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On Mean Convergence of Random Fourier - Hermite Series

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Abstract: The work in this article is an initiative to explore random Fourier - Hermite series in orthogonal Hermite polynomials. We choose the random coefficients in the series to be the Fourier-Hermite coefficients of a symmetric stable process with weight function $U(v,b) = e^{\frac{-v^2}{2}}(1+|v|)^b$, where $b < \frac{1}{2}$. The existence of these random coefficients, which we find to be dependent random variables, is established. The random Fourier-Hermite series is proven to be convergent in the sense of mean if the scalars in the series are the Fourier-Hermite coefficients of a function g in the weighted space $L^2_{W(v,B)}(\mathbb{R})$, where the weights are given by $W(v,B) = e^{\frac{-v^2}{2}}(1+|v|)^B$ with $B > \frac{-1}{2}$ such that $b < B$. The sum functions of the series is obtained to the stochastic integral $\int_{-\infty}^{\infty} g(v)U(v,b)dX(v,\omega)$.

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1. Introduction

Fourier Series in orthogonal functions e^{inu} and other orthogonal polynomials like Hermite polynomials, Jacobi polynomials, etc., has a widespread application in physical sciences. Random Fourier series(RFS) in orthogonal functions e^{inu} is important in signal processing. For the first time, the application of RFS in Hermite polynomial is found in image encryption and decryption in the work of Liu and Liu [11] in 2007, who expected its more application in general signal and image processing. The RFS they used is an RFT with random coefficients chosen from the unit circle in $\mathbb{C}$ randomly. This motivated us to explore the random Fourier - Hermite series(RFHS) with different random coefficients. Since stable processes are a better model for white noise, the random coefficients $D_n(\omega)$ choosen in this article are Fourier - Hermite coefficients(FHC) of a symmetric stable process(SSP) defined as $\int_{-\infty}^{\infty} H_n(u)U(u,b)$ with weights $U(u,b) = e^{\frac{-u^2}{2}}(1+|u|)^b$, $b < \frac{1}{2}$. We establish the existence of these random variables and demonstrate their dependence. It is proved that the random series $\sum_{k=0}^{\infty} d_k \mathcal{D}_k(\omega) H_k(u)$ in Hermite polynomials $H_k(u)$ convergence in mean to the stochastic integral $\int_{-\infty}^{\infty} g(u)U(u,b)dX(u,\omega)$ if the scalars d_k are FHC of a function g in the space $L^2_{W(u,B)}(\mathbb{R})$, defined as $d_k := r_k^2 \int_{-\infty}^{\infty} g(u)e^{-u^2}H_k(u)du$.

2. Preliminaries

Consider $\phi_n(u)$, $n \in \mathbb{N}_0$, $\mathbb{N}_0 := \{0,1,2,\dots\}$ to be a sequence of functions orthonormal concerning a measure $\mathcal{F}(u)$ that is,

$$\int_a^b \phi_n(u)\phi_m(u)d\mathcal{F}(u) = \delta_{nm},$$

where, δ_{mn} is the Kronecher's delta function and let

$$g(u) \sim \sum_{n=0}^{\infty} a_n\phi_n(u) \quad (2.1)$$

be the formal expansion of an arbitrary function in terms of this sequence where $a_n := \int_a^b g(v)\phi_n(v)d\mathcal{F}(v)$. Many researchers have exhaustively explored the convergence characteristics of series of the form (2.1) for a specific set of functions $\phi_n(v)$. Specifically, the exploration is on the following question:

For what values of $p,\ 1 \leq p < \infty$, does the existence of the integral $\int_a^b |g(u)|^p d\mathcal{F}(u)$ imply,

$$\lim_{n\to\infty} \int_a^b |g(u) - \sum_{k=0}^{n} a_k\phi_k(u)|^p d\mathcal{F}(u) = 0? \quad (2.2)$$

