An Approach to Goldie Extending Modules on the Class of Cyclic Submodules

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Abstract: In this article, we provide a class of modules that is comparable to \( G_z \)-extending and \( G_r \)-extending modules. We specify what a module \( M \) is as \( G_p \)-extending if and only if for each cyclic submodule \( A \) of \( M \), there exists a direct summand \( D \) of \( M \) such that \( A \cap D \) is essential in both \( A \) and \( D \). We look into \( G_p \)-extending modules and locate this inference between the other extending properties. We present some characterizations of \( G_p \)-extending condition. We show that the direct sum of \( G_p \)-extending need not be \( G_p \)-extending and deal with decompositions for \( G_p \)-extending concept.

Keywords: cyclic submodules, P-extending modules, Goldie extending modules, \( G_p \)-extending.

1. Introduction

Throughout this paper, all rings are associative with unitary, \( R \) denotes such a ring, and all modules are unital right \( R \)-modules. In the spirit of [1], for a module \( M \), think of the following relations on the set of submodules of \( M \):

\[ A \alpha B \] if and only if there exists a submodule \( C \) of \( M \) such that \( A \leq e C \) and \( B \leq e C \).

\[ A \beta B \] if and only if \( A \cap B \leq e A \) and \( A \cap B \leq e B \). Recall that \( \beta \) is an equivalence relation. "It is clear that a module \( M \) is extending (or CS) if and only if for each submodule \( A \) of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \alpha D \), (see [1,2]). Further a module \( M \) is called Goldie extending module (or G-extending) if and only if for each submodule \( A \) of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \) or equivalently, for each closed submodule \( A \) in \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \) (see [1]). Obviously, every extending module is G-extending.

As a generalization of CS-modules is p-extending (see [3,4]). Recall that a module \( M \) is called p-extending if every cyclic submodule of \( M \) is essential in a direct summand of \( M \)."

"In this paper, we study a module condition including the \( \beta \) relation on the set of all cyclic submodules of a module. We call a module \( M \) is \( G_p \)-extending if for every cyclic submodule \( A \) of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \). A ring \( R \) is \( G_p \)-extending if \( R_R \) is \( G_p \)-extending module. It is clear that the class of \( G_p \)-extending modules property contains the type of G-extending modules. The notion of \( G_p \)-extending generalizes both of G-extending, extending and p-extending modules."

"In section 2, we consider connections between \( G_p \)-extending property, p-extending and G-extending conditions. Moreover, we give sufficient circumstances under which p-extending and \( G_p \)-extending modules are equivalent."
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Section 3, is devoted to the characterizations of $G^p$-extending modules. Since the direct sum of $G^p$-extending modules need not be $G^p$-extending, we focus when a direct sum of $G^p$-extending modules is also $G^p$-extending. Also, we give sufficient conditions under which the direct summand of $G^p$-extending is also $G^p$-extending. These are introduced in section 4. Also, in section 4, we investigate $G^p$-extending essential extensions of a module or ring."

Following [5], $M$ is called UC-module if every submodule of $M$ has a unique closure in $M$.

2. Preliminary results.

"The $G^p$-extending notion is based on two tools, namely an equivalence relation on cyclic submodules of a module $M$. Let us begin by mentioning basic facts about them. First recall the following relations on the set of submodules of $M$ (see [1]).

(i) $A\alpha B$ if and only if there exists a submodule $C$ of $M$ such that $A \leq C$ and $B \leq C$.

(ii) $A\beta B$ if and only if $A \cap B \leq A$ and $A \cap B \leq B$.

Observe that $\alpha$ is reflexive and symmetric, but it may not be transitive. However, $\beta$ is an equivalence relation. Note that for submodules of module $M$, if $A\alpha B$, then $A\beta B$.

Proposition 2.1: A module $M$ is p-extending if and only if for each cyclic submodule $A$ of $M$, there is a direct summand $D$ of $M$ such that $A\alpha D$.

Proof: The proof is routine."
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). In general, the converse implications do not hold.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are clear.

