On Mean Convergence of Random Fourier - Hermite Series

Bharatee Mangaraj[1], Sabita Sahoo[2]

[1] Phd Scholar, Department of Mathematics, Sambalpur University, Odisha, India
[2] Retired Professor, Department of Mathematics, Sambalpur University, Odisha, India

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Abstract: The work in this article is an initiative to explore random Fourier - Hermite series in orthogonal Hermite polynomials. We choose the random coefficients in the series to be the Fourier-Hermite coefficients of a symmetric stable process with weight function $U(v,b) = e^{-v^2/(1+|v|^b)}$, where $b < \frac{1}{2}$. The existence of these random coefficients, which we find to be dependent random variables, is established. The random Fourier-Hermite series is proven to be convergent in the sense of mean if the coefficients in the series are the Fourier-Hermite coefficients of a function $g$ in the weighted space $L^2_{W(v,B)}(\mathbb{R})$, where the weights are given by $W(v,B) = e^{-v^2/(1+|v|^b)}$ with $B > \frac{1}{2}$ such that $b < B$. The sum functions of the series is obtained to the stochastic integral $\int_{\omega}^\infty g(v)U(v,b)dX(v,\omega)$. 

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1. Introduction
Fourier Series in orthogonal functions $e^{inu}$ and other orthogonal polynomials like Hermite, Jacobi polynomials, etc., has a widespread application in physical sciences. Random Fourier series(RFS) in orthogonal functions $e^{inu}$ is important in signal processing. For the first time, the application of RFS in Hermite polynomial is found in image encryption and decryption in the work of Liu and Liu [11] in 2007, who expected its more application in general signal and image processing. The RFS they used is an RFT with random coefficients chosen from the unit circle in $\mathbb{C}$ randomly. This motivated us to explore the random Fourier - Hermite series(RFHS) with different random coefficients. Since stable processes are a better model for white noise, the random coefficients $D_n(\omega)$ chosen in this article are Fourier - Hermite coefficients(FHC) of a symmetric stable process(SSP) defined as $\int_{\omega}^\infty H_n(u)U(u,b)$ with weights $U(u,b) = e^{-u^2/(1+|u|^b)}$, $b < \frac{1}{2}$. We establish the existence of these random variables and demonstrate their dependence. It is proved that the random series $\sum_{k=0}^\infty d_kD_k(\omega)H_k(u)$ in symmetric stable polynomials $H_k(u)$ convergence in mean to the stochastic integral $\int_{\omega}^\infty g(u)U(u,b)dX(u,\omega)$ if the scalars $d_k$ are FHC of a function $g$ in the space $L^2_{W(u,B)}(\mathbb{R})$, defined as $d_k := \frac{r_k^2}{\int_{\omega}^\infty g(u)e^{-u^2}H_k(u)du}$.

2. Preliminaries
Consider $\phi_n(u), \ n \in \mathbb{N}_0$, $\mathbb{N}_0 := \{0,1,2,\ldots\}$ to be a sequence of functions orthonormal concerning a measure $\mathcal{F}(u)$ that is,
\[ \int_a^b \phi_n(u) \phi_m(u) \, d\mathcal{F}(u) = \delta_{nm}, \]
where, \( \delta_{mn} \) is the Kronecker’s delta function and let \( g(u) \sim \sum_{n=0}^{\infty} a_n \phi_n(u) \) (2.1) be the formal expansion of an arbitrary function in terms of this sequence where \( a_n := \int_a^b g(v) \phi_n(v) \, d\mathcal{F}(v) \). Many researchers have exhaustively explored the convergence characteristics of series of the form (2.1) for a specific set of functions \( \phi_n(v) \). Specifically, the exploration is on the following question:

For what values of \( p, 1 \leq p < \infty \), does the existence of the integral \( \int_a^b |g(u)|^p \, d\mathcal{F}(u) \) imply,

\[ \lim_{n \to \infty} \int_a^b |g(u) - \sum_{k=0}^{n} a_k \phi_k(u)|^p \, d\mathcal{F}(u) = 0? \] (2.2)

