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Generalized Modulus of Smoothness and $K - functional$.

Bushra Kadhum Awaad [1*], Eman Samer Bhaya [2]
[1]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babyel, Iraq.
[2]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babyel, Iraq.
[2]Department of Mathematics, College of Education, Al- Zahraa University for Women, Karbala, Iraq.
bushra.k@uokerbala.edu..iq
emanbhaya@itnet.uobabylon.edu.iq
emanbhaya@alzahraa.edu.iq

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Abstract

Many articles introduced about direct and inverse theorems interims of ordinary modulus of smoothness andK-functional. Here we shall define a generalized modulus of smoothness andK-functional, then we prove they are equivalent.

Keywords: K—functional, Modulus etc.

1. Introduction

Many articles such us[1], [3], [6], [8],[9],[10] introduced approximation theorems interims of the ordinary modulus of smoothness, with ordinary symmetric difference. Here we shall define new symmetric difference, then we use it to obtain anew modulus of smoothness and $K - functional$, in anew quasi normed spaces, call it $L_{p,\beta}$. let us define L_p ,$0 < p < 1$, an $[-1,1]$ as the spaces of all measurable functions satisfies :

$$\|f\|_p = \left(\int_{-1}^{1} |f|^p\right)^{\frac{1}{p}} < \infty . \qquad [9]$$

Define $L_{p,\beta}$, measurable function spaces of functions f with domain $[-1,1]$.

$$\sqrt{(1-x)^{\beta}}\, f \in L_P ,$$

It mean

$$\|f\|_{p,\beta} = \left(\int_{-1}^{1} (\left|f(x)(1-x)^{\frac{\beta}{2}})\right|^p dx\right)^{\frac{1}{p}},$$

$$= \left\|f \quad (1-x)^{\frac{\beta}{2}}\right\|_p.$$

Let $E_n(f)_{p,\beta}$, best approximation error of f in $L_{p,\beta}$,using algebraic $< n$ in polynomials of degree $L_{p,\beta}$, and

$$E_n(f)_{p,\beta} = \inf_{p_{n \in IP_n}} \|f - p_n\| , [8]$$

Where,

$\|\mathbb{P}_n$ is $n - 1$, degree algebraic polynomials spaces .

Let us define the operator

$$D_{x,\nu,\mu} = \left(\sqrt{1-x}\right)^{-\nu}(1+x)^{-\mu}\frac{d}{dx}\left(\sqrt{(1-x)}\right)^{\nu+1}\left(\sqrt{(1+x)}\right)^{\mu+1}\frac{d}{dx}. \quad [4]$$

Peetre $K-functional$ on $L_{p,\alpha}$ and $IP_n$is defined by

$$K(f,\delta)_{p,\alpha} = \inf_{g\in IP_n}\left(\|f - g\|_{p,\alpha} + \delta^2\| D_{x,2,2g(x)} \|_{p,\alpha}\right), [4].$$

Let us introduce usual smoothness modulus.

$$\tilde{\omega}_1 (f,\delta)_{p,\beta} = \sup_{|t|\le\delta}\|\tau_t(f) - f\|_{p,\alpha} . \quad (1)$$

Where,

$$\tau_t(f) = \frac{1}{\pi(1-x^2)\cos^4\frac{t}{2}}\int_0^\pi \Big(2\left(\sqrt{1-x^2}\cos t + x\sin t \cos \varphi + \sqrt{1-x^2}(1-\cos t)\sin^2\varphi\right)^2 -$$
$$1 + \left(x\cos t - \sqrt{1-x^2}\sin t\cos\varphi\right)^2\Big) f\left(x\cos t - \sqrt{1-x^2}\sin t\cos\varphi\right) d\varphi .$$

Let $f \in L_{p,\alpha,}$. put $y = \cos t, z = \cos\varphi$, in the $\tau_t(f)$, let $\tau_y(f)$. Then

$$\tau_t(f) = \frac{4}{\pi(1-x^2)(1+y)^2}\int_{-1}^1 B_y(x,z,R) f(R) \frac{dz}{\sqrt{1-z^2}}, \quad [1].$$

where ,

$$R = xy - z\sqrt{1-x^2}\sqrt{1-y^2},$$

$$B_y(x,z,R) = 2\left(\sqrt{1-x^2}y + zx\sqrt{1-y^2} + \sqrt{1-x^2}(1-y)(1-z^2)\right)^2 - (1-R^2).$$

2. Auxiliary Results

To prove our main theorem we need some results, that we make use of them in our proof. Let us begin with Lemma .2.1.