(ii)$\nRightarrow$(i) "Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Q}$, where $p$ is any prime integer. Then $M_{\mathbb{Z}}$ is $G$-extending by [1, corollary (3.3)]. However $M_{\mathbb{Z}}$ is not extending [8, Example 10]."

(iii)$\nRightarrow$(ii) Let $M_2(R)$ be the ring as in [9, Example 13.8]. Then $M_2(R)$ is a von Neumann regular ring which is not a Baer ring. Hence it is neither right nor left CS, by [10, example 2.7], however it's well acknowledged that each von Neumann regular ring is nonsingular, therefore $M_2(R)$ is not is $G$-extending, see [1, Proposition 1.8]. Also, this is an example to show that (iv)$\nRightarrow$(i).

(iii)$\nRightarrow$(iv) Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. Then $M_{\mathbb{Z}}$ is $G^p$-extending but not $P$-extending, see [1, Corollary 3.3].

The condition under which $G^p$-extending and $P$-extending modules are equivalent is stated in the following proposal.

**Proposition 2.4.** "Let $M$ be a module.

1. If $M$ is a UC-module. Then $M$ is $G^p$-extending if and only if $M$ is $P$-extending.
2. If $M$ is a nonsingular module. Then $M$ is $G^p$-extending if and only if $M$ is $P$-extending.
3. If $M$ is an indecomposable module. Then $M$ is $G^p$-extending if and only if $M$ is $P$-extending."

Proof:

(i) "Assume that $M$ is $G^p$-extending and let $A$ be a cyclic submodule of $M$, then there exists a direct $D$ of $M$ such that $A \beta D$. One can easily show that $(A \cap D)\alpha A$ and $(A \cap D)\alpha D$. But $M$ is UC module, therefore $\alpha$ is transitive, hence $A\alpha D$. Thus $M$ is $P$-extending. The converse is clear.

(ii) Let $M$ be a $G^p$-extending and let $A$ be a cyclic submodule of $M$, then there is a direct $D$ of $M$ such that $A \beta D$. It is sufficient to show that $A \leq D$. Since $\frac{A+D}{D} \equiv \frac{A}{A \cap D}$ is singular and $\frac{A+D}{D} \leq \frac{M}{D} \equiv D'$ is nonsingular, hence $A+D=D$ which implies that $A \leq D$. The converse is obvious.

(iii) Let $M$ be a $G^p$-extending and let $A$ be a cyclic submodule of $M$, then there is a direct $D$ of $M$ such that $A \beta D$. Since $M$ is indecomposable, then $D=M$. Thus $M$ is $P$-extending module. The converse is clear."

**Corollary 2.5:** "Let $M$ be an indecomposable module. Then the following statements are equivalent:

1. $M$ is uniform.
2. $M$ is CS.
3. $M$ is $G$-extending.
4. $M$ is $P$-extending."
Example 2.6: “Let \( F \) be a field and \( V \) be a vector space over \( F \) with \( \dim (F \cdot V) = 2 \). Let \( R \) be the trivial extension of \( F \) with \( V \), i.e.,
\[
R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}.
\]
Since \( R \) is indecomposable which is not CS, then \( R \) is not \( G^p \)-extending.”


“In this section, we give equivalent conditions to \( G^p \)-extending property. We start by the following theorem.

\textbf{Theorem 3.1:} An \( R \)-module \( M \) is \( G^p \)-extending if and only if for each cyclic submodule \( A \) of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \) and \( D' \) is a complement of \( A \), where \( M = D \oplus D' \).

\textbf{Proof:} Suppose that \( M \) is \( G^p \)-extending , let \( A \) be a cyclic submodule of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \) and \( D' \) is a complement of \( A \), hence \( A \cap D \leq A \) and \( A \cap D' = 0 \). Now, let \( B \) be a submodule of \( M \) such that \( D \leq B \) and \( A \cap B = 0 \). Since \( A \cap D \leq c D \), then \( B \cap D = 0 \). But \( D' \) is a complement of \( D \), therefore \( B = D' \). Thus, \( D' \) is a complement of \( A \). The converse is clear.”

