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An Approach to Goldie Extending Modules on the Class of Cyclic Submodules

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Abstract: In this article, we provide a class of modules that is comparable to G^z-extending and G^r-extending modules. We specify what a module M is as G^p-extending if and only if for each cyclic submodule A of M, there exists a direct summand D of M such that A∩ D is essential in both A and D. We look into G^p-extending modules and locate this inference between the other extending properties. We present some of characterizations of G^p-extending condition. We show that the direct sum of G^p-extending need not be G^p-extending and deal with decompositions for G^p-extending concept.

Keywords: cyclic submodules, P-extending modules, Goldie extending modules, G^p-extending.

1. Introduction

Throughout this paper, all rings are associative with unitary, R denotes such a ring, and all modules are unital right R- modules. In the spirit of [1] , for a module M, think of the following relations on the set of submodules of M:"

$A\alpha B$ if and only if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$.

$A\beta B$ if and only if $A \cap B \leq_e A$ and $A \cap B \leq_e B$. Recall that β is an equivalence relation. "It is clear that a module M is extending (or CS) if and only if for each submodule A of M, there is a direct summand D of M such that $A\alpha D$, (see [1,2]). Further a module M is called Goldie extending module (or G-extending) if and only if for each submodule A of M , there is a direct summand D of M such that $A\beta D$ or equivalently, for each closed submodule A in M, there is a direct summand D of M such that $A\beta D$ (see [1]). Obviously, every extending module is G-extending.

As a generalization of CS-modules is p-extending (see [3,4]). Recall that a module M is called p-extending if every cyclic submodule of M is essential in a direct summand of M."

"In this paper, we study a module condition including the β relation on the set of all cyclic submodules of a module. We call a module M is G^p-extending if for every cyclic submodule A of M, there is a direct summand D of M such that $A\beta D$. A ring R is G^p-extending if R_R is G^p-extending module. It is clear that the class of G^p-extending modules property contains the type of G-extending modules. The notion of G^p-extending generalizes both of G-extending, extending and p-extending modules."

"In section 2, we consider connections between G^p-extending property, p-extending and G-extending conditions. Moreover, we give sufficient circumstances under which p-extending and G^p-extending modules are equivalent.

Section 3, is devoted to the characterizations of G^p-extending modules. Since the direct sum of G^p-extending modules need not be G^p-extending, we focus when a direct sum of G^p-extending modules is also G^p-extending. Also, we give sufficient conditions under which the direct summand of G^p-extending is also G^p-extending. These are introduced in section 4. Also, in section 4, we investigate G^p-extending essential extensions of a module or ring."

Following [5], M is called UC-module if every submodule of M has a unique closure in M.

2. Preliminary results.

"The G^p-extending notion is based on two tools, namely an equivalence relation on cyclic submodules of a module M. Let us begin by mentioning basic facts about them. First recall the following relations on the set of submodules of M (see [1]).

(i) $A\alpha B$ if and only if there exists a submodule C of M such that $A \leq_e C$ and $B \leq_e C$.

(ii) $A\beta B$ if and only if $A \cap B \leq_e A$ and $A \cap B \leq_e B$.

Observe that α is reflexive and symmetric, but it may not be transitive. However, β is an equivalence relation. Note that for submodules of module M. if $A\alpha B$, then $A\beta B$.

Proposition 2.1: A module M is p-extending if and only if for each cyclic submodule A of M, there is a direct summand D of M such that $A\alpha D$.

Proof: The proof is routine."

"Motivated by proposition 2.1 and Akalan, Birkenmeier, Tercan's use of the β equivalence relation in [1]. As a generalization of Goldie extending modules, we introduce a class of modules which is analogous to that of G^z-extending and G^r-extending modules which are introduced in [6] and [7] respectively."

Definition 2.2: We call a module M is G^p-extending module if for each cyclic submodule A of M, there is a direct summand D of M such that $A\beta D$.

"Note that M is G-extending if and only if for each submodule A of M there is a direct summand D of M such that $A\beta D$. It is clear that the class of G^p-extending contains both of the classes of G-extending and p-extending modules.

Now, we locate the G^p-extending condition with respect to several known generalizations of the extending property."

Proposition 2.3: Make M a module. Let's think about the aforementioned circumstances.

(i) M is CS.

(ii) M is G-extending.

