Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 3s (2024)

Certain Properties on Univalent Functions Related to New Linear and Integral Operators

Lect. Dr . Bassim Kareem Mihsin
General Directorate of Education in Karbala, Iraq
E- mail: bassim_kareem@karbala.edu.iq
basmk3756@gmail.com

Article History:
Received: *06-04-2024*
Revised: *28-05-2024*
Accepted: *12-06-2024*

Abstract:

We are implementing the two new operators, $\mathcal{T}^{\mathfrak{m}}_{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}}f(z)$ and $\mathrm{I}^{\mathfrak{m}}_{\mathfrak{r},\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}}f(z)$ say linear and integral operators respectively, of analaytic functions in open unit disk $\mho$, to pedimenting new results for superordination and subordination. We conclude several sandwich-type results are the master goal for this paper.

Keywords: Analytic functions ,multivalent functions, Hadamart product , Differential subordination , Superordination , Sandwich theorems , dominant , Subordinant .

1. Introduction

Suppose that $\mathcal{B}$ to be class functions intailing the following function:

$$f(z) = a + a_{\mathfrak{n}} z^{\mathfrak{n}} + a_{\mathfrak{n}+1} z^{\mathfrak{n}+1} + \cdots (a \in \mathbb{C}, (\mathfrak{n} \in \mathbb{N} = \{1,2,\ldots\}; z \in \mho), \quad (1,1)$$

where $f(z)$ be analytic to open unit disk $\mho = \{z \in \mathbb{C} : |z| < 1\}$. Assume $M[a,\mathfrak{n}]$ be subclass of the function $f \in \mathcal{G}$. For $a \in \mathbb{C}$ and $\mathfrak{n}$ to be positive integer number ,

If $f \in \mathcal{G}$ defined by (1.1) and $g \in \mathcal{G}$ is given by formula:

$$f(z) = z + \sum_{\mathfrak{n}=2}^{\infty} a_{\mathfrak{n}} z^{\mathfrak{n}} , \; g(z) = z + \sum_{\mathfrak{n}=2}^{\infty} b_{\mathfrak{n}} z^{\mathfrak{n}}$$

By ussing convolution of $f \; and \; g$ to get

$$(f * g)(z) = z + \sum_{\mathfrak{n}=2}^{\infty} a_{\mathfrak{n}} b_{\mathfrak{n}} z^{\mathfrak{n}} = (g * f)(z) .$$

If the functions $f \; and \; g$ be analytic functions in , so f be subordinate to g in $\mho$, for that we can say $f(z) \prec g(z)$, if existing a Schwarz function $\mathrm{w}(z)$ to be analytic within $\mho$ to satisfy the conditions that $|\mathrm{w}(z)| < 1$ ($z \in \mho$) and $\mathrm{w}(0) = 0$ and where $f(z) = g(\mathrm{w}(z))$, ($z \in \mho$) . As additional ,to that if g be univalent function in $\mho$, so we satisfy the equivalence relation link (see [15],[16]and[18])

$$f(\mho) \subset g(\mho), (z \in \mho) \text{ and } f(z) \prec g(z) \Leftrightarrow f(0) = g(0).$$

Definition (1.1)[15]: Assume that $\flat(z)$ to be univalent in $\mho$ and $\partial: \mathbb{C}^3 \times \mho \dashrightarrow \mathbb{C}$. Let $\mathcal{P}(z)$ be analytic in $\mho$ to satisfy the following differential subordination of second – order:

$$\partial\left(\mathcal{P}(z), z(\mathcal{P}(z))', z^2(\mathcal{P}(z))''; z\right) \prec \flat(z), \quad (1.2)$$

therefore the equation (1.2) of differential subordination have the solution $\mathcal{P}(z)$. The formula (1.2) represent the solution of differential subordination which has $\mathrm{q}(z)$ as a dominant , or more, additional to that to be simply dominant, if $\mathcal{P}(z) \prec \mathrm{q}(z)$ to all $\mathcal{P}(z)$ for that will satisfy (1.2).If $\tilde{\mathrm{q}}(z)$ is univalent dominant which satisfys $\tilde{\mathrm{q}}(z) \prec \mathrm{q}(z)$ for every dominant $\mathrm{q}(z)$ for (1.2) , so the formula (1.2) will be satisfied by the best dominant.