The sequence \( \phi_n(u) \) forms a basis for the space of these functions when this equation holds for every \( g(u) \) such that \( \int_a^b |g(u)|^p \, d\mathcal{F}(u) \) exists \[25\]. If \( d\mathcal{F}(u) = W(u) \, du \), \( W(u) \) is the weight function then the sequence \( \phi_n(u)(W(u))^\frac{1}{2} \) is orthonormal on the classical sense and one led to the formal expansion

\[ g(u) \sim \sum_{n=0}^{\infty} b_n \phi_n(u)(W(u))^\frac{1}{2}, \]
where \( b_n := \int_a^b g(v) \phi_n(v)(W(v))^\frac{1}{2} \, dv \). The above question can be read as: for what values of \( p, 1 \leq p < \infty \), does the measurability of \( g(u) \) and the relation \( \int_a^b |g(u)|^p \, du < \infty \) (i.e., \( g \in L^p(\mathbb{R}) \)) imply,

\[ \lim_{n \to \infty} \int_a^b |g(u) - \sum_{k=0}^{n} b_k \phi_k(u)(W(u))^\frac{1}{2}|^p \, du = 0? \]

M. Riesz \[19\] was the first to look into this kind of issue, focussing on the case of trigonometric functions. Later, Schauder \[21\], Kober \[9\], Caton and Hille \[3\] looked at other sets of functions.

In this article the sequence of functions \( \phi_n(u) \) are considered to be the orthogonal Hermite polynomials \( H_n(u) \) with weight \( e^{-u^2} \) satisfy

\[ \int_{-\infty}^{\infty} H_n(u) H_m(u) e^{-u^2} \, du = \sqrt{\pi} 2^n n! \delta_{mn}. \] (2.3)

The \( n^{th} \) degree Hermite polynomials defined as \( H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n}(e^{-u^2}) \) \[23\]. The normalized Hermite functions of degree \( n \in \mathbb{N}_0 \) \[4, 5, 6, 16\] defined as,

\[ \psi_n(u) := r_n H_n(u) e^{-\frac{u^2}{2}}, n \geq 0, u \in \mathbb{R}, \] (2.4)

where \( r_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \) meet the orthonormal condition

\[ \int_{-\infty}^{\infty} \psi_m(u) \psi_n(u) \, du = \delta_{mn}. \] (2.5)

These \( \psi_n(u) \) form a basis in \( L^p(\mathbb{R}), p \geq 2 \) \[24, 13\].

Pollard \[18\] in 1948 showed that, if \( g(u) e^{\frac{-u^2}{2}} \in L^2(\mathbb{R}) \), then

\[ \int_{-\infty}^{\infty} |s_n(u)|^p e^{-u^2} \, du \leq C \int_{-\infty}^{\infty} |g(u)|^2 e^{-u^2} \, du, \]
where, \( s_n \) is the \( n^{th} \) partial sum of the Hermite polynomial series \( \sum_{k=0}^{\infty} d_k H_k(u) \) for \( d_k := r_k^2 \int_{-\infty}^{\infty} g(u) H_k(u) e^{-u^2} \),

\( k \in \mathbb{N}_0 \). This suggests that

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\[ \| s_n - g \|_2 \to 0 \] (2.6)
as \( n \to \infty \) with \( \| \cdot \|_2 \) denoting the usual \( L^2 \) norm on \( \mathbb{R} \).

This conclusion was extended to a larger class of functions \( L^p(\mathbb{R}), \frac{4}{3} < p < 4 \) by Askey and Wainger [1] in 1965. For measurable function \( g \) such that \( g(u)e^{-\frac{u^2}{2}} \in L^p(\mathbb{R}), \frac{4}{3} < p < 4 \), they proved the inequality
\[ \| s_n(u)e^{-\frac{u^2}{2}} \|_p \leq C \| g(u)e^{-\frac{u^2}{2}} \|_p, \]
where \( s_n = \sum_{k=0}^{\infty} a_k \psi_k(u) \) and \( a_k = \int_{-\infty}^{\infty} f(v) \psi_k(v) dv. \) This implies the mean Convergence
\[ \| g(u) - \sum_{k=0}^{n} a_k \psi_k(u) \|_p \to 0 \] (2.7)
as \( n \to \infty \) where \( \| g \|_p = \left( \int_{-\infty}^{\infty} |g|^p du \right)^{\frac{1}{p}}. \)