“The next result gives another characterization to \( G^p \)-extending modules.

\textbf{Proposition 3.2:} Let \( M \) be an \( R \)-module, the following conditions are equivalent:

(i) \( M \) is \( G^p \)-extending.

(ii) For all cyclic submodule \( A \) of \( M \) , there exists a submodule \( X \) of \( M \) and a direct summand \( D \) of \( M \) such that \( X \leq c A \) and \( X \leq c D \).

(iii) For every cyclic submodule \( A \) of \( M \) there exists a complement \( B \) of \( A \) and a complement \( C \) of \( B \) such that \( A \beta C \) and each homomorphism \( f : C \oplus B \rightarrow M \) extends to a homomorphism \( g : M \rightarrow M \).

\textbf{Proof:} (i) \( \Rightarrow \) (ii) Assume that \( M \) is \( G^p \)-extending and let \( A \) be a cyclic submodule of \( M \), there is a direct summand \( D \) of \( M \) such that \( A \beta D \), hence \( A \cap D \leq c A \) and \( A \cap D \leq c D \). Take \( X = A \cap D \), we get the result.

(ii) \( \Rightarrow \) (iii) Let \( A \) be a cyclic submodule of \( M \). By (ii), there exists a submodule \( X \) of \( M \) and a direct summand \( D \) of \( M \) such that \( M = D \oplus D' \), \( X \leq c A \) and \( X \leq c D \). Take \( D = C \) and \( D' = B \).

(iii) \( \Rightarrow \) (i) Let \( A \) be a cyclic submodule of \( M \). From (iii), there exists a complement \( B \) of \( A \) and a complement \( C \) of \( B \) such that \( A \beta C \) and every homomorphism \( f : C \oplus B \rightarrow M \) extends to a homomorphism \( g : M \rightarrow M \) and by [11, Lemma 3.97], \( D \) is a direct summand of \( M \), hence \( M \) is \( G^p \)-extending.”
Theorem 3.3: A module $M$ is $G^P$-extending if and only if for every direct summand $A$ of the injective hull $E(M)$ of $M$ with $A \cap M$ is cyclic submodule of $M$, there is a direct summand $D$ of $M$ such that $(A \cap M) \beta D$.

Proof: "Let $A$ be a cyclic submodule of $M$ and let $B$ be a complement of $A$, then $A \oplus B \leq M$. Since $M \leq E(M)$, then $A \oplus B \leq E(M)$ implies $E(M) = E(A) \oplus E(B)$. It can be seen that $E(A) \cap M$ is cyclic submodule in $M$. By our assumption, there is a direct summand $D$ of $M$ such that $(E(A) \cap M) \beta D$. But we have $(A \cap M) \beta (E(A) \cap M)$, hence $A \beta D$. The converse implication is clear."

Theorem 3.4: Suppose $M$ is an $R$-module. The assertions that follow are identical.

(i) $M$ is $G^P$-extending module.

(ii) A decomposition exists for each cyclic submodule $A$ of the module $M$. $M = D \oplus D'$, such that $(D'+A) \beta M$.

(iii) Fore very cyclic submodule $A$ of $M$, there is a decomposition $M = D \oplus D'$, such that $(D'+A) \beta M$.

Proof: (i) $\Rightarrow$ (ii) "Let $M$ be a $G^P$-extending and let $A$ be a cyclic submodule of $M$, there exists direct summand $D$ of $M$ such that $A \beta D$, then $M = D \oplus D'$, $D' \leq M$. Since $\{A, D'\}$ is an independent family, then $(A+D') \beta M$, see [12, Proposition 1.4]."

(ii) $\Rightarrow$ (iii) "Let $A$ be a cyclic submodule of $M$. By (ii), there is a decomposition $M = D \oplus D'$, such that $(D'+A) \beta M$. Claim that $M = D + A$ and $D + A = M \cap (D'+A)$. Since $M = D \oplus D'$, then $M = D + A$ and $D = M \cap (D'+A)$. Hence $A = D + A$ is a direct summand of $M$ and $K \beta M$.