Definition (1.2)([15] *also see* [16]) : Assume that $\partial: \mathbb{C}^3 \times \mathrm{U} \rightarrow \mathbb{C}$ and assume the function $\flat(z)$ which be analytic in U. If $\partial\left(\mathcal{P}(z), z(\mathcal{P}(z))', z^2(\mathcal{P}(z))''; z\right)$ and the univalent function $\mathcal{P}(z)$ within U where $\mathcal{P}(z)$ satisfy the following differential Superordination of second–order :

$$\flat(z) \prec \partial\left(\mathcal{P}(z), z(\mathcal{P}(z))', z^2(\mathcal{P}(z))''; z\right), \tag{1.3}$$

then the equation (1.3) have the differential superordination solution of (1.3) say $\mathcal{P}(z)$. Equation (1.3) leads to the solution of subordinant $\mathrm{q}(z)$, where must be analytic function or we can say that subordinant will be more simple when $\mathrm{q}(z) \prec \mathcal{P}(z)$ for every $\mathcal{P}(z)$ halds (1.3). The function $\tilde{\mathrm{q}}(z)$ be univalent subordinant which satisfy $\mathrm{q}(z) \prec \tilde{\mathrm{q}}(z)$ to all subordinants $\mathrm{q}(z)$ of (1.3) ,will be best subordinant .

In [16], Miller and Macanu have obtained sufficient conditions to functions $\flat, \mathrm{q}$ *and* ∂ for where the implicationts given by:

$$\mathrm{q}(z) \prec \mathcal{P}(z) \Rightarrow \flat(z) \prec \partial(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \tag{1.4}$$

by taking the results (see [1,2,4,5,6,7,,8,919]) for obtaining sufficient conditions to normalized analytic functions for satisfing:

$$\mathrm{q}_1(z) \prec \frac{zf'(z)}{f(z)} \prec \mathrm{q}_2(z) ,$$

such that the two tbe univalent function in U say$\mathrm{q}_1(z)$ *and* $\mathrm{q}_2(z)$ where $\mathrm{q}_1(0) = \mathrm{q}_2(0) = 1$.

Addition to that, El-Ashwah and Aouf [23] ,Ali et al. [3] ,Atshan and Hadi [6] , Atshan and Ali [4] ,and Gochhayat [25] derived some superordination and subordination results to analytic functions in U . Recently ,Al-Ameedee et al. [1] ,Atshan et al. [4,5] and Gochhayat [24] got sandwich results to some classes of analytic functions

The function $\phi_{\mathbb{t}}(z, \mathbb{t}, \mathbb{s})$ define the following series:

$$\phi_{\mathbb{t}}(z, \mathbb{t}, \mathbb{s}) = \sum_{n=0}^{\infty} \frac{z^n}{\left(\frac{\mathbb{r}+n\mathbb{s}}{\mathbb{s}}\right)^{\mathbb{t}}} .$$

Where $z \in \mathrm{U}, \mathbb{r} \in \mathbb{C} \setminus z_0^- = \{0, -1, -2, \ldots\}, \mathbb{t} \in \mathbb{C}, Re(\mathbb{t}) > 1, z \in \partial\mathrm{U}, \mathbb{s} \in N \setminus \{1\}$.

The following normalized function $\mathcal{J}_{\mathbb{r},\mathbb{s}}^{\mathbb{t}}(z)$ defined as:

$$\mathcal{J}_{\mathbb{r},\mathbb{s}}^{\mathbb{t}}(z) = \left(\frac{\mathbb{r}}{\mathbb{s}} + 1\right)^{\mathbb{t}} \left[\phi_{\mathbb{t}}(z, \mathbb{t}, \mathbb{s}) - \left(\frac{\mathbb{r}}{\mathbb{s}}\right)^{-\mathbb{t}}\right] = z + \sum_{n=0}^{\infty} \left(\frac{\mathbb{r}+\mathbb{s}}{\mathbb{r}+n\mathbb{s}}\right)^{\mathbb{t}} a_n z^n, z \in \mathrm{U} , \tag{1. 5}$$

see more [22].

Swamy [21] defined linear operator $\mathcal{F}_{\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z)$ for $\mathbb{m} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{u} \in R$ $, \mathbb{v} \geq 0, \mathbb{u} + \mathbb{v} > 0$ and define by:

$$\mathcal{F}_{\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\mathbb{u}+n\mathbb{v}}{\mathbb{u}+\mathbb{v}}\right)^{\mathbb{m}} a_n z^n . \tag{1.6}$$

See [10,11,12,13,14,17,20,21].

Definition (1.3): Suppose $f \in \mathcal{G}, z \in \mho, \mathbb{r} \in \mathbb{C} \setminus z_0^- = \{0,-1,-2,...\}, \mathbb{t} \in \mathbb{C}$, $, \mathbb{m} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{u} \in R$ $, \mathbb{s} \in N \setminus \{1\}, \mathbb{v} \geq 0, \mathbb{u} + \mathbb{v} > 0, Re(\mathbb{t}) > 1$ *where* $z \in \partial\mho$ and we define new operator:

$\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z): \mathcal{G} \to \mathcal{G}$, where