The sequence $\phi_n(u)$ forms a basis for the space of these functions when this equation holds for every $g(u)$ such that $\int_a^b |g(u)|^p d\mathcal{F}(u)$ exists [25]. If $d\mathcal{F}(u) = W(u)du$, $W(u)$ is the weight function then the sequence $\phi_n(u)(W(u))^{\frac{1}{2}}$ is orthonormal on the classical sense and one led to the formal expansion

$$g(u) \sim \sum_{n=0}^{\infty} b_n\phi_n(u)(W(u))^{\frac{1}{2}},$$

where $b_n := \int_a^b g(v)\phi_n(v)(W(v))^{\frac{1}{2}}dv$. The above question can be read as: for what values of $p,\ 1 \leq p < \infty$, does the measurability of $g(u)$ and the relation $\int_a^b |g(u)|^p du < \infty$ (i. e., $g \in L^p(\mathbb{R})$) imply,

$$\lim_{n\to\infty} \int_a^b |g(u) - \sum_{k=0}^{n} b_k\phi_k(u)(W(u))^{\frac{1}{2}}|^p du = 0?$$

M. Riesz [19]was the first to look into this kind of issue, focussing on the case of trigonometric functions. Later, Schauder [21], Kober [9], Caton and Hille [3] looked at other sets of functions.

In this article the sequence of functions $\phi_n(u)$ are considered to be the orthogonal Hermite polynomials $H_n(u)$ with weight e^{-u^2} satisfy

$$\int_{-\infty}^{\infty} H_m(u)H_n(u)e^{-u^2}du = \sqrt{\pi}2^{\frac{n}{2}}n!\,\delta_{mn}. \quad (2.3)$$

The n^{th} degree Hermite polynomials defined as $H_n(u) = (-1)^n e^{u^2}(\frac{d}{du})^n\{e^{-u^2}\}$ [23]. The normalized Hermite functions of degree $n \in \mathbb{N}_0$ [4, 5, 6, 16] defined as,

$$\psi_n(u) := r_n H_n(u)e^{-\frac{u^2}{2}}, n \geq 0, u \in \mathbb{R}, \quad (2.4)$$

where $r_n = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}$ meet the orthonormal condition

$$\int_{-\infty}^{\infty} \psi_m(u)\psi_n(u)du = \delta_{mn}. \quad (2.5)$$

These $\psi_n(u)$ form a basis in $L^p(\mathbb{R}),\ p \geq 2$ [24, 13].

Pollard [18] in 1948 showed that, if $g(u)e^{\frac{-u^2}{2}} \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} |s_n(u)|^p e^{-u^2}du \leq C\int_{-\infty}^{\infty} |g(u)|^2 e^{-u^2}du,$$

where, s_n is the n^{th} partial sum of the Hermite polynomial series $\sum_{k=0}^{\infty} d_k H_k(u)$ for

$$d_k := r_k^2 \int_{-\infty}^{\infty} g(u)H_k(u)e^{-u^u},$$

$k \in \mathbb{N}_0$. This suggests that

$$\| s_n - g \|_2 \to 0 \tag{2.6}$$

as $n \to \infty$ with $\|.\|_2$ denoting the usual L^2 norm on $\mathbb{R}$.

This conclusion was extended to a larger class of functions $L^p(\mathbb{R})$, $\frac{4}{3} < p < 4$ by Askey and Wainger [1] in 1965. For measurable function g such that $g(u)e^{\frac{-u^2}{2}} \in L^p(\mathbb{R})$, $\frac{4}{3} < p < 4$, they proved the inequality

$$\| s_n(u)e^{-\frac{u^2}{2}} \|_p \leq C \| g(u)e^{-\frac{u^2}{2}} \|_p,$$

where $s_n = \sum_{k=0}^{n} a_k \psi_k(u)$ and $a_k = \int_{-\infty}^{\infty} f(v)\psi_k(v)dv$. This implies the mean Convergence

$$\| g(u) - \sum_{k=0}^{n} a_k \psi_k(u) \|_p \to 0 \tag{2.7}$$

as $n \to \infty$ where $\| g \|_p = \{\int_{-\infty}^{\infty} |g|^p du\}^{\frac{1}{p}}$.