(iii) M is G^p-extending.

(iv) M is p-extending.

"Then (i) ⟹(ii) ⟹(iii) and (i) ⟹ (iv)⟹(iii). In general, the converse implications do not hold."

Proof: (i) ⟹(ii) ⟹(iii) and (i) ⟹ (iv)⟹(iii) are clear.

(ii)⇏(i)"Let M be the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Q}$, where p is any prime integer. Then $M_{\mathbb{Z}}$ is G-extending by [1, corollary (3.3)]. However $M_{\mathbb{Z}}$ is not extending [8 , Example 10]."

(iii)⇏(ii) Let $M_2(R)$be the ring as in [9, Example 13.8]. Then $M_2(R)$ is a von Neumann regular ring which is not a Baer ring. Hence it is neither right nor left CS, by [10, example 2.7] , however it's well acknowledged that each von Neumann regular ring is nonsingular, therefore $M_2(R)$ is not is G-extending, see [1, Proposition 1.8]. Also, this is an example to show that (iv)⇏(i).

(iii)⇏(iv) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. Then $M_{\mathbb{Z}}$ is G^p-extending but not P-extending, see [1, Corollary 3.3].

The condition under which G^p-extending and p-extending modules are equivalent is stated in the following proposal.

Proposition 2.4:" Let M be a module.

(i) If M is a UC- module. Then M is G^p-extending if and only if M is P-extending.

(ii) If M is a nonsingular module. Then M is G^p-extending if and only if M is P-extending.

(iii) If M is an indecomposable module. Then M is G^p-extending if and only if M is P-extending."

Proof:

(i) "Assume that M is G^p-extending and let A be a cyclic submodule of M, then there exists a direct D of M such that $A\beta D$. One can easily show that $(A \cap D)\alpha A$ and $(A \cap D)\alpha D$. But M is UC module, therefore α is transitive, hence $A\alpha D$. Thus M is P-extending. The converse is clear.

(ii) Let M be a G^p-extending and let A be a cyclic submodule of M, then there is a direct D of M such that $A\beta D$. It is sufficient to show that $A \leq D$. Since $\frac{A+D}{D} \cong \frac{A}{A \cap D}$ is singular and $\frac{A+D}{D} \leq \frac{M}{D} \cong D'$ is nonsingular, hence $A+D = D$ which implies that $A \leq D$. The converse is obvious.

(iii) Let M be a G^p-extending and let A be a cyclic submodule of M, then there is a direct D of M such that $A\beta D$. Since M is indecomposable, then $D=M$. Thus M is P-extending module. The converse is clear."

Corollary 2.5: "Let M be an indecomposable module. Then the following statements are equivalent:

(i) M is uniform.

(ii) M is CS.

(iii) M is G-extending.

(iv) M is P-extending.

(v) M is G^p-extending."

Example 2.6:

"Let F be a field and V be a vector space over F with dim (F_V)=2. Let R be the trivial extension of F with V, i.e,

$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$. Since R_R is indecomposable which is not CS, then R_R is not G^p-extending."

3. Characterizations of G^p-extending.

"In this section, we give equivalent conditions to G^p-extending property.

We start by the following theorem.

Theorem 3.1: An R- module M is G^p-extending if and only if for each cyclic submodule A of M, there is a direct summand D of M such that $A\beta D$ and D' is a complement of A, where $M = D \oplus D'$.

Proof: Suppose that M is G^p-extending , let A be a cyclic submodule of M, there is a direct summand D of M such that $A\beta D$. Let $M = D \oplus D'$, for some submodule D' of M. Since $A \cap D \leq_e A$, then $A \cap D' = 0$. Now, let B be a submodule of M such that $D' \leq B$ and $A \cap B = 0$. Since $A \cap D \leq_e D$, then $B \cap D = 0$. But D' is a complement of D, therefore B=D'. Thus, D' is a complement of A. The converse is clear."

"The next result gives another characterization to G^p-extending modules.

Proposition 3.2: Let M be an R- module, the following conditions are equivalent:

(i) M is G^p-extending.

(ii) For all cyclic submodule A of M , there exists a submodule X of M and a direct summand D of M such that $X \leq_e A$ and $X \leq_e D$.

(iii) For every cyclic submodule A of M there exists a complement B of A and a complement C of B such that $A\beta C$ and each homomorphism $f: C \oplus B \to M$ extends to a homomorphism $g: M \to M$.