$\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) = \mathcal{J}_{\mathbb{r},\mathbb{s}}^{\mathbb{t}}(z) * \mathcal{F}_{\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z)$

$$\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\mathbb{r}+\mathbb{s}}{\mathbb{r}+\mathbb{s}n}\right)^{\mathbb{t}}\left(\frac{\mathbb{u}+n\mathbb{v}}{\mathbb{u}+\mathbb{v}}\right)^{\mathbb{m}} a_n z^n. \tag{1.7}$$

We have from (1.7) that :

$$(\mathbb{u}+\mathbb{v})\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}+1}f(z) = \mathbb{u}\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) - \mathbb{v}z\left(\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z)\right)', \mathbb{v} > 0 . \tag{1.8}$$

We observe that $\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z): \mathcal{G} \to \mathcal{G}$ is an integral operator and for f given by (1.2) we have:

$$\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\mathbb{r}+\mathbb{s}}{\mathbb{r}+\mathbb{s}n}\right)^{\mathbb{t}}\left(\frac{\mathbb{u}+\mathbb{v}}{\mathbb{u}+n\mathbb{v}}\right)^{\mathbb{m}} a_n z^n \ , z \in \mho . \tag{1.9}$$

It follows form (1.9) that:

$$(\mathbb{u}+\mathbb{v})\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z) = \mathbb{u}\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}+1}f(z) + \mathbb{v}z\left(\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}+1}f(z)\right)' . \tag{1.10}$$

Now,in this work has been dedicated to derive several superordination, subordination and sandwich results of differential containing new operators $\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z)$ and $\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}}^{\mathbb{m}}f(z)$.

2. Preliminaries

Constructing our major results ,some following lemmas will be needed with its references , (see also [23]) .

Definition (2.1)[15]: called by $\mathbb{Q}$ which represent all f functions,they must be analytic and one–to–one on $\overline{\mho} \setminus E(f)$, where $E(f) = \left\{\zeta \in \partial\mho: \lim_{z\to\zeta} f(z) = \infty\right\}$ and $\overline{\mho} = \mho \cup \{z \in \partial\mho\}$, in addition to that $f'(\zeta) \neq 0$ of $\zeta \in \partial\mho \setminus E(f)$. More that, we consider a subclass of $\mathbb{Q}$ to $f(0) = a$ is called $\mathbb{Q}(a)$ *where* $\mathbb{Q}(1) = \mathbb{Q}_1$ *and* $\mathbb{Q}(0) = \mathbb{Q}_0$.

Lemma (2.2) [15]: Assume the function $\mathrm{q}(z)$ be univalent and convex in $\mho$ assume $\beta \in \mathbb{C} / \{0\}$ $\alpha \in \mathbb{C}$, $, \beta \neq 0$, and suppose

$$Re\left\{1 + \frac{z\mathrm{q}''(z)}{\mathrm{q}'(z)}\right\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\} .$$

If $P(z)$ is analytic function in $\mho$, and

$$\alpha\mathcal{P}(z) + \beta z\big(\mathcal{P}(z)\big)' \prec \alpha q(z) + \beta z\big(q(z)\big)', \tag{2.1}$$

hence $q(z)$ will be best dominant and $\mathcal{P}(z) \prec q(z)$.

Lemma (2.3) [15]: Let $q(z)$ is belongs in $\mho$ with $q(0) = 1$,where $q(z)$ is convex and univalent. Let $Re(\beta) > 0$ and $\beta \in \mathbb{C}$. If $\mathcal{P}(z) \in M[q(0),1] \cap \mathbb{Q}$ and $\mathcal{P}(z) + \beta z\big(\mathcal{P}(z)\big)'$ is univalent in $\mho$, then $q(z) + \beta z(q(z))' \prec \mathcal{P}(z) + \beta z(\mathcal{P}(z))'$, which implies that $q(z) \prec \mathcal{P}(z)$ *and* $q(z)$ will be best subordinant **.**

3. Subordination Results:

Theorem (3.1): Let $q(z)$ is belongs in $\mho$ with $q(0) = 1$,where $q(z)$ is convex and univalent. Let $\xi \in \mathbb{C}^*$,$\mu, \mathbb{v} > 0, \mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$ andSuppose that $q(z)$ satisfies :

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, Re\left(\frac{\mu(\mathbb{u}+\mathbb{v})}{\xi\mathbb{v}}\right)\right\} . \tag{3.1}$$

Let $f \in \mathcal{G}$ holds the subordination

$$\mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v}) \prec q(z) + \frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} zq'(z), \tag{3.2}$$

Where

$$\mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v}) = (1-\xi)\left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} + \xi\left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} + \left(\frac{\mathcal{T}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}\right), \tag{3.3}$$

then

$$\left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \prec q(z), \tag{3.4}$$

and the equation (3.2) have the best dominant say q .