In 1970, Muckenhoupt [14] generalized the Askey and Wainger [1] result for \( p \in [1, \infty) \). He proved inequalities of the form
\[ \| s_n(u)U(u) \|_p \leq C \| g(u)W(u) \|_p, \]
where \( U(u), W(u) \) are suitable weight functions. It lead to prove
\[ \| (s_n(u) - g(u))U(u) \|_p \to 0, \]
for every \( g \in L^p_{W(u)} \) i.e., \( g(u)W(u) \in L^p \). If \( U(u) = W(u) = e^{-\frac{u^2}{2}} \), he obtained the result of Askey and Wainger [1] for \( \frac{4}{3} < p < 4 \). If \( U(u, b) = e^{-\frac{u^2}{2}(1 + |u|)} \) and \( W(u, B) = e^{-\frac{u^2}{2}(1 + |u|)^B} \) for different suitable numbers \( b \) and \( B \) such that \( b < B \), he obtained this result for \( 1 \leq p \leq \frac{4}{3} \) and \( p \geq 4 \). \( U(u, b) \) and \( W(u, B) \) are dense in \( L^p(\mathbb{R}) \)[14].

The random series considered in this article is expressed as
\[ \sum_{k=0}^{\infty} d_k R_k(\omega)H_k(u) \] (2.8)
where \( d_k \) represents scalars and \( R_k \) denotes random variables.

The work of Nayak et al. [15] and Pattanayak and Sahoo[17] are followed to choose the random variables \( R_k(\omega), k \in \mathbb{N}_0 \) and to study the convergence of the random series (2.8). Suitable real numbers \( b \) and \( B \) are chosen such that \( b < B \), which implies,
\[ \| s_n(u) - g(u)U(u, b) \|_2 \to 0, \]
by the result of Muckenhoupt (Theorem 6, [14]). In the first step, the existence of the stochastic integral
\[ \int_{-\infty}^{\infty} g(u)W(u, B)dX(u, \omega) \]
is established for \( g \in L^2_{W(u, B)}(\mathbb{R}) \). Since \( W(u, B) \) is continuous for \( -\frac{1}{2} < B \), \( H_k(u)W(u, B) \in L^2(\mathbb{R}) \) and the integral \( \int_{-\infty}^{\infty} H_k(u)W(u, B)dX(u, \omega) \) exists. This integral is a random variable. Denote it as \( D_k(\omega) \). Choose these \( D_k(\omega) \) as the random coefficients in the series (2.8). The convergence of the series (2.8) in mean if the scalars
\[ d_k = r_k^2 \int_{-\infty}^{\infty} g(u)e^{-\frac{u^2}{2}}H_k(u)du. \]
are the FHC of a function \( g \) in the weighted \( L^2_{W(u, B)}(\mathbb{R}) \) space with weights \( W(u, B) \) of the form \( e^{-\frac{u^2}{2}(1 + |u|)^B} \) for a suitable \( B \). The stochastic integral
\[
\int_{-\infty}^{\infty} g(u, v)e^{-\frac{v^2}{2}}(1 + |v|)^{\beta} \, dX(v, \omega),
\]

is seen to be the sum function of this series.

Throughout the sections 3 and 4 below, \(X(u, \omega)\) is considered to be the SSP of index \(\mu = 2\) and the weight functions \(U(u, b) = \exp(-\frac{1}{2} u^2)(1 + |u|)^{b}\) and \(W(u, B) = \exp(-\frac{1}{2} u^2)(1 + |u|)^{b}\) where \(b < \frac{1}{2}\) and \(B > -\frac{1}{2}\) such that \(b < B\).

### 3. Existence of the stochastic integral

The following result is required to prove the existence of the integral \(\int_{-\infty}^{\infty} g(u)W(u, B)dX(u, \omega)\).