In 1970, Muckenhoupt [14] generalized the Askey and Wainger [1] result for $p \in [1, \infty)$. He proved inequalities of the form

$$\| s_n(u)U(u) \|_p \leq \mathcal{C} \| g(u)W(u) \|_p,$$

where $U(u)$, $W(u)$ are suitable weight functions. It lead to prove

$$\| (s_n(u) - g(u))U(u) \|_p \to 0,$$

for every $g \in L^p_{W(u)}$ i.e., $g(u)W(u) \in L^p$. If $U(u) = W(u) = e^{\frac{-u^2}{2}}$, he obtained the result of Askey and Wainger [1] for $\frac{4}{3} < p < 4$. If $U(u,b) = e^{\frac{-u^2}{2}}(1 + |u|)^b$ and $W(u,B) = e^{\frac{-u^2}{2}}(1 + |u|)^B$ for different suitable numbers b and B such that $b < B$, he obtained this result for $1 \leq p \leq \frac{4}{3}$ and $p \geq 4$. $U(u,b)$ and $W(u,B)$ are dense in $L^p(\mathbb{R})$[14].

The random series considered in this article is expressed as

$$\sum_{k=0}^{\infty} d_k \mathcal{R}_k(\omega) H_k(u) \tag{2.8}$$

where d_k represents scalars and $\mathcal{R}_k$ denotes random variables.

The work of Nayak et al. [15] and Pattanayak and Sahoo[17] are followed to choose the random variables $\mathcal{R}_k(\omega)$, $k \in \mathbb{N}_0$ and to study the convergence of the random series (2.8). Suitable real numbers b and B are chosen such that $b < B$, which implies,

$$\| (s_n(u) - g(u))U(u,b) \|_2 \to 0,$$

by the result of Muckenhoupt (Theorem 6, [14]). In the first step, the existence of the stochastic integral

$$\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$$

is established for $g \in L^2_{W(u,B)}(\mathbb{R})$. Since $W(u,B)$ is continuous for $-\frac{1}{2} < B$, $H_k(u)W(u,B) \in L^2(\mathbb{R})$ and the integral $\int_{-\infty}^{\infty} H_k(u)W(u,B)dX(u,\omega)$ exists. This integral is a random variable. Denote it as $\mathcal{D}_k(\omega)$. Choose these $\mathcal{D}_k(\omega)$ as the random coefficients in the series (2.8). The convergence of the series (2.8) in mean if the scalars

$$d_k := r_k^2 \int_{-\infty}^{\infty} g(u)e^{-u^2} H_k(u)du.$$

are the FHC of a function g in the weighted $L^2_{W(u,B)}(\mathbb{R})$ space with weights $W(u,B)$ of the form $e^{\frac{-u^2}{2}}(1 + |u|)^B$ for a suitable B. The stochastic integral

$$\int_{-\infty}^{\infty} g(u,v)e^{\frac{-v^2}{2}}(1+|v|)^B dX(v,\omega), \tag{2.9}$$

is seen to be the sum function of this series.

Throughout the sections 3 and 4 below, $X(u,\omega)$ is considered to be the SSP of index $\mu = 2$ and the weight functions $U(u,b) = exp(-\frac{1}{2}u^2)(1+|u|)^b$, $W(u,B) = exp(-\frac{1}{2}u^2)(1+|u|)^B$ where $b < \frac{1}{2}$ and $B > -\frac{1}{2}$ such that $b < B$.

3. Existence of the stochastic integral

The following result is required to prove the existence of the integral $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$.

Lemma 1: *[17] Suppose* $X(u,\omega)$ *is of index* μ, $\mu \in (1,2]$ *and* $g \in L^p[a,b]$, $p \geq \mu$, $s \in \mathbb{R}$ *then*

$$E(|\int_a^b g(u)dX(u,\omega)|) \leq \frac{4}{\pi(\mu-1)}\int_a^b |g(u)|^\mu du + \frac{2}{\pi}\int_{|s|>1} \frac{1-exp(-|s|^\mu \int_a^b |g(u)|^\mu du)}{s^2} ds.$$

Theorem 2: *If* $X(u,\omega)$ *is of index* 2 , *and* $g(u) \in L^2_{W(u,B)}(\mathbb{R})$, *then the integral* $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ *exists in mean* .