Proof: Put

$$\mathcal{S}(z) = \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu}, z \in \mho. \tag{3.5}$$

Hence differentiate (3.5) logarithmically according to z, and taking identity (1.8) in resultant equation , to get

$\frac{z\mathcal{S}'(z)}{\mathcal{S}(z)} = \mu\left(\frac{(\mathbb{u}+\mathbb{v})}{\mathbb{v}}\right)\left(\frac{z\mathcal{T}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)} - 1\right)$,and which can be written as

$$\frac{\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathcal{S}'(z) = \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu}\left(\frac{\mathcal{T}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)} - 1\right)$$

Thus the equation of subordination which is represented by (3.2) be equivalent by

$$\mathcal{S}(z) + \frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathcal{S}'(z) \prec q(z) + \frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} zq'(z) .$$

By apply Lemma (2.2)2.1 when $\sigma = \frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})}$, the proof of theorem(3.1) is complete.

Now, in theorem above,we put , $\mathrm{q}(z) = \frac{1+Az}{1+Bz}$,so we get the following corollary.

Corollary (3.1): Assume$\xi, A, B \in \mathbb{C}$, $A \neq B, |B| < 1, \mu > 0, Re(\xi) > 0$ *and* $\mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$,if $f \in \mathcal{G}$ satisfy the recent subordination case :

$\mathfrak{D}(\mathbb{mm}, \xi, \mu, \mathbb{u}, \mathbb{v}) \prec \left(\frac{1+Az}{1+Bz}\right) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})}\left(\frac{(A+B)z}{(1+Bz)^2}\right)$, such that $\mathfrak{D}(\mathbb{mm}, \xi, \mu, \mathbb{u}, \mathbb{v})$ known by (3.3) ,then

$$\left(\frac{\mathcal{T}^{\mathbb{mm}}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \prec \frac{1+Az}{1+Bz} ,$$

and will be best dominant say $\frac{1+Az}{1+Bz}$.

Now ,put $\mathbb{mm} = 0$ in the Theorem above ,to get a new result.

Corollary (3.2): Let $\mathrm{q}(z)$ is belongs in U with $\mathrm{q}(0) = 1$,where $\mathrm{q}(z)$ is univalent. Let $\xi \in \mathbb{C}^*$,$\mu, \mathbb{v} > 0, \mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$ andSuppose that $\mathrm{q}(z)$ satisfies :

Let $f \in \mathcal{G}$ holds the subordination

$$\mathfrak{D}_1(0, \xi, \mu, \mathbb{u}, \mathbb{v}) \prec \mathrm{q}(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathrm{q}'(z),$$

Where

$$\mathfrak{D}_1(0, \xi, \mu, \mathbb{u}, \mathbb{v}) = (1-\xi)\left(\frac{\mathcal{T}^{0}_{\mathbb{r},\mathbb{s},\mathbb{t}} f(z)}{z}\right)^{\mu} + \xi\left(\frac{\mathcal{T}^{0}_{\mathbb{r},\mathbb{s},\mathbb{t},,} f(z)}{z}\right)^{\mu} + \left(\frac{\mathcal{T}^{1}_{\mathbb{r},\mathbb{s},\mathbb{t},,\mathbb{u},\mathbb{v}} f(z)}{f(z)}\right),$$

then

$$\left(\frac{\mathcal{T}^{0}_{\mathbb{r},\mathbb{s},\mathbb{t},} f(z)}{z}\right)^{\mu} \prec \mathrm{q}(z),$$

and the equation (3.2) have the best dominant say q .

Now ,put $\mathbb{u} = \mathbb{v} = 1$ in the Theorem above ,to get a new result.

Corollary (3.3): Let $\mathrm{q}(z)$ is belongs in U with $\mathrm{q}(0) = 1$,where $\mathrm{q}(z)$ is univalent. Let $\xi \in \mathbb{C}^*$,$\mu > 0$, and Suppose that (3.1) holds. Let $f \in \mathcal{G}$ holds the subordination

$$\mathfrak{D}_2(\mathbb{mm}, \xi, \mu, 1,1) \prec \mathrm{q}(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathrm{q}'(z),$$

Where

$$\mathfrak{D}_2(\mathbb{mm}, \xi, \mu, 1,1) = (1-\xi)\left(\frac{\mathcal{T}^{\mathbb{mm}}_{\mathbb{r},\mathbb{s},\mathbb{t},,1,1} f(z)}{z}\right)^{\mu} + \xi\left(\frac{\mathcal{T}^{\mathbb{mm}}_{\mathbb{r},\mathbb{s},\mathbb{t},,1,1} f(z)}{z}\right)^{\mu} + \left(\frac{\mathcal{T}^{\mathbb{mm}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},,1,1} f(z)}{\mathcal{T}^{\mathbb{mm}}_{\mathbb{r},\mathbb{s},\mathbb{t},,1,1} f(z)}\right),$$

then

$$\left(\frac{\mathcal{T}^{\mathbb{mm}}_{\mathbb{r},\mathbb{s},\mathbb{t},,1,1} f(z)}{z}\right)^{\mu} \prec \mathrm{q}(z),$$

and the equation (3.2) have the best dominant say q .