Proof: (i) ⟹(ii) Assume that M is G^p-extending and let A be a cyclic submodule of M, there is a direct summand D of M such that $A\beta D$, hence $A \cap D \leq_e A$ and $A \cap D \leq_e D$. Take $X = A \cap D$, we get the result.

(ii)⟹(iii) Let A be a cyclic submodule of M. By (ii), there exists a submodule X of M and a direct summand D of M such that $M = D \oplus D'$, $X \leq_e A$ and $X \leq_e D$. Take D = C and D' = B.

(iii)⟹(i) Let A be a cyclic submodule of M. From (iii), there exists a complement B of A and a complement C of B such that $A\beta C$ and every homomorphism $f: C \oplus B \to M$ extends to a homomorphism $g: M \to M$and by [11, Lemma 3.97], D is a direct summand of M, hence M is G^p-extending."

Theorem 3.3 : A module M is G^p-extending if and only if for every direct summand A of the injective hull E(M) of M with $A \cap M$ is cyclic submodule of M, there is a direct summand D of M such that $(A \cap M)\beta D$.

Proof: "Let A be a cyclic submodule of M and let B be a complement of A, then $A \oplus B \leq_e M$. Since $M \leq_e$ E(M), then $A \oplus B \leq_e$ E(M) implies $E(M) = E(A) \oplus E(B)$. It can be seen that $E(A) \cap M$ is cyclic submodule in M. By our assumption, there is a direct summand D of M such that $(E(A) \cap M)\beta D$. But we have$(A \cap M)\beta(E(A) \cap M)$, hence $A\beta D$. The converse implication is clear."

Theorem 3.4:" Suppose M is an R-module. The assertions that follow are identical.

(i) M is G^p-extending module.

(ii) A decomposition exists for each cyclic submodule A of the module M. $M = D \oplus D'$, such that $(D'+A)\ \beta M$.

(ii) Fore very cyclic submodule A of M, there is a decomposition $\frac{M}{A} = \frac{L}{A} \oplus \frac{K}{A}$ such that L is a direct summand of M and $K \beta M$."

Proof: (i)$\Rightarrow$(ii)"Let M be a G^p-extending and let A be a cyclic submodule of M, there exists direct summand D of M such that $A\ \beta D$, then $M = D \oplus D'$, $D' \leq M$. Since $\{A, D'\}$ is an independent family, then $(A+D')\ \beta M$, see [12, Proposition 1.4]."

(ii)$\Rightarrow$(iii) "Let A be a cyclic submodule of M. By (ii), there is a decomposition $M = D \oplus D'$, such that $(D'+A)\ \beta M$. Claim that $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$. Since $M = D \oplus D'$, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D}{A} + \frac{D'+A}{A}$ and $\frac{D+A}{A} \cap \frac{D'+A}{A} = \frac{D \cap (D'+A)}{A} = \frac{A+(D \cap D')}{A}$ $=A$, hence $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$. Take $K = D'+A$ and $L = D+A$, so we get the result."

(iii)$\Rightarrow$(i)"To show that M is G^p-extending, let A be a cyclic submodule of M. By (iii), there is a decomposition $\frac{M}{A} = \frac{L}{A} \oplus \frac{K}{A}$ such that L is a direct summand of M and $K\beta M$. It is enough to show that $A\ \beta L$. Let $i : L \to M$ be the injection map. Since $K\beta M$, then $i^{-1}(K)\ \beta i^{-1}(M)$, that is $(L \cap K)\beta D$. One can easily show that $L \cap K = A$, so M is G^p-extending module."

"*Proposition 3.5:* Let M be an R-module. Then M is G^P-extending module if and only if for every cyclic submodule A of M, there exists an idempotent $f \in$ End (M) such that $A\ \beta f(M)$."

4. Decompositions.

"There are nonsingular modules $M = M_1 \oplus M_2$ in which M_1 and M_2 are P-extending, but M is not P-extending (e.g, Let $R = \mathbb{Z}[x]$ be a polynomial ring of integers and let $M = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$). Note that $\mathbb{Z}[x]$ is G −extending, by [1] and hence G^p-extending but M is not P-extending which is nonsingular, thus by proposition 2.4 M is not G^p-extending. Next, we give various conditions under which the direct sum of G^p-extending is G^p-extending."