Proof: We are aware that $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. So for $g \in L^2_{W(u,B)}$ there exist a sequence of functions $\{h_k\}$ in $C_c(\mathbb{R})$ such that $(g(u)W(u,B) - h_k) \in L^2(\mathbb{R})$ and $\| gW(u,B) - h_k \|_2$ approaches to 0 as $k \to 0$.

Consider two functions h_m and h_n from this sequence $\{h_k\}$.

Without loss of generality assume that the compact support of h_m and h_n can be in $[a,b]$ and [c, d] respectively. So h_m and h_n can be considered to be in $L^2[a,b]$ and $L^2[c,d]$.

Let $[p,q]$ be the smallest closed sub-interval of $\mathbb{R}$ which contains $[a,b] \cup [c,d]$.

Now both h_m and h_n can be considered to be in $L^2[p,q]$. Since h_m and h_n can be continuous, the stochastic integrals

$$\int_p^q h_m(u)dX(u,\omega) = \int_a^b h_m(u)dX(u,\omega)$$
$$= \int_{-\infty}^{\infty} h_m(u)dX(u,\omega)$$

and

$$\int_p^q h_n(u)dX(u,\omega) = \int_a^b h_n(u)dX(u,\omega)$$
$$= \int_{-\infty}^{\infty} h_n(u)dX(u,\omega)$$

exists in the sense of mean[17].

Denote

$$Y_m(\omega) := \int_p^q h_m(u)dX(u,\omega)$$

and

$$Y_n(\omega) := \int_p^q h_m(u)dX(u,\omega).$$

Now applying Lemma 1 for $\mu = 2$, we get

$$E|Y_n(\omega) - Y_m(\omega)|$$

$$= E(|\int_p^q h_n(u)dX(u,\omega) - \int_p^q h_m(u)dX(u,\omega)|)$$
$$= E(|\int_p^q (h_n(u) - h_m(t))dX(u,\omega)|)$$
$$\leq \frac{4}{\pi}\int_p^q |h_n(u) - h_m(u)|^2 du + \frac{2}{\pi}\int_{|s|>1} \frac{1-\exp(-|s|^2\int_p^q |h_n(u)-h_m(u)|^2 du)}{s^2} ds$$
$$\leq \frac{4}{\pi}\int_{-\infty}^{\infty} |h_n(u) - h_m(u)|^2 du + \frac{2}{\pi}\int_{|s|>1} \frac{1-\exp(-c|s|^2\int_{-\infty}^{\infty} |h_n(u)-h_m(u)|^2 du)}{s^2} ds.$$

The integrand in the 2^{nd} integral is dominated by the integrable function $\frac{1}{s^2}$ over $(-\infty,-1]$ and $[1,\infty)$. Since $\| h_n(u) - h_m(u) \|_2 = \int_{-\infty}^{\infty} |h_n(u) - h_m(u)|^2 du$ approaches 0 as $m,n \to \infty$, the $2nd$ integral converges to 0 by DCT and we obtained

$$\lim_{m,n\to\infty} E|Y_n(\omega) - Y_m(\omega)| = 0.$$

$Y_n(\omega)$ is a Cauchy sequence in the sense of mean. Hence there exists a random variable $Y(\omega)$ such that $E|Y_n(\omega) - Y(\omega)| = 0$. This $Y(\omega)$ is independent of the choice of the sequence of functions h_n. In fact, if another sequence f_n in $Cc(\mathbb{R})$ converges to g i.e.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(u) - g(u)W(u,B)|^2 du = 0 \ \ as \ \ n \ \to \ \infty.$$

Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(u) - h_n(u)|^2 du$$
$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(u) - g(u)W(u,B) + g(u)W(u,B) - h_n(u)|^2 du$$
$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(u) - g(u)W(u,B)|^2 du + \lim_{n\to\infty} \int_{-\infty}^{\infty} |g(u)W(u,B) - h_n(u)|^2 du$$

which converges to 0. Thus we obtain

$$\lim_{n\to\infty} E(|\int_{-\infty}^{\infty} f_n(u)dX(u,\omega) - Y(\omega)|)$$
$$= \lim_{n\to\infty} E(|\int_{-\infty}^{\infty} f_n(u)dX(u,\omega) - \int_{-\infty}^{\infty} h_n(u)dX(u,\omega) + \int_{-\infty}^{\infty} h_n(u)dX(u,\omega) - Y(\omega)|)$$
$$= \lim_{n\to\infty} E(|\int_{-\infty}^{\infty} f_n(u)dX(u,\omega) - \int_{-\infty}^{\infty} h_n(u)dX(u,\omega)|) + \lim_{n\to\infty} E(|\int_{-\infty}^{\infty} h_n(u)dX(u,\omega) - Y(\omega)|)$$
$$= 0 \ \ by \ \ Lemma \ \ 1.$$

Hence the stochastic integral $\int_{-\infty}^{\infty} h_n(u)dX(u,\omega)$ converges uniquely to $Y(\omega)$, in the sense of mean. Define this random variable $Y(\omega)$ to be the stochastic integral, $Y(\omega) = \int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$.

This theorem implies the existence of the integral $\int_{-\infty}^{\infty} H_k(u)W(u,B)dX(u,\omega)$ for $B > \frac{-1}{2}$. The random variables $\mathcal{D}_k(\omega) = \int_{-\infty}^{\infty} H_k(u)W(u,B)dX(u,\omega)$ are found to be dependent. It is established by showing the fact that the characteristic function(CF) of $(\mathcal{D}_k(\omega) + \mathcal{D}_l(\omega))$ is not equal to the product of CF of $\mathcal{D}_k(\omega)$ and the CF $\mathcal{D}_l(\omega)$. The CF of $\mathcal{D}_k(\omega)$ is computed in the following theorem.

Theorem 3 *The CF of* $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ *is* $exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$ *for* $g(u) \in L^2_{W(u,B)}(\mathbb{R})$.

Proof: As we know $Cc(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist a sequence of functions $\{h_k\}$ in $Cc(\mathbb{R})$ for $g \in L^2(\mathbb{R})$ such that $\| h_k - gW(u,B) \|_2 \to 0$. Further it is known that the stochastic integrals $\int_{-\infty}^{\infty} h_k dX(u,\omega)$ and $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ exists by Theorem 2. Denote these random variables as $Y_k := \int_{-\infty}^{\infty} h_k(u)dX(u,\omega)$ and $Y := \int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ in mean.

Y_k converges to Y in mean $\Rightarrow$ Y_k converges to Y in law $\Rightarrow$ distribution of Y_k weakly converges to distribution of Y[12] .

Now the CF of $Y_k := \exp(-c|s|^2 \int_{-\infty}^{\infty} |h_k(u)|^2 du)$.

For $1 \le p < \infty$, it is true that ([20], page no. 75)

$$\int_{-\infty}^{\infty} ||h_k(u)|^2 - |g(u)|^2| du \le 4R \int_{-\infty}^{\infty} |h_k(u) - g(u)|^2 du \to 0,$$

and since $W(u,B)$ are dense in $L^2(\mathbb{R})$, this implies

$$\int_{-\infty}^{\infty} |h_k(u)|^2 du \ \textit{approaches to} \ \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du$$

$$\Rightarrow$$

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |h_k(u)|^2 du) \ \textit{approaches to} \ \exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du).$$

LHS is the CF of Y_k, which converges to the continuous function on the RSH.

By continuity theorem([12], Theorem 1.3.7, page no. 15), RHS is the CF of the limiting function of Y_k, which is Y. This proves that the CF of $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ is $\exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$, implies the pointwise convergence of $C_k(s)$ to $\exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$ as $k \to \infty$[2].