For $z = \cos\varphi$ and $R = x\cos t - z\sqrt{1-x^2}\sin t, f \in L_{p,\beta}[-1,1]$, we get

$$\left\| \frac{1}{1-x^2}\int_{-1}^1 (1-R^2)|f(R)|\frac{dz}{\sqrt{1-z^2}}\right\|_{p,\beta} \le C\,|f| \quad (2.1)$$

Proof

Using simple calculating of integrals we obtain the result in (2.1).

Lemma.2.2. [1]

$$\left|B_y(x,z,R)\right| \le 19(1-R^2)$$

Where,

$R = x\cos t - z\sqrt{1-x^2}\sin t, z = \cos\varphi$ and $x = \cos\theta_1$.

Lemma. 2.3.

If $f \in L_{p,\beta}[-1,1]$, we get

$$\| \tau_t(f)\|_{p,\beta} \leq \frac{19C(p)}{\pi\cos^4\frac{t}{2}} \|f\|_{p,\beta},$$

Where $c > 0$.

Proof.

$$\| \tau_t(f)\|_{p,\beta} = \left\|\frac{1}{\pi\cos^4\frac{t}{2}}\frac{1}{1-x^2}\int_{-1}^{1} B_{\cos t}(x,z,R)\, f(R)\frac{dz}{\sqrt{1-z^2}}\right\|_{p,\beta}$$

$$\| \tau_t(f)\|_{p,\beta} = \frac{1}{\pi\cos^4\frac{t}{2}}\left\|\frac{1}{1-x^2}\int_{-1}^{1} B_{\cos t}(x,z,R)\, f(R)\frac{dz}{\sqrt{1-z^2}}\right\|_{p,\beta}$$

$$\| \tau_t(f)\|_{p,\beta} = \frac{1}{\pi\cos^4\frac{t}{2}}\left(\int_{-1}^{1}\left|\frac{1}{1-x^2}\int_{-1}^{1}\int_{-1}^{1} B_{\cos t}(x,z,R)\, f(R)\frac{dz}{\sqrt{1-z^2}}\right|^p (1-x)^{\frac{\beta}{2}}\,dx\right)^{\frac{1}{p}}.$$

$$\| \tau_t(f)\|_{p,\beta} \leq \frac{1}{\pi\cos^4\frac{t}{2}}\left(\int_{-1}^{1}\left(\frac{1}{1-x^2}\int_{-1}^{1}|B_{\cos t}(x,z,R)||f(R)|\frac{dz}{\sqrt{1-z^2}}\right)^p (1-x)^{\frac{\beta}{2}}\,dx\right)^{\frac{1}{p}}.$$

Using Lemma (2.2), we get

$$\| \tau_t(f)\|_{p,\beta} \leq \frac{1}{\pi\cos^4\frac{t}{2}}\left(\int_{-1}^{1}\left(\frac{1}{1-x^2}\int_{-1}^{1} 19(1-R^2)\,|f(R)|\frac{dz}{\sqrt{1-z^2}}\right)^p (1-x)^{\frac{\beta}{2}}\,dx\right)^{\frac{1}{p}}.$$

Using Lemma (2.1) ,to obtain

$$\| \tau_t(f)\|_{p,\beta} \leq \frac{19C(p)}{\pi\cos^4\frac{t}{2}}\|f\|_{p,\beta}.$$

Lemma.2.4. [2].

If $y \in (-1,1)$, and $x \in (-1,1)$, for almost every then

$$\tau_y\left(D_{x,2,2}f\right) = D_{x,2,2}\tau_y(f).$$

Lemma.2.5.[3].

let f' be an absolutely continuous with domain $[a,b] \subset (-1,1)$,with $D_{x,2,2}f \in L_{1,2.}$, So almost every $x \in (-1,1)$ and every $y \in (-1,1)$

$$\tau_y(f) - f = \int_1^y (1-\nu)^{-1}(1-\nu)^{-5}\int_1^\nu (1+u)^4\,\tau_u\left(D_{x,2,2}f,x\right)du\,d\nu\ .$$

$$\tau_y(f) - \tau_0(f) = -\int_1^y (1-\nu)^{-1}(1-\nu)^{-5}\int_\nu^{-1} (1+u)^4\,\tau_u\left(D_{x,2,2}f,x\right)du\,d\nu\ .$$

Lemma.2.6.[3].

let f' be absolutely continuous with domain $[a,b] \subset (-1,1)$,with $D_{x,2,2}f(x) \in L_{1,2.}$ So almost every $x \in (-1,1)$ and every $t \in (-\pi,\pi)$.