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Proposition 4.1: Let $M = M_1 \oplus M_2$ be a distributive module if M_1 and M_2 are G^p-extending modules, then M is G^p-extending.

Proof: "Let A be a cyclic submodule of M. Since M is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since A is cyclic in M, then $A \cap M_1$ and $A \cap M_2$ are cyclic in M_1 and M_2 respectively. But M_1 and M_2 are G^p-extending modules, therefore there are direct summand D_1 of M_1 and D_2 of M_2 such that $(A \cap D_1) \beta D_1$ and $(A \cap D_2) \beta D_2$ hence $A\ \beta(D_1 \oplus D_2)$, by [12 , Proposition 1.4]. Thus, M is G^p-extending module."

The following statements are also easily proved by using a similar argument.

Proposition 4.2 : "Let $M = M_1 \oplus M_2$ be a duo module if M_1 and M_2 are G^p-extending modules, then M is G^p-extending."

Proposition 4.3: Let M_1 and M_2 be G^p-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is G^p-extending module.

Proposition 4.4: "Let $M = M_1 \oplus M_2$ be an R- module with M_1 being G^p-extending and M_2 is semisimple. Suppose that for any cyclic submodule A of M, $A \cap M_1$ is a direct summand of A, then M is G^p-extending.

Proof: Let A be a cyclic submodule of M, then it is easy to see that $A + M_1 = M_1 \oplus [(A + M_1) \cap M_2]$. Since M_2 is semisimple, then $(A + M_1) \cap M_2$ is a direct summand of M_2 and therefore $A + M_1$ is a direct summand of M. By our assumption, $A \cap M_1$ is a direct summand of A, then $A = (A \cap M_1) \oplus A'$, for some submodule A' of A. One can easily show that $A \cap M_1$ is cyclic in M_1. But M_1 is G^p-extending, then there is a direct summand D of M_1 such that $(A \cap M_1)\ \beta\ D$ is hence $A = ((A \cap M_1) \oplus A') \beta (M_1 + A)$. Thus, M is G^p-extending."

Proposition 4.5: "Let $M = M_1 \oplus M_2$ such that M_1 is G^p-extending and M_2 is injective module. Then M is G^p-extending if and only if for every cyclic submodule A of M such that $A \cap M_2 \neq 0$ there is a direct summand D of M such that $A \beta D$.

Proof: Suppose that for every cyclic submodule A of M such that $A \cap M_2 \neq 0$ there exists direct summand D of M such that $A \beta D$. Let A be a cyclic submodule of M such that $A \cap M_2 = 0$. By [2], there is a submodule M' of M containing A such that $M = M' \oplus M_2$. Since $M' \cong \frac{M}{M2} \cong M_1$ is G^p-extending and A is cyclic submodule of M', then there is a direct summand K of M' such that $A \beta K$. Thus, M is G^p-extending. The converse is obvious."

We now list several circumstances in which a direct summand of a module that extends G^p-extending is G^p-extending.

Proposition 4.6:"Let A be a direct summand of a G^p-extending module M , if the intersection of A with any direct summand of M is a direct summand of A, then A is G^p-extending module."

Proof: "Let X be a cyclic in A, then X is cyclic in M. But M is G^p-extending, therefore there exists a direct summand D of M such that $X \beta D$. It can be seen that $X \beta\ (A \cap D)$. By our assumption $A \cap D$ is a direct summand of A. Thus, A is G^p-extending."

Proposition 4.7:"Let A be a cyclic submodule of a G^p-extending module M ."

(i) If for each $e^2 = e \in End(M_R)$, there exists $f^2 = f \in End(A_R)$ such that $A \cap eM \leq_e fA$, then A is G^p-extending.

(ii) If for each $e^2 = e \in End(M_R)$, there exists $f^2 = f \in End(A_R)$ such that $eM\beta fM$ and $fA \subseteq A$, then A is G^p-extending.

Proof:

(i) "Let Y be a cyclic submodule of A. Hence Y is a cyclic submodule of M. By proposition 3.2, there is $X \leq_e Y$ and $e^2 = e \in End(M_R)$ such that $X \leq_e eM$. Then $X \leq_e eM \cap A \leq_e fA$, for some $f^2 = f \in End(M_R)$.Thus, A is G^p-extending."