The following theorem proves that the random variables $\mathcal{D}_n(\omega)$ are dependent.

Theorem 4 *The random variables* $\mathcal{D}_n(\omega) = \int_{-\infty}^{\infty} H_n(u)W(u,B)dX(u,\omega)$ *are dependent.*

Proof: By Theorem 3, the CF of $\mathcal{D}_n(\omega)$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(x)W(u,B)|^2 du).$$

Hence, the CF of $(\mathcal{D}_n(\omega) + \mathcal{D}_m(\omega))$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(u)W(u,B) + H_m(u)W(u,B)|^2 du),$$

whereas the product of CF of $\mathcal{D}_n(\omega)$ and the CF of $\mathcal{D}_m(\omega)$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(u)W(u,B)|^2 du) \exp(-c|s|^2 \int_{-\infty}^{\infty} |H_m(u)W(u,B)|^2 du)$$

$$= \exp(-c|s|^2 \int_{-\infty}^{\infty} (|H_n(u)W(u,B)|^2 + |H_m(x)W(u,B)|^2) du).$$

Since CF of $(\mathcal{D}_n(\omega) + \mathcal{D}_m(\omega))$ is not equal to the product of CF of $\mathcal{D}_n(\omega)$ and CF of $\mathcal{D}_m(\omega)$, $\mathcal{D}_n(\omega)$ are dependent random variables.

4. Convergence of random Fourier - Hermite series $\sum_{k=0}^{\infty} d_k \mathcal{D}_k(\omega) H_k(u)$

To prove the convergence of RFHS, we employ the following inequality.

Lemma 5 *Let g be any function in $L^2_{W(u,B)}(\mathbb{R})$ then*

$$E\left(\left|\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)\right|\right) \leq \frac{4}{\pi}\int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du + \frac{2}{\pi}\int_{|s|>1} \frac{1-\exp(-|s|^2\int_{-\infty}^{\infty}|g(u)W(u,B)|^2 du)}{s^2} ds.$$

Its proof requires the following two results.

Lemma 6 *[22] A stable random variable $X(u,\omega)$ always satisfies the inequality $E|X|^i < \infty$ for all $i \in (0,\mu)$, $0 < \mu \leq 2$.*

Lemma 7 *[7] If Ψ is the CF of a random variable X and $F(X)$ is the distribution function of X then, $E|X| = \int_{-\infty}^{\infty} |X| dF(X) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-Re\Psi(s)}{s^2} ds$.*

Proof of Lemma 5: We know that, by Theorem 2, $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ exists in mean. Now using Lemma 6 and 7, we have

$$E\left(\left|\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)\right|\right) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-Re\Psi(s)}{s^2} ds$$
$$= \frac{2}{\pi}\int_{|s|\leq 1} \frac{1-Re\Psi(s)}{s^2} ds + \frac{2}{\pi}\int_{|s|>1} \frac{1-Re\Psi(s)}{s^2} ds.$$

Here

$$\int_{|s|\leq 1} \frac{1-Re\Psi(s)}{s^2} ds = \int_{-1}^{1} \frac{1-\exp(-|s|^2\int_{-\infty}^{\infty}|g(u)W(u,B)|^2 du)}{s^2} ds$$
$$\leq \int_{-1}^{1} \frac{|s|^2\int_{-\infty}^{\infty}|g(u)W(u,B)|^2 du}{s^2} ds \quad (\because\ 1 - e^{-u} < u \text{ for } u > 0)$$
$$= 2\int_{0}^{1} ds \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du$$
$$= 2\int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du$$

Hence we have

$$E\left(\left|\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)\right|\right) \leq \frac{4}{\pi}\int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du + \frac{2}{\pi}\int_{|s|>1} \frac{1-\exp(-|s|^2\int_{-\infty}^{\infty}|g(u)W(u,B)|^2 du)}{s^2} ds.$$