**Lemma 1:** [17] Suppose \(X(u, \omega)\) is of index \(\mu \in (1, 2]\) and \(g \in L^p[a, b]\), \(p \geq \mu\), \(s \in \mathbb{R}\) then
\[
E(|\int_{a}^{b} g(u) dX(u, \omega)|) \leq \frac{4}{\pi(\mu - 1)} \int_{a}^{b} |g(u)|^{\mu} du + \frac{2}{\pi} \int_{|s|>1} \frac{1 - \exp(-|s|^2)}{|s|^2} \int_{a}^{b} |g(u)|^{\mu} du \, ds.
\]

**Theorem 2:** If \(X(u, \omega)\) is of index 2, and \(g(u) \in L^2_{W(u, B)}(\mathbb{R})\), then the integral \(\int_{-\infty}^{\infty} g(u)W(u, B)dX(u, \omega)\) exists in mean.

**Proof:** We are aware that \(C_c(\mathbb{R})\) is dense in \(L^2(\mathbb{R})\). So for \(g \in L^2_{W(u, B)}\) there exist a sequence of functions \(\{h_k\}\) in \(C_c(\mathbb{R})\) such that \((g(u)W(u, B) - h_k) \in L^2(\mathbb{R})\) and \(\|gW(u, B) - h_k\|_2\) approaches to 0 as \(k \to 0\).

Consider two functions \(h_m\) and \(h_n\) from this sequence \(\{h_k\}\).

Without loss of generality assume that the compact support of \(h_m\) and \(h_n\) can be in \([a, b]\) and \([c, d]\) respectively. So \(h_m\) and \(h_n\) can be considered to be in \(L^2[a, b]\) and \(L^2[c, d]\).

Let \([p, q]\) be the smallest closed sub-interval of \(\mathbb{R}\) which contains \([a, b]\) and \([c, d]\).

Now both \(h_m\) and \(h_n\) can be considered to be in \(L^2[p, q]\). Since \(h_m\) and \(h_n\) can be continuous, the stochastic integrals
\[
\int_{p}^{q} h_m(u) dX(u, \omega) = \int_{a}^{b} h_m(u) dX(u, \omega) = \int_{-\infty}^{\infty} h_m(u) dX(u, \omega)
\]
and
\[
\int_{p}^{q} h_n(u) dX(u, \omega) = \int_{a}^{b} h_n(u) dX(u, \omega) = \int_{-\infty}^{\infty} h_n(u) dX(u, \omega)
\]
exists in the sense of mean[17].

Denote
\[
Y_m(\omega) = \int_{p}^{q} h_m(u) dX(u, \omega)
\]
and
\[
Y_n(\omega) = \int_{p}^{q} h_n(u) dX(u, \omega).
\]

Now applying Lemma 1 for \(\mu = 2\), we get
\[
E|Y_n(\omega) - Y_m(\omega)|.
\]
\[ E \left( \int_{0}^{a} h_{m}(u) dX(u, \omega) - \int_{0}^{b} h_{m}(u) dX(u, \omega) \right) \]
\[ = E \left( \int_{0}^{a} (h_{m}(u) - h_{m}(t)) dX(u, \omega) \right) \]
\[ \leq \frac{4}{\pi} \int_{0}^{\infty} |h_{m}(u) - h_{m}(t)|^2 du + \frac{2}{\pi} \int_{|s| > 1} \frac{1 - \exp(-|s|^2 f_{m}(u) - h_{m}(u))^2 du}{\pi^2} ds \]
\[ \leq \frac{4}{\pi} \int_{0}^{\infty} \frac{1}{s^2} |h_{m}(u) - h_{m}(t)|^2 du + \frac{2}{\pi} \int_{|s| > 1} \frac{1}{s^2} \exp(-c|s|^2 f_{m}(u) - h_{m}(u))^2 du ds. \]

The integrand in the 2\text{nd} integral is dominated by the integrable function \( \frac{1}{s^2} \) over \((-\infty, -1]\) and \([1, \infty)\). Since \( \| h_{m}(u) - h_{m}(t) \|_2 = \int_{-\infty}^{\infty} |h_{m}(u) - h_{m}(t)|^2 du \) approaches 0 as \( m, n \to \infty \), the 2\text{nd} integral converges to 0 by DCT and we obtained
\[ \lim_{m,n \to \infty} E |Y_{n}(\omega) - Y_{m}(\omega)| = 0. \]

