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Multiple Integrals Involving Pragathi-Satyanarayana's I-Function, Generalized Gamma and Generalized Hypergeometric Functions

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Abstract: This paper evaluates the general multiple integrals involving Pragathi-Satyanarayana's I-function, generalized gamma function and generalized hypergeometric function. The result is perceived as innovative and possesses the ability to generate the previous findings. Furthermore, a collection of corollaries will be revealed at the end.

Keywords: Generalized Gamma function, Pragathi-Satyanarayana's I-function, Mellin-Barnes integral contour and generalized hyper geometric function.

1. Introduction

Saxena [6] defined the I-function as:

$$I_{p_i,q_i,r}^{m,n}(z)=I_{p_i,q_i,r}^{m,n}\left(z\left|\begin{matrix}(a_j,A_j)_{1,n},(a_{ji},A_{ji})_{n+1,p_i}\\ \cdot\\ (b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right)=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma(1-a_j+A_js)\prod_{j=1}^{m}\Gamma(b_j-B_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-A_{ji}s)\right]}z^s\,ds \tag{1.1}$$

This function is a generalization of the H-function [4]. It converges if:

$$\Omega_i>0,\ |\arg(z)|<\frac{\pi}{2}\Omega_i,\ i=1,\dots,r \tag{1.2}$$

$$\Omega_i\geq 0,\ |\arg(z)|<\frac{\pi}{2}\Omega_i \text{ and } \mathrm{Re}(\zeta_i)+1<0 \tag{1.3}$$

Where

$$\Omega_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}G_j-\max_{1\leq i\leq r}\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}G_{ji}\right) \tag{1.4}$$

and

$$\zeta_i=\sum_{j=1}^{m}g_j-\sum_{j=1}^{n}a_j+\left(\sum_{j=m+1}^{q_i}g_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{p_i-q_i}{2},\ i=1,2,\dots,r. \tag{1.5}$$

Earlier to Saxena, Arjun K Rathie [3] defined a generalized H-function[4], this function was defined as-

$$I(z) = I_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)_{1,q} \end{matrix} \right. \right) = \frac{1}{2\pi\omega}\int_L \Omega_{p,q}^{m,n}(s)\, z^s\, ds \tag{1.6}$$

Where $$\Omega_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma^{B_j}(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma^{A_j}(1 - a_j + \alpha_j s)}{\prod_{j=n+1}^{p}\Gamma^{A_j}(a_j - \alpha_j s)\prod_{j=m+1}^{q}\Gamma^{B_j}(1 - b_j + \beta_j s)} \tag{1.7}$$

The quantities A_j and B_j are positive reals. The integral (1.6) converges when $|\arg(z)| < \frac{1}{2}\Delta$, ($\Delta > 0$) and $$\Delta = \sum_{j=1}^{m} B_j\beta_j - \sum_{j=m+1}^{q} B_j\beta_j + \sum_{j=1}^{n} A_j\alpha_j - \sum_{j=n+1}^{p} A_j\alpha_j\ . \tag{1.8}$$

Now we have a new function defined by Pragathi and Satyanarayana [7]. This function is a unification of the above mentioned I – functions. It's called Pragathi-Satyanarayan' I-function. We have this integral representation as:

$$\psi_{p_i,q_i,r}^{m,n}(z) = \psi_{p_i,q_i,r}^{m,n}\left(z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}; (a_{ji}, \alpha_{ji}; A_{ji})_{n+1,p_i} \\ (b_j, \beta_j; B_j)_{1,m}; (b_{ji}, \beta_{ji}; B_{ji})_{m+1,q_i} \end{matrix} \right. \right) =$$

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{n}\Gamma^{A_j}(1 - a_j + \alpha_j s)\prod_{j=1}^{m}\Gamma^{B_j}(b_j - \beta_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}(1 - b_{ji} + \beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}(a_{ji} - \alpha_{ji}s)\right]}\, z^s\ ds \tag{1.9}$$

The general convergence criterion of the above integral is defined in [7].