The following theorem establishes the convergence of the series $\sum d_k \mathcal{D}_k(\omega) H_k(u)$, to the integral

$$\int_{-\infty}^{\infty} g(u,v)U(v,b)dX(v,\omega), \tag{4.1}$$

in the sense of mean, if

$$d_k := r_k^2 \int_{-\infty}^{\infty} g(v)H_k(v)e^{-v^2} dv \tag{4.2}$$

are the FHC of $g \in L^2_{W(v,B)}(\mathbb{R})$. Here $\mathcal{D}_k(\omega)$ are defined as,

$$\mathcal{D}_k(\omega) := \int_{-\infty}^{\infty} H_k(v)W(v,B)dX(v,\omega). \tag{4.3}$$

Its proof requires the following lemma, which is the statement of Theorem 1 and Theorem 6 of Muckenhoupt [14] for $p = 2$.

Lemma 8 *[14] Let* $g \in L^2_{W(v,\ B)}(\mathbb{R})$ *then,*

$$\int_{-\infty}^{\infty} |s_n(g,\ u)U(u,\ b)|^2 du \le \mathcal{C} \int_{-\infty}^{\infty} |g(u)W(u,\ B)|^2 du$$

and

$$\| (s_n(u) - g(u))U(u,\ b) \|_2 \to 0. \tag{4.4}$$

Theorem 9 *For all measurable functions* $g \in L^2_{W(u,B)}(\mathbb{R})$ *, the series* $\sum_{k=0}^{\infty} d_k \mathcal{D}_k(\omega) H_k(u)$ *converges in mean to the integral (4.1).*

Proof: For $g \in L^2_{W(u,B)}(\mathbb{R})$, let the Fourier - Hermite series expansion of g be $\sum_{k=-\infty}^{\infty} d_k H_k(u)$ [1]. Let its partial sum be $s_n(u) := \sum_{k=0}^{n} d_k H_k(u)$. Let $\mathcal{S}_n(u,\omega) = \sum_{k=0}^{n} d_k \mathcal{D}_k(\omega) H_k(u)$ be the n^{th} partial sum of the RFHS $\sum_{k=0}^{\infty} d_k \mathcal{D}_k(\omega) H_k(u)$.

Now,

$$\begin{aligned}\mathcal{S}_n(u,\omega) &= \sum_{k=0}^{n} d_k \mathcal{D}_k(\omega) H_k(u) \\ &= \sum_{k=0}^{n} d_k \left(\int_{-\infty}^{\infty} H_k(v) U(v,b) dX(v,\omega)\right) H_k(u) \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=0}^{n} d_k H_k(v) H_k(u)\right) U(v,b) dX(v,\omega)\end{aligned}$$

Since, the series $\sum_{k=0}^{n} d_k H_k(x) H_k(v)$ exists, let $s_n(u,v)$ be the n^{th} partial sum of the series $\sum_{k=0}^{\infty} d_k H_k(u) H_k(v)$. This implies,

$$\mathcal{S}_n(u,\omega) = \int_{-\infty}^{\infty} s_n(u,v) U(v,b) dX(v,\omega).$$

By Lemma 2, we know that, $\int_{-\infty}^{\infty} g(u,v) U(v,b) dX(v,\omega)$ exists in mean.

Now,

$$\begin{aligned}&E\left(\left|S_n(u,\omega) - \int_{-\infty}^{\infty} g(u,v) U(v,b) dX(v,\omega)\right|\right) \\ &= E\left(\left|\int_{-\infty}^{\infty} s_n(u,v) U(v,b) dX(v,\omega) - \int_{-\infty}^{\infty} g(u,v) U(v,b) dX(v,\omega)\right|\right) \\ &= E\left(\left|\int_{-\infty}^{\infty} (s_n(u,v) U(v,b) - g(u,v) U(v,b) dX(v,\omega)\right|\right) \\ &\le \frac{4}{\pi} \int_{-\infty}^{\infty} |(s_n(u,v) - g(u,v)) U(v,b)|^2 dv \\ &+ \frac{2}{\pi} \int_{|s|>1} \frac{1 - \exp(-|s|^2 \int_{-\infty}^{\infty} |(s_n(u,\ v) - g(u,\ v)) U(v,\ b)|^2 dv)}{s^2} ds\end{aligned}$$

Lemma 8 and the dominance of $\frac{1}{s^2}$ lead both of these integrals to tend to zero. Thus, the theorem is established.