\( Y_{n}(\omega) \) is a Cauchy sequence in the sense of mean. Hence there exists a random variable \( Y(\omega) \) such that \( E |Y_{n}(\omega) - Y(\omega)| = 0. \) This \( Y(\omega) \) is independent of the choice of the sequence of functions \( h_{n}. \)

In fact, if another sequence \( f_{n} \) in \( C(\mathbb{R}) \) converges to \( g \) i.e.
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{n}(u) - g(u)W(u, B)|^2 du = 0 \text{ as } n \to \infty. \]

Then
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{n}(u) - h_{n}(u)|^2 du \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{n}(u) - g(u)W(u, B) + g(u)W(u, B) - h_{n}(u)|^2 du \]
\[ = \lim_{n \to \infty} \int_{-\infty}^{\infty} |f_{n}(u) - g(u)W(u, B)|^2 du + \lim_{n \to \infty} \int_{-\infty}^{\infty} |g(u)W(u, B) - h_{n}(u)|^2 du \]
which converges to 0. Thus we obtain
\[ \lim_{n \to \infty} E \left( \int_{-\infty}^{\infty} f_{n}(u) dX(u, \omega) - Y(\omega) \right) \]
\[ = \lim_{n \to \infty} E \left( \int_{-\infty}^{\infty} f_{n}(u) dX(u, \omega) - \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) + \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) - \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) - Y(\omega) \right) \]
\[ = \lim_{n \to \infty} E \left( \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) - \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) \right) \]
\[ = 0 \text{ by Lemma 1.} \]

Hence the stochastic integral \( \int_{-\infty}^{\infty} h_{n}(u) dX(u, \omega) \) converges uniquely to \( Y(\omega) \), in the sense of mean.

Define this random variable \( Y(\omega) \) to be the stochastic integral, \( Y(\omega) = \int_{-\infty}^{\infty} g(u)W(u, B) dX(u, \omega). \)

This theorem implies the existence of the integral \( \int_{-\infty}^{\infty} H_{k}(u)W(u, B) dX(u, \omega) \) for \( B > \frac{-1}{2}. \)

The random variables \( D_{k}(\omega) = \int_{-\infty}^{\infty} H_{k}(u)W(u, B) dX(u, \omega) \) are found to be dependent. It is established by showing the fact that the characteristic function(CF) of \( (D_{k}(\omega) + D_{l}(\omega)) \) is not equal to the product of CF of \( D_{k}(\omega) \) and the CF \( D_{l}(\omega). \) The CF of \( D_{k}(\omega) \) is computed in the following theorem.
Theorem 3  The CF of $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ is $\exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$ for $g(u) \in L^2_{W(u,B)}(\mathbb{R})$.

Proof: As we know $Cc(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist a sequence of functions $\{h_k\}$ in $Cc(\mathbb{R})$ for $g \in L^2(\mathbb{R})$ such that $\|h_k - gW(u,B)\|_2 \to 0$. Further it is known that the stochastic integrals $\int_{-\infty}^{\infty} h_k dX(u,\omega)$ and $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ exists by Theorem 2. Denote these random variables as $Y_k := \int_{-\infty}^{\infty} h_k(u)dX(u,\omega)$ and $Y := \int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ in mean.

$Y_k$ converges to $Y$ in mean $\Rightarrow Y_k$ converges to $Y$ in law $\Rightarrow$ distribution of $Y_k$ weakly converges to distribution of $Y$[12].

Now the CF of $Y_k := \exp(-c|s|^2 \int_{-\infty}^{\infty} |h_k(u)|^2 du)$.

For $1 \leq p < \infty$, it is true that ([20], page no. 75)

$$\int_{-\infty}^{\infty} |h_k(u)|^2 - |g(u)|^2 |du \leq 4R \int_{-\infty}^{\infty} |h_k(u) - g(u)|^2 du \to 0,$$

and since $W(u,B)$ are dense in $L^2(\mathbb{R})$, this implies

$$\int_{-\infty}^{\infty} |h_k(u)|^2 du \approaches \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du \Rightarrow \exp(-c|s|^2 \int_{-\infty}^{\infty} |h_k(u)|^2 du) \approaches \exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du).$$

LHS is the CF of $Y_k$, which converges to the continuous function on the RSH.