By the same way we can take Theorem (3.1), to prove the following theorems by using the identity(1.10).

Theorem (3.2): Let $\mathfrak{q}(z)$ is belongs in $\mho$ with $\mathfrak{q}(0) = 1$,where $\mathfrak{q}(z)$ is convex and univalent. Let $\xi \in \mathbb{C}^*$,$\mathbb{v} > 0$, $\mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$ andSuppose that $\mathfrak{q}(z)$ satisfies :

$$Re\left\{1 + \frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right\} > \max\left\{0, -Re\left(\frac{\mu}{\xi}\right)\right\} . \quad (3.6)$$

Let $f \in \mathcal{G}$ holds the subordination

$$\mathfrak{L}(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi) \prec \mathfrak{q}(z) + \frac{\xi}{\mu} z\mathfrak{q}'(z), \quad (3.7)$$

where

$$\mathfrak{L}(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi) = (1 - \xi)\left(\frac{\mathbb{u}+\mathbb{v}}{\mathbb{v}}\right)\left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu} + \xi\left(\frac{\mathbb{u}+\mathbb{v}}{\mathbb{v}}\right)\left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu} + \left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}} f(z)}{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}\right), \quad (3.8)$$

then

$$\left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu} \prec \mathfrak{q}(z), \quad (3.9)$$

and the equation (3.7) have the best dominant say $\mathfrak{q}$.

Proof: Put

$$\mathcal{S}(z) = \left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}} f(z)}{z}\right)^{\mu}, z \in \mho. \quad (3.10)$$

Hence differentiate (3.10) logarithmically according to z, and taking identity (1.10) in resultant equation , to get

$\frac{z\,\mathcal{S}'(z)}{\mathcal{S}(z)} = \mu\left(\frac{(\mathbb{u}+\mathbb{v})}{\mathbb{v}}\right)\left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}} f(z)}{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)} - 1\right)$,and which can be written as

$$\frac{\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathcal{S}'(z) = \left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu}\left(\frac{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}} f(z)}{I_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)} - 1\right) .$$

Thus the equation of subordination which is represented by (3.7) be equivalent by

$$\mathcal{S}(z) + \frac{\xi}{\mu} z\mathcal{S}'(z) \prec \mathfrak{q}(z) + \frac{\xi}{\mu} z\mathfrak{q}'(z) .$$

By apply Lemma (2.2) when $\sigma = \frac{\xi}{\mu}$, the proof of theorem(3.2) is complete therefore ,we put $\mathbb{v} = 1$ in above theorem to obtain the results below.

Corollary(3.4): Let $\mathfrak{q}(z)$ is belongs in $\mho$ with $\mathfrak{q}(0) = 1$,where $\mathfrak{q}(z)$ is univalent . Let $\xi \in \mathbb{C}^*$,$\mu > 0$, $\mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$ andSuppose that $\mathfrak{q}(z)$ satisfies :

$$Re\left\{1 + \frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right\} > \max\left\{0, -Re\left(\frac{\mu(\mathbb{u}+\mathbb{v})}{\xi\mathbb{v}}\right)\right\} . \quad (3.11)$$

Let $f \in \mathcal{G}$ holds the subordination

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 3s (2024)

$$\mathfrak{L}_1(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi) \prec \mathrm{q}(z) + \frac{\xi}{\mu} z\mathrm{q}'(z), \tag{3.12}$$

Where

$$\mathfrak{L}_1(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi) = (1-\xi)\left(\frac{\mathbb{u}+\mathbb{v}}{\mathbb{v}}\right)\left(\frac{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} + \xi\left(\frac{\mathbb{u}+\mathbb{v}}{\mathbb{v}}\right)\left(\frac{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} + \left(\frac{\mathrm{I}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}\right) \tag{3.13}$$

then

$$\left(\frac{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \prec \mathrm{q}(z), \tag{3.14}$$

and the equation (3.7) have the best dominant say q .

4. Superordination Results :

Theorem (4.1) : Assume that the function q are univalent and convex in U $with$ $\mathrm{q}(0) = 1, Re(\xi) > 0, \xi \in \mathbb{C}, \mu, \mathbb{v} > 0, \mathbb{u} \in R$ such that $\mathbb{u} + \mathbb{v} > 0$. if $f \in \mathcal{G}$, where

$$\left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},1,1} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0), 1] \cap \mathrm{Q} \quad . \tag{4.1}$$

If $\mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v})$is univalent function in U as defined by (3.3),and satisfies the superordination case below;

$$\mathrm{q}(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\, \mathrm{q}'(z) \prec \mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v}), \tag{4.2}$$

$$\text{then } \mathrm{q(z)} \prec \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},1,1} f(z)}{z}\right)^{\mu}, \tag{4.3}$$

and $\mathrm{q(z)}$ will be best subordination .