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References

[1] Askey, R. and Wainger, S., *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math., **87** (1965), 695–708.

[2] Bachman, G., Narici, L. and Beckenstein, E., *Fourier and Wavelet Analysis*, Springer - verlag, New York,Inc. (2000).
[3] Caton, W. B., Hille, E., *Laguerre polynomials and Laplace integrals*, Duke Math. J. **12(2)**, (1945), 217–242.
[4] Celeghini, E., *Theory of image and Quantum Mechanics*, a common paradigm, J. Phys. Conf. Ser., **626**, (2015), 012–047.
[5] Celeghini, E., Del Olmo, M. A., *Coherent Orthogonal polynomials*, Ann. Phys., **335**, (2013), 78–85.
[6] Celeghini, E., Del Olmo, M. A., *Quantum physics and signal processing in rigged Hilbert spaces by means of special functions, Lie algebras and Fourier and Fourier-like transforms*, J. Phys. Conf. Ser., **597**, (2015), 012–022.
[7] Chow, Y. S. and Teicher, H., *Probability Theory; Independence, Interchangrability*, Martingales, Springer - Verlag, New York, (1978).
[8] Indritz, J., *An inequality for Hermite polynomials*, Proc. of the Amer. Math. Soc., **12(6)**, (1961), 981–983;
[9] Kober, H., *A Note on approximation by rational functions*, Proceedings of the Edinburgh Math. Soc., **7**, (1946), 123–133;
[10] Kwapien, S. and Woyczynski, W. A., *Random series and stochastic integrals: single and multiple*, Birkhauser, (1992).
[11] Liu, Z., Liu, S., *Randomization of Fourier transform*, Opt. Lett. **32**, (2007), 478–480.
[12] Lukacs, E., *Stochastic convergence*, Second Ed., Academic Press, (1975).
[13] Muckenhoupt, B., *Mean convergence of Hermite and Laguerre series I*, Trans. Amer. Math. Soc. **147**, (1970), 419–431;
[14] Muckenhoupt, B., *Mean convergence of Hermite and Laguerre series. II*, Trans. Amer. Math. Soc. **147**, (1970), 433–460;
[15] Nayak, C., Pattanayak, S. and Mishra, M. N., *Random Fourier-Stieltjes series associated with stable process*, Tohoku Math. J., **39** (1), (1987), 1–15;
[16] Olver, F. W., Lozier, D. W., Boisvert, R. F., Clark, C. W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Newyork, NY, USA, (2010).
[17] Pattanayak, S., Sahoo, S., *On summability of random Fourier-Stieltjes series*, J. Int. Acad. Phys. Sci., **9**, (2005), 9–17;
[18] Pollard, H., *The mean convergence of orthogonal series II*, Trans. Amer. Math. Soc. **63** (1948), 355–367.
[19] Riesz, M., *Surles fonctions conjugee′s*, Mathematische Zeitschrift, Vol **27**(1927), 218–244.
[20] Rudin, W., *Real and complex analysis*, $3rd$ edition, McGraw-Hill, New York, (1987).
[21] Schauder, J., *Eine Eigenschaft der Haarschen orthogonalsystem*, ibid, **28**, (1928), 317–320.
[22] Shiryayev, A. N., *Probability*, Springer Verlag, New York, (1984).
[23] Szegő, G., *Orthogonal polynomials*, Colloquium publications, Amer. Math. Soc. **23**, (1939).
[24] Wiener, N., *The Fourier integral and certain of its applications*, Dover, Newyork, (1933).
[25] Wing, G. M., *The mean convergence of orthogonal series*, Amer. J. Math., **72**, (1950), 792–808.