By continuity theorem([12], Theorem 1.3.7, page no. 15), RHS is the CF of the limiting function of $Y_k$, which is $Y$. This proves that the CF of $\int_{-\infty}^{\infty} g(u)W(u,B)dX(u,\omega)$ is $\exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$, implies the pointwise convergence of $C_k(s)$ to $\exp(-c|s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)$ as $k \to \infty[2]$.

The following theorem proves that the random variables $\mathcal{D}_n(\omega)$ are dependent.

Theorem 4  The random variables $\mathcal{D}_n(\omega) = \int_{-\infty}^{\infty} H_n(u)W(u,B)dX(u,\omega)$ are dependent.

Proof: By Theorem 3, the CF of $\mathcal{D}_n(\omega)$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(x)W(u,B)|^2 du).$$

Hence, the CF of $(\mathcal{D}_n(\omega) + \mathcal{D}_m(\omega))$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(u)W(u,B) + H_m(u)W(u,B)|^2 du),$$

whereas the product of CF of $\mathcal{D}_n(\omega)$ and the CF of $\mathcal{D}_m(\omega)$ is

$$\exp(-c|s|^2 \int_{-\infty}^{\infty} |H_n(u)W(u,B)|^2 du) \exp(-c|s|^2 \int_{-\infty}^{\infty} |H_m(u)W(u,B)|^2 du) = \exp(-c|s|^2 \int_{-\infty}^{\infty} (|H_n(u)W(u,B)|^2 + |H_m(x)W(u,B)|^2 du)).$$

Since CF of $(\mathcal{D}_n(\omega) + \mathcal{D}_m(\omega))$ is not equal to the product of CF of $\mathcal{D}_n(\omega)$ and CF of $\mathcal{D}_m(\omega)$, $\mathcal{D}_n(\omega)$ are dependent random variables.
4. Convergence of random Fourier - Hermite series $\sum_{k=0}^{\infty} d_k \mathcal{D}_k(\omega) H_k(u)$

To prove the convergence of RFHS, we employ the following inequality.

**Lemma 5** Let $g$ be any function in $L^2_{W(u,B)}(\mathbb{R})$ then
\[
E(|\int_{-\infty}^{\infty} g(u) W(u,B) dX(u,\omega))| \leq \frac{4}{\pi} \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du + \frac{2}{\pi} \int_{[s]>1} \frac{1-\exp(-s^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)}{s^2} ds.
\]

Its proof requires the following two results.

**Lemma 6** [22] A stable random variable $X(u,\omega)$ always satisfies the inequality $E|X|^{i} < \infty$ for all $i \in (0,\mu)$, $0 < \mu \leq 2$.

**Lemma 7** [7] If $\Psi$ is the CF of a random variable $X$ and $F(X)$ is the distribution function of $X$ then, $E|X| = \int_{-\infty}^{\infty} |X| dF(X) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1-\Re \Psi(s)}{s^2} ds$.

**Proof of Lemma 5:** We know that, by Theorem 2, $\int_{-\infty}^{\infty} g(u)W(u,B) dX(u,\omega)$ exists in mean. Now using Lemma 6 and 7, we have
\[
E(|\int_{-\infty}^{\infty} g(u) W(u,B) dX(u,\omega))| = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1-\Re \Psi(s)}{s^2} ds = \frac{2}{\pi} \int_{|s|\leq 1} \frac{1-\Re \Psi(s)}{s^2} ds + \frac{2}{\pi} \int_{[s]>1} \frac{1-\Re \Psi(s)}{s^2} ds.
\]

Here
\[
\int_{|s|\leq 1} \frac{1-\Re \Psi(s)}{s^2} ds = \int_{1}^{\infty} \frac{1-\exp(-s^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)}{s^2} ds
\]
\[
\leq \int_{1}^{\infty} |s|^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du ds ( \because 1 - e^{-u} < u \text{ for } u > 0)
\]
\[
= 2 \int_{0}^{\infty} ds \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du
\]
\[
= 2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du
\]

Hence we have
\[
E(|\int_{-\infty}^{\infty} g(u) W(u,B) dX(u,\omega))| \leq \frac{4}{\pi} \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du + \frac{2}{\pi} \int_{[s]>1} \frac{1-\exp(-s^2 \int_{-\infty}^{\infty} |g(u)W(u,B)|^2 du)}{s^2} ds.
\]