$\tau_t\,(f)-f=\int_0^t\left(\sin\frac{v}{2}\right)^{-1}\left(\cos\frac{v}{2}\right)^{-9}\int_0^v\tau_u\left(D_{x,2,2}f,x\right)\ \sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^9 dudv$.

$\tau_t\,(f)\ -\ \tau_{\frac{\pi}{2}}\,(f)\ \ =-\ \int_{\frac{\pi}{2}}^t\left(\sin\frac{v}{2}\right)^{-1}\left(\cos\frac{v}{2}\right)^{-9}\int_v^\pi\tau_u\left(D_{x,2,2}f,x\right)\ \sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^9\ dudv$.

Lemma.2.7.

If $f\in L_{p,\beta}$, then

$$E_n(f)_{p,\beta}\ \leq C(p)\frac{1}{n^2}\ \|D(f)\|_{p,\beta}\,.$$

Proof.

By using $E_n(f)_{p,\beta}\ \ \leq\ \|f-Q\|_{p,\beta}$,

where Q is algebraic polynomial of degree less than $n-1$,
and

$Q(x)=\frac{1}{\gamma_m}\int_0^\pi T_{2;\cos t}(f,x)\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^{2q+4}\ \sin^5 tdt$

where

$$\gamma_m=\int_0^\pi\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^{2q+4}\ \sin^5 tdt\ ,$$

Then

$$E_n(f)_{p,\beta}\ \leq\left\|f-\frac{1}{\gamma_m}\int_0^\pi T_{2;\cos t}(f,x)\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^{2q+4}\ \sin^5 tdt\right\|_{p,\beta}.$$

$=\left\|\frac{1}{\gamma_m}\int_0^\pi\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^{2q+4}\ \sin^5 tdt\ \ f(x)-\frac{1}{\gamma_m}\int_0^\pi T(f)\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^{2q+4}\ \sin^5 tdt\right\|_{p,\beta}$

$$E_n(f)_{p,\beta}\leq C(p)\|f-T\|_{p,\beta}.$$

Using the

$$\|\tau_t\,(g)\ -\ g\,\|_{p,\beta}\ \leq\ C(p)\frac{1}{\cos^4\frac{t}{2}}\ t^2\ \left\|D_{x,2,2g}\ \right\|_{p,\beta},[2].$$

To obtain

$$E_n(f)_{p,\beta}\ \leq\ C(p)\ \frac{1}{n^2}\ \ \|D(f)\|_{p,\beta}\,.$$

3. The Main Result

After we define our new modulus of smoothness, and $k-functional$. we shall relate then to others. That is we shall introduce the theorem:

Theorem.3.1.

For $f \in L_{p,\beta}$,$0 < p < 1$, we have

$$C_1(p)\, K(f,\delta)_{p,\beta} \le \tilde{\omega}(f,\delta)_{p,\beta} \le C_2(p) \frac{1}{\cos^4\frac{\delta}{2}}\, K(f,\delta)_{p,\beta} .$$

Where, $C_1(p)$ and $C_2(p)$ are positive constants depending on p only.

Proof .

Let $\mathbb{P}_n$be the set of n th degree algebraic polynomials we shall prove (3.1)

$$\|\tau_t(g) - g\|_{p,\beta} \le C(p) \frac{1}{\cos^4\frac{t}{2}}\, t^2 \left\|D_{x,2,2g}\right\|_{p,\beta},$$

Where, $C(p)$ is constant depend on p only.

If $0 < t \le \frac{\pi}{2}$, then lemma.2.6.implies

$I_1 = \|\tau_t(g) - g\|_{p,\beta}$.

$= \left\|\int_0^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_0^v \tau_u\left(D_{x,2,2}g, x\right) \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^9 du dv\right\|_{p,\beta}$.

Applying lemma .2.3 $\left[\| \tau_t(f)\|_{p,\beta} \le \frac{19C(p)}{\pi\cos^4\frac{t}{2}} \|f\|_{p,\beta}\right]$, we get

$$I_1 \le \int_0^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_0^v \tau_u\left(D_{x,2,2}g,\right) \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^9 du dv.$$

$$I_1 \le C(p)\left\|D_{x,2,2}g\right\|_{p,\beta} \int_0^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_0^v \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^5 du dv.$$

The inequality

$$\int_0^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_0^v \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^5 du dv \le C(p) t^2.$$

For $0 < t \le \frac{\pi}{2}$, we obtain

$I_1 \le C(p) t^2 \left\|D_{x,2,2}g\right\|_{p,\beta} \le C(p) \frac{1}{\cos^4\frac{t}{2}}\, t^2 \left\|D_{x,2,2}g\right\|_{p,\beta}$.