(iii) $\Rightarrow$ (i) "To show that $M$ is $G^P$-extending, let $A$ be a cyclic submodule of $M$. By (iii), there is a decomposition $M = D \oplus D'$, such that $(D'+A) \beta M$. Let $i : L \to M$ be the injection map. Since $K \beta M$, then $i^{-1}(K) \beta i^{-1}(M)$, that is $(L \cap K) \beta D$. One can easily show that $L \cap K = A$, so $M$ is $G^P$-extending module."

"Proposition 3.5: Let $M$ be an $R$-module. Then $M$ is $G^P$-extending module if and only if for every cyclic submodule $A$ of $M$, there exists an idempotent $f \in \text{End}(M)$ such that $A \beta f(M)$.

4. Decompositions.

"There are nonsingular modules $M = M_1 \oplus M_2$ in which $M_1$ and $M_2$ are $P$-extending, but $M$ is not $P$-extending (e.g. Let $R = \mathbb{Z}[x]$ be a polynomial ring of integers and let $M = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$). Note that $\mathbb{Z}[x]$ is $G$-extending, by [1] and hence $G^P$-extending but $M$ is not $P$-extending which is nonsingular, thus by proposition 2.4 $M$ is not $G^P$-extending. Next, we give various conditions under which the direct sum of $G^P$-extending is $G^P$-extending."

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Proposition 4.1: Let $M = M_1 \oplus M_2$ be a distributive module if $M_1$ and $M_2$ are $G^p$-extending modules, then $M$ is $G^p$-extending.

Proof: "Let $A$ be a cyclic submodule of $M$. Since $M$ is distributive, then $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since $A$ is cyclic in $M$, then $A \cap M_1$ and $A \cap M_2$ are cyclic in $M_1$ and $M_2$ respectively. But $M_1$ and $M_2$ are $G^p$-extending modules, therefore there are direct summand $D_1$ of $M_1$ and $D_2$ of $M_2$ such that $(A \cap D_1) \beta D_1$ and $(A \cap D_2) \beta D_2$ hence $A \beta (D_1 \oplus D_2)$, by [12, Proposition 1.4]. Thus, $M$ is $G^p$-extending module."

The following statements are also easily proved by using a similar argument.

Proposition 4.2: "Let $M = M_1 \oplus M_2$ be a duo module if $M_1$ and $M_2$ are $G^p$-extending modules, then $M$ is $G^p$-extending."

Proposition 4.3: Let $M_1$ and $M_2$ be $G^p$-extending modules such that $annM_1 + annM_2 = R$, then $M_1 \oplus M_2$ is $G^p$-extending module.

Proposition 4.4: "Let $M = M_1 \oplus M_2$ be an $R$-module with $M_1$ being $G^p$-extending and $M_2$ is semisimple. Suppose that for any cyclic submodule $A$ of $M$, $A \cap M_1$ is a direct summand of $A$, then $M$ is $G^p$-extending.

Proof: Let $A$ be a cyclic submodule of $M$, then it is easy to see that $A + M_1 = M_1 \oplus [(A + M_1) \cap M_2]$. Since $M_2$ is semisimple, then $(A + M_1) \cap M_2$ is a direct summand of $M_2$ and therefore $A + M_1$ is a direct summand of $M$. By our assumption, $A \cap M_1$ is a direct summand of $A$, then $A = (A \cap M_1) \oplus A'$, for some submodule $A'$ of $A$. One can easily show that $A \cap M_1$ is cyclic in $M_1$. But $M_1$ is $G^p$-extending, then there is a direct summand $D$ of $M_1$ such that $(A \cap M_1) \beta D$ is hence $A = ((A \cap M_1) \oplus A') \beta (M_1 + A)$. Thus, $M$ is $G^p$-extending."