(ii) "Let Y be a cyclic submodule of A, then Y is cyclic in M. Then there exists $e^2 = e \in End(M_R)$ such that $Y\beta eM$. Hence $Y\beta fM$. Since $fA \subseteq A$, A is G^p-extending."

Proposition 4.8:"Let K be a projection invariant cyclic submodule of M. If M is G^p-extending, then there exists $M_1 \leq M$ such that $M = M_1 \oplus K$ and K is G^p-extending.

Proof: There exists $e^2 = e \in End(M_R)$ such that $K\beta eM$. But $K = eK \oplus (1-e)K$, $eK = K \cap eM$, and $(1-e)K = K \cap (1-e)M$ because K is projection invariant, then $eK \leq_e eM$ and $eK \leq_e K$. Hence $K \cap (1-e)M = 0$. So $K = eK \leq_e eM$. Since K is cyclic in M, then $K=eM$. Let $M_1 = (1-e)M$. Therefore $M = M_1 \oplus K$. Observe that, by Proposition 4.7 (ii), K is G^p-extending."

Theorem 4.9: Let M be a G^p-extending module. If M has SIP or satisfies the C_3 condition, then any cyclic direct summand of M is G^p-extending.

Proof: "Let $M = N \oplus N'$ for some submodules N, N' of M where N is cyclic in M. Using Proposition 4.8(i) , where N is taken to be cyclic in M and applying the SIP gives that N is a G^p-extending.

Now assume that M satisfies the C_3 condition. Let $\pi: M \to N$ be the canonical projection. Let K be any cyclic submodule of N, then K is cyclic in M. By hypothesis, there exists a direct summand L of M such that $K \cap L \leq_e K$ and $\cap L \leq_e L$. Since M satisfies C_3 condition, $N' \oplus L$ is a direct summand of M. It can be seen that $N' \oplus L = N' \oplus \pi(L)$(see [11, Lemma 2.71]). Hence $\pi(L)$ is a direct summand of N. For any $0 \neq y \in \pi(L)$, $y = \pi(x)$ for some $0 \neq x \in L$. There exists an $r \in R$ such that $0 \neq xr \in K \cap L$. So $xr = k = x_1$, where $k \in K$ and $x_1 \in L$. Now $0 \neq xr = \pi(x)r = k = \pi(x_1) \in K \cap \pi(L)$. It follows that $K \cap \pi(L) \leq_e \pi(L)$. It is clear that $\pi(L) = N \cap \left(N' \oplus \pi(L)\right) = N \cap (N' \oplus L)$. Hence $K \cap \pi(L) = K \cap (N' \oplus L) \leq_e K$. Thus, N is G^p-extending."

"Next, we investigate G^p-extending essential extensions of a module or ring. Let us begin with the following useful result which provides relative injectivity or certain direct summands of a Goldie extending module (or nonsingular G^p-extending module)."

"Let N, M be modules. N is said to be M-ejective if, for each $K \leq M$ and each homomorphism $f: K \longrightarrow N$, there exists a homomorphism $g: M \to N$ and $X \leq_e K$ such that $g(x) = f(x)$, for all $x \in X$, see [1]."

Proposition 4.10 : "Let R be any ring , M_1 a semisimple right R- module, and M_2 a right R- module with zero socle such that $M = M_1 \oplus M_2$ is a Goldie extending UC- module. Then M_1 is M_2 ejective."

Proof: "Obviously, $M_1 = Soc(M)$. Let N be any submodule of M_2, and let $\varphi: N \to M_1$ be a homomorphism. Let $L = \{x - \varphi(x): x \in N\}$. Then L is a submodule of M and $L \cap M_1$=0. There exists submodules K, K' of M such that $M = K \oplus K', K \cap L \leq_e L$ and $K \cap L \leq_e K$. It is clear that K is a closure of $K \cap L$ in M. By assumption, $L \leq K$. Since $K \cap L \cap M_1 = L \cap M_1 = 0, K \cap L \cap Soc(M) = Soc(L) = 0$. It follows that $Soc(K) = K \cap M_1 = 0$. Hence $M_1 = Soc(M) \subseteq K'$. Thus, $K' = M_1 \oplus (K' \cap M_2)$ and $M = K \oplus M_1 (K' \cap M_2)$. Let $\pi: M \to M_1$ denote the canonical projection with kernel $K \oplus (K' \cap M_2)$. Let θ be the restriction of π to M_2. Then $\theta: M_2 \to M_1$. Let x be any element of N. Since $x\big(x - \varphi(x)\big) + \varphi(x), \theta(x) = \varphi(x)$. It follows that M_1 is M_2- injective."