If $\frac{\pi}{2} \le t < \pi$, so by Lemma (2.6).

$\left[\tau_t(f) - \tau_{\frac{\pi}{2}}(f) = -\int_{\frac{\pi}{2}}^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_v^\pi \tau_u\left(D_{x,2,2}f\right) \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^9 du dv.\right]$We get

$I_2 = \left\|\tau_t(f) - \tau_{\frac{\pi}{2}}(f)\right\|_{p,\beta} = \left\|\int_{\frac{\pi}{2}}^t \left(\sin\frac{v}{2}\right)^{-1} \left(\cos\frac{v}{2}\right)^{-9} \int_v^\pi \tau_u\left(D_{x,2,2}g, x\right) \sin\frac{u}{2} \left(\cos\frac{u}{2}\right)^9 du dv\right\|_{p,\beta}$.

Applying Lemma. 2.3 $\left[\| \quad \tau_t\,(f)\|_{p,\beta} \le \frac{19\mathrm{C}(p)}{\pi\cos^4\frac{t}{2}}\|f\|_{p,\beta}\right]$, we get

$I_2 \le \mathrm{C}(p)\left\|D_{x,2,2}g\right\|_{p,\beta}\int_{\frac{\pi}{2}}^{t}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_{\nu}^{\pi}\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{5}\,dud\nu.$

Assume $\frac{\pi}{2} \le t < \pi$, so

$$\int_{\frac{\pi}{2}}^{t}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_{\nu}^{\pi}\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{5}\,dud\nu \le \mathrm{C}(p)\frac{1}{\cos^4\frac{t}{2}},$$

It the follows that

(3.2) $\qquad I_2 \le \mathrm{C}(p)\frac{1}{\cos^4\frac{t}{2}}\left\|D_{x,2,2}g\right\|_{p,\beta} \le \mathrm{C}(p)\frac{1}{\cos^4\frac{t}{2}}t^2\left\|D_{x,2,2}g\right\|_{p,\beta}.$

Since

$$\|\tau_t\,(g)-g\|_{p,\beta} \le \left\|\tau_t\,(g)-\tau_{\frac{\pi}{2}}(g)\right\|_{p,\beta} + \left\|\tau_{\frac{\pi}{2}}(g)-g\right\|_{p,\beta}$$

Using (3.2)and (3.1)we obtain, for

0< $t \le \frac{\pi}{2}$, that

$$\|\tau_t\,(g)-g\|_{p,\beta} \le \mathrm{C}(p)\,\frac{1}{\cos^4\frac{t}{2}}t^2\left\|D_{x,2,2}g\right\|_{p,\beta}.$$

For $\frac{\pi}{2} < t \le \pi$,

we proved inequality (3.1) for $0 < t \le \pi$.

$$\tau_{\cos t}(g) = \tau_{\cos -t}(g,),$$

Let $f \in L_{p,\beta}$ and $0 \le |t| \le \delta < \pi$. so for any $g \in L_{p,\beta}$.

Using Lemma 2.3 we get

$\|\tau_t\,(f)-f\|_{p,\beta} \le \mathrm{C}(p)\big(\|\tau_t\,(f-g,x)\|_{p,\beta} + \|\tau_t\,(g)-g\|_{p,\beta} + \|g-f\|_{p,\beta}\big).$

$\|\tau_t\,(f)-f\|_{p,\beta} \le \mathrm{C}(p)\frac{1}{\cos^4\frac{t}{2}}\|f-g\|_{p,\beta} + \|\tau_t(g)-g\|_{p,\beta}.$

By using(3.1), to get

$$\|\tau_t\,(f)-f\|_{p,\beta} \le \mathrm{C}(p)\frac{1}{\cos^4\frac{t}{2}}\|f-g\|_{p,\beta} + t^2\left\|D_{x,2,2}g\right\|_{p,\beta}.$$

To prove the second inequality, assume function

$g_\delta = \frac{1}{\mathcal{K}(\delta)}\int_0^{\delta}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_0^{\nu}\tau_u(f,x)\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{9}\,dud\nu,$

Where

$$\mathcal{K}(\delta)=\int_0^{\delta}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_0^{\nu}\tau_u(f)\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{9}\,dud\nu.$$

Let $0<\delta<\frac{\pi}{2}$, then

(3.3) $$C(p)\delta^2\leq\mathcal{K}(\delta)\leq C(p)\delta^2.$$

Applying Lemma.2.3, to get $\|g_\delta\|_{p,\beta}\leq$
$\frac{1}{\mathcal{K}(\delta)}\int_0^{\delta}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_0^{\nu}\|\tau_u(f)\|_{p,\beta}\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{9}\,dud\nu\leq C(p)\frac{1}{\cos^4\frac{\delta}{2}}\|f\|_{p,\beta}.$

That is $g_\delta\in L_{p,\beta}$.