Proof: Put

$$\mathcal{S}(z) = \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu}, z \in \mathrm{U}. \tag{3.4}$$

Hence differentiate (3.4) logarithmically according to z, and taking identity (1.8) in resultant equation , to get

$\frac{z\,\mathcal{S}'(z)}{\mathcal{S}(z)} = \mu\left(\frac{(\mathbb{u}+\mathbb{v})}{\mathbb{v}}\right)\left(\frac{z\mathcal{T}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)} - 1\right)$,and which can be written as

$$\frac{\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathcal{S}'(z) = \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu}\left(\frac{\mathcal{T}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)} - 1\right)$$

Thus the equation of subordination which is represented by (4.2) be equivalent by

$$\mathcal{S}(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathcal{S}'(z) \prec p(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} zp'(z) .$$

By apply Lemma (2.2), the proof of theorem(4.1) is complete.

Now, in theorem above,we put , $\mathbb{m} = 0$,so we get the following corollary.

Corollary (4.1) : Let $\mathrm{q}(z)$ is belongs in U with $\mathrm{q}(0)=1$,where $\mathrm{q}(z)$ is convex and univalent, $Re(\xi)>0, \xi \in \mathbb{C}, \mu, \mathbb{v}>0, \mathbb{u} \in R$. if $f \in \mathcal{G}$, where

$$\left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t}}^{\mathbb{m}} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0),1] \cap \mathrm{Q} \quad . \tag{4.5}$$

If $\mathfrak{D}_1(0,\xi,\mu,\mathbb{u},\mathbb{v})$is univalent function as defined by (3.3),and satisfies the superordination case below;

$$\mathrm{q}(z)+\frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\, \mathrm{q}'(z) \prec \mathfrak{D}_1(0,\xi,\mu,\mathbb{u},\mathbb{v}), \tag{4.6}$$

$$\text{then } \mathrm{q}(\mathrm{z}) \prec \left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t}}^{\mathbb{m}} f(z)}{z}\right)^{\mu}, \tag{4.7}$$

and $\mathrm{q}(\mathrm{z})$ will be best subordination .

Now, in theorem above,we put , $\mathbb{v}=1$,so to obtain corollary below.

Corollary (4.2) :Let $\mathrm{q}(z)$ is belongs in U with $\mathrm{q}(0)=1$,where $\mathrm{q}(z)$ is convex and univalent, $Re(\xi)>0, \xi \in \mathbb{C}, \mu>0$,such that $\mathbb{u}+\mathbb{v}>0$. if $f \in \mathcal{G}$, where

$$\left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1}^{\mathbb{m}} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0),1] \cap \mathrm{Q} \quad . \tag{4.8}$$

If $\mathfrak{D}(\mathbb{m},\xi,\mu,\mathbb{u},1)$is univalent function as defined by (3.3),and satisfies the superordination case below;

$$\mathrm{q}(z)+\frac{\xi\mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\, \mathrm{q}'(z) \prec \mathfrak{D}_3(\mathbb{m},\xi,\mu,\mathbb{u},1), \tag{4.9}$$

where

$$\mathfrak{D}_3(\mathbb{m},\xi,\mu,\mathbb{u},1)=(1-\xi)\left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u}1}^{\mathbb{m}} f(z)}{z}\right)^{\mu}+\xi\left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1}^{\mathbb{m}} f(z)}{z}\right)^{\mu}+\left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1}^{\mathbb{m}+1} f(z)}{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1}^{\mathbb{m}} f(z)}\right)$$

$$\text{then } \mathrm{q}(\mathrm{z}) \prec \left(\frac{\mathcal{T}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1}^{\mathbb{m}} f(z)}{z}\right)^{\mu}, \tag{4.10}$$

and $\mathrm{q}(\mathrm{z})$ will be best subordination .

Theorem (4.2) : Assume that the function q are univalent and convex in U *with* $\mathrm{q}(0)=1, Re(\xi)>0, \xi \in \mathbb{C}, \mu, \mathbb{v}>0$,such that $\mathbb{u}+\mathbb{v}>0$. if $f \in \mathcal{G}$, where

$$\left(\frac{\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0),1] \cap \mathrm{Q} \quad . \tag{4.11}$$

If $\mathfrak{L}(\mu,\mathbb{m},\mathbb{u},\mathbb{v},\xi)$is univalent function as defined by (3.8),and satisfies the superordination case below;

$$\mathrm{q}(z)+\frac{\xi}{\mu} z\, \mathrm{q}'(z) \prec \mathfrak{L}(\mu,\mathbb{m},\mathbb{u},\mathbb{v},\xi), \tag{4.12}$$

then

$$\mathrm{q}(\mathrm{z}) \prec \left(\frac{\mathrm{I}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}}^{\mathbb{m}+1} f(z)}{z}\right)^{\mu}, \tag{4.13}$$

and $\mathrm{q}(\mathrm{z})$ will be best subordination .

thus by taking Lemma (2.3) ,we obtain result that required .