The following theorem establishes the convergence of the series $\sum d_k \mathcal{D}_k(\omega) H_k(u)$, to the integral
\[
\int_{-\infty}^{\infty} g(u,v) U(v,b) dX(v,\omega),
\]
in the sense of mean, if
\[
d_k = \int_{-\infty}^{\infty} g(v) H_k(v) e^{-v^2} dv
\]
are the FHC of $g \in L^2_{W(u,B)}(\mathbb{R})$. Here $\mathcal{D}_k(\omega)$ are defined as,
\[
\mathcal{D}_k(\omega) = \int_{-\infty}^{\infty} H_k(v) W(v,B) dX(v,\omega).
\]
Its proof requires the following lemma, which is the statement of Theorem 1 and Theorem 6 of Muckenhoupt [14] for $p = 2$.

**Lemma 8** [14] Let $g \in L^2_{W(v,B)}(\mathbb{R})$ then,
\[
\int_{-\infty}^{\infty} |s_n(g,u)U(u, b)|^2 du \leq C \int_{-\infty}^{\infty} |g(u)W(u, B)|^2 du
\]
and
\[
\| (s_n(u) - g(u))U(u, b) \|_2 \to 0. \tag{4.4}
\]

**Theorem 9** For all measurable functions $g \in L^2_{W(u,B)}(\mathbb{R})$, the series $\sum_{k=0}^{\infty} d_k D_k(\omega)H_k(u)$ converges in mean to the integral (4.1).

**Proof:** For $g \in L^2_{W(u,B)}(\mathbb{R})$, let the Fourier - Hermite series expansion of $g$ be $\sum_{k=0}^{\infty} d_k H_k(u)$ [1]. Let its partial sum be $s_n(u) = \sum_{k=0}^{n} d_k H_k(u)$. Let $S_n(u, \omega) = \sum_{k=0}^{n} d_k D_k(\omega)H_k(u)$ be the $n^{th}$ partial sum of the RFHS $\sum_{k=0}^{\infty} d_k D_k(\omega)H_k(u)$.

Now,
\[
S_n(u, \omega) = \sum_{k=0}^{n} d_k D_k(\omega)H_k(u)
= \sum_{k=0}^{n} d_k (\int_{-\infty}^{\infty} H_k(v)U(v, b)dX(v, \omega))H_k(u)
= \int_{-\infty}^{\infty} (\sum_{k=0}^{n} d_k H_k(v)H_k(u))U(v, b)dX(v, \omega)
\]
Since, the series $\sum_{k=0}^{n} d_k H_k(x)H_k(v)$ exists, let $s_n(u,v)$ be the $n^{th}$ partial sum of the series $\sum_{k=0}^{\infty} d_k H_k(u)H_k(v)$. This implies,
\[
S_n(u, \omega) = \int_{-\infty}^{\infty} s_n(u,v)U(v, b)dX(v, \omega).
\]
By Lemma 2, we know that, $\int_{-\infty}^{\infty} g(u,v)U(v, b)dX(v, \omega)$ exists in mean.
Now,
\[
E(|S_n(u, \omega) - \int_{-\infty}^{\infty} g(u,v)U(v, b)dX(v, \omega)|)
= E(\int_{-\infty}^{\infty} (s_n(u,v)U(v, b)dX(v, \omega) - \int_{-\infty}^{\infty} g(u,v)U(v, b)dX(v, \omega))|)
= E(\int_{-\infty}^{\infty} (s_n(u,v)U(v, b) - g(u,v))U(v, b)dX(v, \omega))|)
\leq \frac{4}{\pi} \int_{-\infty}^{\infty} |(s_n(u,v) - g(u,v))U(v, b)|^2 dv
+ 2 \int_{|s|>1} \frac{1 - \exp(-|s|^2)}{s^2} \int_{-\infty}^{\infty} |(s_n(u,v) - g(u,v))U(v, b)|^2 dv dS
\]
Lemma 8 and the dominance of $\frac{1}{s^2}$ lead both of these integrals to tend to zero. Thus, the theorem is established.

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**References**

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