Put $g\;=-\int_0^x(1-y^2)^{-3}\int_y^1(1-z^2)^2\left(f(z)-\frac{m_1}{m_2}\right)dzdy,$

Where

$m_1=\int_{-1}^1(1-z^2)^2f(z)dz$, $m_2=\int_{-1}^1(1-z^2)^2\,dz.$

$$D_{x,2,2g}(x)=f-\frac{m_1}{m_2},$$

$D_{x,2,2g}(x)\;=\left(\sqrt{1-x}\right)^{-2}(1+x)^{-2}\frac{dg}{dx}\left(\sqrt{1-x}\right)^{3}\left(\sqrt{1+x}\right)^{3}\frac{dg}{dx},$

We have

$$g_\delta\;=\frac{1}{\mathcal{K}(\delta)}\int_0^{\delta}\left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_0^{\nu}\tau_u\left(D_{x,2,2g},x\right)\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^{9}\,dud\nu+\frac{m_1}{m_2}.$$

Using Lemma.2.6.we get

(3.4) $$g_\delta\;=\;\frac{1}{\mathcal{K}(\delta)}(\tau_\delta(g)-g)+\frac{m_1}{m_2}.$$

Applying the operator $D_{x,2,2g}$ for (3.4) and Lemm2.4,
$\left(\tau_y\left(D_{x,2,2}f,\right)=\;D_{x,2,2}\tau_y(f).\right)$,

we get

$D_{x,2,2g\delta}(x),\,g_\delta(x)\;\;=\frac{1}{\mathcal{K}(\delta)}\left(D_{x,2,2}g(\tau_\delta)\;-\;D_{x,2,2g}(x)\right)+\frac{m_1}{m_2}.$

$D_{x,2,2g\delta}(x)\;\;=\frac{1}{\mathcal{K}(\delta)}\left(\tau_\delta\left(D_{x,2,2g},x\right)-f\right)-\frac{m_1}{m_2}+\frac{m_1}{m_2}.$

(3.5) $$D_{x,2,2g\delta}(x)\;=\frac{1}{\mathcal{K}(\delta)}\;\;(\tau_\delta(f)-f).$$

Therefore, by Lemmas. 2.3. and 2.4 ,also $g_\delta\in L_{p,\beta}$.

Then using (3.5) equality and inequality (3.3) , we get

(3.6) $\left\|D_{x,2,2g\delta}(x)\right\|_{p,\beta} \leq \mathrm{C}(p)\frac{1}{\delta^2}\|\tau_\delta(f)-f\|_{p,\beta}.$

Then using(3.6), to get

$$(3.7) \qquad \left\|D_{x,2,2g\delta}(x)\right\|_{p,\beta} \leq \mathrm{C}(p)\frac{1}{\delta^2}\tilde{\omega}(f,\delta)_{p,\beta.}$$

And

$$(3.8) \qquad \|f-g_\delta\|_{p,\beta}$$
$$\leq \frac{1}{\mathcal{K}(\delta)}\int_0^\delta \left(\sin\frac{\nu}{2}\right)^{-1}\left(\cos\frac{\nu}{2}\right)^{-9}\int_0^\nu \|f-\tau_u(f)\|_{p,\beta}\sin\frac{u}{2}\left(\cos\frac{u}{2}\right)^9 dud\nu \leq \tilde{\omega}(f,\delta)_{p,\beta}.$$

For $0<\delta\leq\frac{\pi}{2}$, we have proved that

$$I(\delta)=\|f-g_\delta\|_{p,\beta}+\delta^2\left\|D_{x,2,2g\delta}(x)\right\|_{p,\beta} \leq \mathrm{C}(p)\tilde{\omega}(f,\delta)_{p,\beta}.$$

For $\frac{\pi}{2}<\delta\leq\pi$, we get $\delta^2<\pi^2.1$, and, $1<\frac{\pi}{2}$,

$$K(f,\delta)_{p,\beta} \leq \pi^2\left(\|f-g_1\|_{p,\beta}+1^2\left\|D_{x,2,2g1}(x)\right\|_{p,\beta}\right)$$

Using (3.7) $and(3.8)$,to obtain

$$K(f,\delta)_{p,\beta} \leq \mathrm{C}(p)\,\tilde{\omega}(f,1)_{p,\beta} \leq \tilde{\omega}(f,1)_{p,\beta}.$$

Conclusions

K functional and modulus of smoothness are equivalent on weighted spaces.

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