Proposition 4.5: "Let $M = M_1 \oplus M_2$ such that $M_1$ is $G^p$-extending and $M_2$ is injective module. Then $M$ is $G^p$-extending if and only if for every cyclic submodule $A$ of $M$ such that $A \cap M_2 \neq 0$ there is a direct summand $D$ of $M$ such that $A \beta D$.

Proof: Suppose that for every cyclic submodule $A$ of $M$ such that $A \cap M_2 \neq 0$ there exists direct summand $D$ of $M$ such that $A \beta D$. Let $A$ be a cyclic submodule of $M$ such that $A \cap M_2 = 0$. By [2], there is a submodule $M'$ of $M$ containing $A$ such that $M = M' \oplus M_2$. Since $M \cong \frac{M}{M_2} \cong M_1$ is $G^p$-extending and $A$ is cyclic submodule of $M'$, then there is a direct summand $K$ of $M'$ such that $A \beta K$. Thus, $M$ is $G^p$-extending. The converse is obvious."

We now list several circumstances in which a direct summand of a module that extends $G^p$-extending is $G^p$-extending.

Proposition 4.6: "Let $A$ be a direct summand of a $G^p$-extending module $M$, if the intersection of $A$ with any direct summand of $M$ is a direct summand of $A$, then $A$ is $G^p$-extending module."

Proof: "Let $X$ be a cyclic in $A$, then $X$ is cyclic in $M$. But $M$ is $G^p$-extending, therefore there exists a direct summand $D$ of $M$ such that $X \beta D$. It can be seen that $X \beta (A \cap D)$. By our assumption $A \cap D$ is a direct summand of $A$. Thus, $A$ is $G^p$-extending."
Proposition 4.7: "Let $A$ be a cyclic submodule of a $GP$-extending module $M$."

(i) If for each $e^2 = e \in \text{End}(M_R)$, there exists $f^2 = f \in \text{End}(A_R)$ such that $A \cap eM \leq e f A$, then $A$ is $GP$-extending.

(ii) If for each $e^2 = e \in \text{End}(M_R)$, there exists $f^2 = f \in \text{End}(A_R)$ such that $eM \beta f M$ and $f A \subseteq A$, then $A$ is $GP$-extending.

Proof:

(i) "Let $Y$ be a cyclic submodule of $A$. Hence $Y$ is a cyclic submodule of $M$. By proposition 3.2, there is $X \leq e Y$ and $e^2 = e \in \text{End}(M_R)$ such that $X \leq e M$. Then $X \leq e M \cap A \leq e f A$, for some $f^2 = f \in \text{End}(M_R)$. Thus, $A$ is $GP$-extending."

(ii) "Let $Y$ be a cyclic submodule of $A$, then $Y$ is cyclic in $M$. Then there exists $e^2 = e \in \text{End}(M_R)$ such that $Y \beta e M$. Hence $Y \beta f M$. Since $fA \subseteq A$, $A$ is $GP$-extending."

Proposition 4.8: "Let $K$ be a projection invariant cyclic submodule of $M$. If $M$ is $GP$-extending, then there exists $M_1 \leq M$ such that $M = M_1 \oplus K$ and $K$ is $GP$-extending.

Proof: There exists $e^2 = e \in \text{End}(M_R)$ such that $K \beta e M$. But $K = e K \oplus (1 - e) K = K \cap (1-e) M$ because $K$ is projection invariant, then $e K \leq e M$ and $e K \leq e K$. Hence $K \cap (1-e) M = 0$. So $K = e K \leq e M$. Since $K$ is cyclic in $M$, then $K = e M$. Let $M_1 = (1-e) M$. Therefore $M = M_1 \oplus K$. Observe that, by Proposition 4.7 (ii), $K$ is $GP$-extending."

Theorem 4.9: Let $M$ be a $GP$-extending module. If $M$ has SIP or satisfies the $C_3$ condition, then any cyclic direct summand of $M$ is $GP$-extending.