Now, in theorem above,we put , $\mathbb{v} = 1$,so to obtain corollary below.

Corollary (4.3) :Let $\mathrm{q}(z)$ is belongs in U with $\mathrm{q}(0) = 1$,where $\mathrm{q}(z)$ is convex and univalent, $Re(\xi) > 0, \xi \in \mathbb{C}, \mu > 0, \mathbb{u}$ *be* real number. if $f \in \mathcal{G}$, where

$$\left(\frac{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0), 1] \cap \mathrm{Q} \quad . \tag{4.14}$$

If $\mathfrak{L}_1(\mu, \mathbb{m}, \mathbb{u}, 1, \xi)$is univalent function as defined by (3.8),and satisfies the superordination case below;

$$\mathrm{q}(z) + \frac{\xi}{\mu} z\, \mathrm{q}'(z) \prec \mathfrak{L}_1(\mu, \mathbb{m}, \mathbb{u}, 1, \xi), \tag{4.15}$$

then
$$\mathrm{q}(\mathrm{z}) \prec \left(\frac{\mathrm{I}^{\mathbb{m}+1}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},1} f(z)}{z}\right)^{\mu}, \tag{4.16}$$

and $\mathrm{q}(\mathrm{z})$ will be best subordination .

thus by taking Lemma (2.3) ,we obtain result that required .

5. Sandwich Results

Joining Theorems (3.1) and (4.1) , in order to get sandwich theorem (5.1) .

Theorem (5.1) : Assume the two convex functions in U say$\mathrm{q}_1(z)$ *and* $\mathrm{q}_2(z)$ together with $\mathrm{q}_1(0) = \mathrm{q}_2(0) = 1$.Suppose that $Re(\xi) > 0, \xi \in \mathbb{C}, \mu, \mathbb{v} > 0$, $\mathbb{u}$ be real number such that $\mathbb{u} + \mathbb{v} > 0$. .I$f$ $f \in \mathcal{G}$,where

$$\left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0), 1] \cap \mathrm{Q}\,,$$

and $\mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v})$ which is given by (3.3) be univalent function and holds

$$\mathrm{q}_1(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathrm{q}'_1(z) \prec \mathfrak{D}(\mathbb{m}, \xi, \mu, \mathbb{u}, \mathbb{v}) \prec \mathrm{q}_2(z) + \frac{\xi \mathbb{v}}{\mu(\mathbb{u}+\mathbb{v})} z\mathrm{q}'_2(z), \tag{5.1}$$

implies $\mathrm{q}_1(z) \prec \left(\frac{\mathcal{T}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \prec \mathrm{q}_2(z)$,with $\mathrm{q}_1(z)$ best subordinant and $\mathrm{q}_2(z)$ best dominant (5.1) respectively.

Joining Theorems (3.2) and (4.2) , in order to get sandwich theorem (5.2) .

Theorem (5.2) : Assume the two univalent convex functions in U say$\mathrm{q}_1(z)$ *and* $\mathrm{q}_2(z)$ together with $\mathrm{q}_1(0) = \mathrm{q}_2(0) = 1$. Suppose that $Re(\xi) > 0, \xi \in \mathbb{C}, \mu, \mathbb{v} > 0$, $\mathbb{u}$ real number such that $\mathbb{u} + \mathbb{v} > 0$.

.If $f \in \mathcal{G}$,where$\left(\frac{\mathrm{I}^{\mathbb{m}}_{\mathbb{r},\mathbb{s},\mathbb{t},\mathbb{u},\mathbb{v}} f(z)}{z}\right)^{\mu} \in M[\mathrm{q}(0),1] \cap \mathrm{Q}$,

Assume univalent function in U say $\mathfrak{L}(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi)$,

then

$$\mathrm{q}_1(z) + \frac{\xi}{\mu} \mathrm{q}^2{}_1(z) \prec \mathfrak{L}(\mu, \mathbb{m}, \mathbb{u}, \mathbb{v}, \xi) \prec \mathrm{q}_2(z) + \frac{\xi}{\mu} z\mathrm{q}'_2(z) \,. \tag{5.2}$$

implies $q_1(z) \prec \left(\frac{I^{m}_{r,s,t,u,v}f(z)}{z}\right)^{\mu} \prec q_2(z)$, and with $q_1(z)$ best subordinant and $q_2(z)$ best dominant (5.2) respectively .