Corollary 4.11: (i)"Let $M = \oplus_{i=1}^{n} M_i$, where each M_i is uniform. If $E(M_i) \ncong E\big(M_j\big)$ for all $i \neq j$, then M is G^p-extending.

(ii) Let S be a simple module and $M_1, M_2 \leq E(S)$. If there exists a homomorphism $h: M_2 \to S$ such that $h(S) \neq 0$, then $M = M_1 \oplus M_2$ is G^p-extending.

Proof: (i) From [1, Corollary 4.11] , M is Goldie extending. Thus Proposition 2.3 gives that M is G^p-extending.

(ii) By [1, Corollary 4.14], M_1 is M_2- ejective and so it is G-extending. Now, by proposition 2.3 M is G^p-extending."

Example 4.12: (i) "Let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}_p) \oplus \mathbb{Q}$ and let T be the polynomial ring $\mathbb{Z}[x]$. Then $M_{\mathbb{Z}}$ is included in corollary 4.11(i). On the other hand , it is well known that T^2 is not G^p-extendingT-module. Hence, we obtain that the condition $E(M_i) \ncong E(M_j)$ for all $i \neq j$, is not superfluous in corollary 4.11(i).

(ii) Let K be a field and R=$K[x, y]$, the commutative local Frobenious K-algebra (see[1, Example 4.15]) defined by the relations $xy = x^2 - y^2 = 0$. Then R_R is a uniform injective module with simple submodule Kx^2. Let $M_2 = xR = \{k_1 x + k_{.2}\, x^2 : k_i \in K\}$, and let h be the R-homomorphism, $h: xR \to Kx^2$, defined by $h(k_1 x + k_2 x^2) = k_2 x^2$. Then $h(Kx^2) \neq 0$. Thus, by Corollary 4.11(ii), $M = M_1 \oplus xR$ is G^p-extending for any $M_1 \leq R_R$."

" Next example exhibits that G^p-extending property is not closed under essential extensions of a module."

<u>*Example 4.13:*</u> "Let F be any field and $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$. Then $Soc(R_R) \leq_e R_R$. Obviously $Soc(R)$ is a G^p-extending right R- module. However, it is well known that R_R is not G^p-extending (see [13, Theorem 3.4])."

" In contrast to essential extensions of a module which satisfies G^p-extending condition, we have the following overring of a ring R if Sis an overring of R such that R_R essential in S_R."

Theorem 4.14:"Let S be a right essential overring of R (i.e., $R_R \leq_e S_R$). If R_R is G^p-extending, then S_R and S_S are G^p-extending.

Proof: Let Y_R be any cyclic submodule of S_R. That much is clear to see.$X = Y \cap R$ is cyclic submodule of R_R . By Proposition 3.2, there exists $K_R \leq R_R$ and $e^2 = e \in R$ such that $K_R \leq_e X_R$ and $K_R \leq_e eR_R$.

Notice that$K_R \leq_e Y_R$. Now, let us show that $K_R \leq_e eS_R$. Let $0 \neq es \in eS$. There exists $r_1 \in R$ such that $0 \neq esr_1 \in R$. Hence $0 \neq esr_1 \in eR$, so there exists $r_2 \in R$ such that $0 \neq esr_1r_2 \in K$. Thus $K_R \leq_e eS_R$. By Proposition 3.2, S_R is G^p-extending. A similar demonstration illustrates that $KS_S \leq_e Y_S$ and $KS_S \leq_e eS_S$. Therefore S_S is G^p-extending.

Corollary 4.15: Let $T = T_m(R)$ and $M = M_m(R)$. If T_T is G^p-extending, then M_T and M_M are G^p-extending.

Proof: This outcome is a result of Theorem 4.14 and the realityM_T is a rational extension of T_T."

"It is not known so far whether direct summands of Goldie extending module enjoy with the property. Like the former case the authors desire to obtain whether the G^p-extending property is inherited by its direct summands or not?

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