Proof: "Let $M = N \oplus N'$ for some submodules $N, N'$ of $M$ where $N$ is cyclic in $M$. Using Proposition 4.8(i), where $N$ is taken to be cyclic in $M$ and applying the SIP gives that $N$ is a $GP$-extending.

Now assume that $M$ satisfies the $C_3$ condition. Let $\pi: M \to N$ be the canonical projection. Let $K$ be any cyclic submodule of $N$, then $K$ is cyclic in $M$. By hypothesis, there exists a direct summand $L$ of $M$ such that $K \cap L \leq e K$ and $\cap L \leq e L$. Since $M$ satisfies $C_3$ condition, $N' \oplus L$ is a direct summand of $M$. It can be seen that $N' \oplus L = N' \oplus \pi(L)$ (see [11, Lemma 2.71]). Hence $\pi(L)$ is a direct summand of $N$. For any $0 \neq y \in \pi(L)$, $y = \pi(x)$ for some $0 \neq x \in L$. There exists an $r \in R$ such that $0 \neq x r \in K \cap L$. So $x r = k = x_1$, where $k \in K$ and $x_1 \in L$. Now $0 \neq x r = \pi(x) r = k = \pi(x_1) \in K \cap \pi(L)$. It follows that $K \cap \pi(L) \leq \pi(L)$. It is clear that $\pi(L) = N \cap (N' \oplus \pi(L)) = N \cap (N' \oplus L)$. Hence $K \cap \pi(L) = K \cap (N' \oplus L) \leq e K$. Thus, $N$ is $GP$-extending."

"Next, we investigate $GP$-extending essential extensions of a module or ring. Let us begin with the following useful result which provides relative injectivity or certain direct summands of a Goldie extending module (or nonsingular $GP$-extending module)."

"Let $N, M$ be modules. $N$ is said to be $M$-jective if, for each $K \leq M$ and each homomorphism $f : K \to N$, there exists a homomorphism $g : M \to N$ and $X \leq e K$ such that $g(x) = f(x)$, for all $x \in X$, see [1]."

Proposition 4.10: "Let $R$ be any ring, $M_1$ a semisimple right $R$-module, and $M_2$ a right $R$-module with zero socle such that $M = M_1 \oplus M_2$ is a Goldie extending UC-module. Then $M_1$ is $M_2$ ejective."
Proof: "Obviously, $M_1 = Soc(M)$. Let $N$ be any submodule of $M_2$, and let $\varphi: N \to M_1$ be a homomorphism. Let $L = \{x - \varphi(x): x \in N\}$. Then $L$ is a submodule of $M$ and $L \cap M_1 = 0$. There exists submodules $K, K'$ of $M$ such that $M = K \oplus K'$, $K \cap L \subseteq L$ and $K \cap L \subseteq K$. It is clear that $K$ is a closure of $K \cap L$ in $M$. By assumption, $L \leq K$. Since $K \cap L \cap M_1 = L \cap M_1 = 0$, $K \cap L \cap Soc(M) = Soc(L) = 0$. It follows that $Soc(K) = K \cap M_1 = 0$. Hence $M_1 = Soc(M) \subseteq K'$. Thus, $K' = M_1 \oplus (K' \cap M_2)$ and $M = M_1 \oplus M_1(K' \cap M_2)$. Let $\pi: M \to M_1$ denote the canonical projection with kernel $K \oplus (K' \cap M_2)$. Let $\theta$ be the restriction of $\pi$ to $M_2$. Then $\theta: M_2 \to M_1$. Let $x$ be any element of $N$. Since $x(x - \varphi(x)) + \varphi(x), \theta(x) = \varphi(x)$. It follows that $M_1$ is $M_2$-injective."

Corollary 4.11: (i) Let $M = \bigoplus_{i=1}^{n} M_i$, where each $M_i$ is uniform. If $E(M_i) \not\cong E(M_j)$ for all $i \neq j$, then $M$ is $G^{P}$-extending.

(ii) Let $S$ be a simple module and $M_1, M_2 \leq E(S)$. If there exists a homomorphism $h: M_2 \to S$ such that $h(S) \not= 0$, then $M = M_1 \oplus M_2$ is $G^{P}$-extending.