6. Conclusions

The main aim of our present work is dedecated to give a new results connected by new operators, $\mathcal{T}^{m}_{r,s,t,u,v}f(z)$ and $I^{m}_{r,s,t,u,v}f(z)$ say linear and integral operators respectively for univalent function in open unit disc U ,by using differential superordinations and subordinations. The introduced results have properties of differential subordinations which are analogous to differential superordination properties in sandwich theorm,in supplement to that the results of this work include with a new ideas, which can be applied on analytic and multivalent functions theory.

Refrences

[1] S. A. Al- Ameedee ,W. G. Atshan and F. A. Al-Maamori, On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics, 23(4)(2020), 803-809 .

[2] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori ,Some new results of differential subordiantions for higher-order derivatives of multivalent functions, Journal of Physics : Conference Series ,1804 (2021) 012111 ,1-11.

[3] R. M. Ali, V. Ravichandran and N. Seenivasagan , Differential subordination and superordination of analytic functions defined by the multiplier, Math. transformation Inequal . Appl., 12 (2009), 123-139.

[4] W. G. Atshan and A. A. R. Ali , On some sandwich theorems of analytic functions involving Noor –Sălăgean operator ,Advances in Mathematics: Scientific Journal , 9(10)(2020), 8455-8467.

[5] W. G. Atshan , A. H. Battor and A. F. Abaas, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics , 24(3)(2021) , 579-591.

[6] W. G. Atshan and R. A. Hadi , Some differential subordination and superordination results of p-valent functions defined by differential operator, Journal of Physics: Conference Series ,1664 (2020) 012043 ,1-15 .

[7] W. G. Atshan and S. R. Kulkarni , On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator ,Journal of Inequalities in Pure and Applied Mathematics , 10(2)(2009) ,Article 53 , 11 pp.

[8] T. Bullboacă ,Classes of first - order differential superordinations ,Demonstration Math ., 35(2) (2002) , 287 - 287 .

[9] T. Bullboacă , Differential subordinations and superordinations , Recent results, House of scientific Book Publ.Cluj-Napoca (2005).

[10] J. Choi and H. M . Srivastava, Certain families of series associated with the Hurwitz – Lerch Zeta function, Appl. Math. Comput ., 170(2005) ,399-409 .

[11] C. Ferreira and J. L. Lopez ,Asymptotic expansions of the Hurwitz – Lerch Zeta function ,J. Math. Anal ., 298(2004) ,210-224 .

[12] S. D. Lin and H.M. Srivastava, Some families of the Hurwitz-Lerch Zeta function and associated fractional derivative and other integral representation , Appl.Math. Comp. ,154(2004) ,725 -733 .

[13] S. D. Lin and H.M. Srivastava and P. Y. Wang , some expansion formulas for a class of generalized Hurwitz-Lerch Zeta function , Integral Transforms Spec.Funct. 17(2006) ,817 -827 .

[14] Q. M. Luo and H. M. Srivastava , some generalization of the Apodtol –Bernoulli and Apostol-polnomials , J. Math. Anal. ,308(2005) ,290-302 .

[15] S. S. Miller and P. T. Mocanu , Differential subordinations: Theory and Applications Mathematics ,225 ,Marcel Dekker,New York and Basel, (2000).

[16] S. S. Miller and P. T. Mocanu, Subordinations of differential superodinations, Complex Variables, 48(10) (2003), 815-826.

[17] G. S. Sălăgean , Subclasses of univalent functions , Lecture Notes in Math., 1013 ,Springer- Verlag ,(1983) ,362-372 .

[18] T. N. Shanmugam , V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions , Aust. J. Math. Anal. Appl., 3 (1) (2006),1-11.

[19] T. N. Shanmugam, V. Rvichandran and S. Sivasubramanian ,Differential sandwich theorems for subclasses of analytic functions ,Aust .J. Math .Anal. ,3 ,Article 8(2006) .,1 -11 .

[20] H. M. Srivastava and A. A. Attiya ,An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination,Integral Transforms Spec.Funct.,18,(2007),207-216.

[21] S. R. Swamy ,Inclusion properties of certion subclasses of Analytic functions , International Mathematical Forum ,Vol. 7, No .36 ,(2012), 1751 -1760 .

[22] S. R. Swamy,Sandwich theorems for p-valent functionsDefined by certain integral operator, Int. J. Math.Arch.,4(3),101-107,2013.

[23] R. M. El-Ashwah and M. K. Aouf , Differential subordination and superordination for certain subclasses of p-valent functions , Mathematical and Computer Modelling ,51(5-6)(2010) ,349-360.

[24] P. Gochhayat , Sandwich-type theorems of some subclasses of multivalent functions involving Dziok-Srivastava operator , Acta Univ. Apulensis,32(2012),31-47 .

[25] P. Gochhayat , Sandwich-type results for a class of functions defined by a generalized differential operator, Math. Vesink,65(2)(2013),178-186 .