

$$\frac{t}{\sup_{\kappa\in[0,1]} \frac{1}{3\varrho}|\iota(\kappa)-\eta(\kappa)|} \leq \frac{t}{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))-\varsigma(s,\eta(s))ds\right|}$$

So

$$\frac{3t}{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}|\iota(\kappa)-\eta(\kappa)|} \leq \frac{t}{\sup_{\kappa\in[0,1]} \frac{1}{\varrho}\left|\int_0^1 \Omega(k,s)\varsigma(s,\iota(s))-\varsigma(s,\eta(s))ds\right|}$$

Define $\Upsilon: (0,1] \to [0,+\infty)$ by $\Upsilon(t) = \frac{1}{\iota} - 1$.

Then Υ is a strictly non-decreasing, continuous function and $\Upsilon(1) = 0$.

For $H = \frac{1}{3}$, we obtain $\Upsilon\left(\frac{1}{\zeta_{\%}(\Gamma_\iota,\Gamma_\eta,t)} - 1\right) \leq H\Upsilon\left(\frac{1}{\zeta_{\%}(\iota,\eta,t)} - 1\right)$.

Therefore theorem (**3**.1) implies Γ has a unique fixed point and hence the integral equation (**4**.1) has a solution $\iota^{\circledast} \in C^2([0,1])$.

Conclusion:

In this paper, we defined a new space called modular revised fuzzy metric space and stated some examples of this space. Also, we formulate and prove some new fixed point results under this space. In addition, we provided some examples and an application for showing the validity of our results.

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