Proof: (i) From [1, Corollary 4.11], $M$ is Goldie extending. Thus Proposition 2.3 gives that $M$ is $G^{P}$-extending.

(ii) By [1, Corollary 4.14], $M_1$ is $M_2$-ejective and so it is $G$-extending. Now, by proposition 2.3 $M$ is $G^{P}$-extending."

Example 4.12: (i) Let $M$ be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}_p) \oplus \mathbb{Q}$ and let $T$ be the polynomial ring $\mathbb{Z}[x]$. Then $M_\mathbb{Z}$ is included in corollary 4.11(i). On the other hand, it is well known that $T^2$ is not $G^{P}$-extending $T$-module. Hence, we obtain that the condition $E(M_i) \not\cong E(M_j)$ for all $i \neq j$, is not superfluous in corollary 4.11(i).

(ii) Let $K$ be a field and $R=K[x, y]$, the commutative local Frobenious $K$-algebra (see [1, Example 4.15]) defined by the relations $xy = x^2 - y^2 = 0$. Then $R_R$ is a uniform injective module with simple submodule $Kx^2$. Let $M_2 = xR = \{k_1x + k_2x^2: k_i \in K\}$, and let $h$ be the $R$-homomorphism, $h: xR \to Kx^2$, defined by $h(k_1x + k_2x^2) = k_2x^2$. Then $h(Kx^2) \not= 0$. Thus, by Corollary 4.11(ii), $M = M_1 \oplus xR$ is $G^{P}$-extending for any $M_1 \leq R_R$."

"Next example exhibits that $G^{P}$-extending property is not closed under essential extensions of a module."

Example 4.13: "Let $F$ be any field and $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$. Then $Soc(R) \leq_e R_R$. Obviously $Soc(R)$ is a $G^{P}$-extending right $R$-module. However, it is well known that $R_R$ is not $G^{P}$-extending (see [13, Theorem 3.4])."

"In contrast to essential extensions of a module which satisfies $G^{P}$-extending condition, we have the following overring of a ring $R$ if $S$ is an overring of $R$ such that $R_{R_S}$ essential in $S_{R_S}$.

Theorem 4.14: "Let $S$ be a right essential overring of $R$ (i.e., $R_{R_S} \leq_e S_{R_S}$). If $R_{R_S}$ is $G^{P}$-extending, then $S_{R_S}$ and $S_{R}$ are $G^{P}$-extending.

Proof: Let $Y_R$ be any cyclic submodule of $S_{R_S}$. That much is clear to see. $X = Y \cap R$ is cyclic submodule of $R_{R_S}$. By Proposition 3.2, there exists $K_R \leq R_R$ and $e^2 = e \in R$ such that $K_R \leq_e X_R$ and $K_R \leq_e e R_R$."
Notice that $K_R \leq_e Y_R$. Now, let us show that $K_R \leq_e eS_R$. Let $0 \neq es \in eS$. There exists $r_1 \in R$ such that $0 \neq esr_1 \in R$. Hence $0 \neq esr_1 \in eR$, so there exists $r_2 \in R$ such that $0 \neq esr_1r_2 \in K$. Thus $K_R \leq_e eS_R$. By Proposition 3.2, $S_R$ is $G^p$-extending. A similar demonstration illustrates that $KS_S \leq_e Y_S$ and $KS_S \leq_e eS_S$. Therefore $S_S$ is $G^p$-extending.

**Corollary 4.15:** Let $T = T_m(R)$ and $M = M_m(R)$. If $T_T$ is $G^p$-extending, then $M_T$ and $M_M$ are $G^p$-extending.

**Proof:** This outcome is a result of Theorem 4.14 and the reality $M_T$ is a rational extension of $T_T$.

"It is not known so far whether direct summands of Goldie extending module enjoy with the property. Like the former case the authors desire to obtain whether the $G^p$-extending property is inherited by its direct summands or not?

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**References**