2. Required Multiple integrals

In this section, we remark two generalized multiple integral formulae. In lemma 1, we use the formula about the generalized Gamma-function. In lemma 2, we have a unified multiple integrals involving the generalized hypergeometric function [1, 5].

Lemma 1. Prudnikov et al. ([2], Ch 3.3.3, 2 page 588) equality is given by

$$\int_{x_1 \ge 0} \cdots \int_{x_n \ge 0} \Pi_{k=1}^{n} x_k^{\upsilon_k - 1} dx_1 \ldots dx_n = \Gamma\left[\begin{matrix} \frac{\upsilon_1}{\beta_1}, \ldots, \frac{\upsilon_n}{\beta_n} \\ \vdots \\ \sum_{k=1}^{n}\frac{\upsilon_k}{\beta_k} + 1 \end{matrix}\right] \prod_{k=1}^{n} \frac{a_k^{u_k}}{b_k} \tag{2.1}$$
$$\left(\frac{x_1}{\alpha_1}\right)^{\beta_1} + \ldots + \left(\frac{x_n}{\alpha_n}\right)^{\beta_n} \le 1$$

Provided $a_k, \beta_k, \upsilon_k\ (k = 1, \ldots, n) > 0$.

Lemma 2. Prudnikov et al. ([2], Ch3.3.5, 14 page 595) equality is given by

$$\int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k}(1 - x_k)^{\beta_k - \alpha_k - 1}\left(1 - \prod_{k=1}^{n} x_k z\right)^{-\upsilon} dx_1 \ldots dx_n =$$

$$\Gamma\begin{bmatrix}\alpha_1,...,\alpha_n,\beta_1-\alpha_1,...,\beta_n-\alpha_n\\ ..\\ \beta_1,...,\beta_n\end{bmatrix}{}_{n+1}F_n\left[\begin{matrix}\upsilon,\alpha_1,...,\alpha_n\\ \beta_1,...,\beta_n\end{matrix}\middle|\; z\right] \quad (2.2)$$

Provided $Re(\upsilon), Re(\alpha_k), Re(\beta_k) > 0,$ (k=1, …, n) and $|\arg(1-z)|<\pi$.

3. Main Multiple integrals

With reference to above lemmas, we have the generalized integrals as –

Theorem 1. Prove that

$$\int_{\substack{x_1\ge 0\\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+.....+\left(\frac{x_n}{a_n}\right)^{\beta_n}\le 1}}\cdots\int_{x_n\ge 0}\Pi_{k=1}^{n}\, x_k^{\upsilon_k-1}\,\psi_{p_i,q_i,r}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\ldots dx_n =$$

$$\Pi_{k=1}^{n}\frac{a_k}{\beta_k}\cdot\;\psi_{p_i+n,q_i+1,r}^{m,n+n}\left(z\left|\begin{matrix}A_n,\left(a_j,\alpha_j;A_j\right)_{1,n};\left(a_{ji},\alpha_{ji};A_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j;B_j\right)_{1,m};\left(b_{ji},\beta_{ji};B_{ji}\right)_{m+1,q_i},B_n\end{matrix}\right.\right) \quad (3.1)$$

Provided $\alpha_k,\,\beta_k,\mu_k,\upsilon_k>0,\ \frac{\upsilon_k}{\beta_k}+\frac{\mu_k}{\beta_k}\min_{1\le j\le m}\mathrm{Re}(B_j\,\frac{b_j}{\beta_j})>0$, ($k$=1, …, n) and $\Omega>0,\quad |\arg(z)|<\frac{\pi}{2}\Omega,$

where $\Omega=\sum_{j=1}^{m}\beta_j B_j+\sum_{j=1}^{n}\alpha_j A_j-\max_{1\le i\le r}\left[\sum_{j=1+m}^{q_i}\beta_{ji}B_{ji}+\sum_{j=1+n}^{p_i}\alpha_{ji}A_{ji}\right]$ (or) $|\arg(z)|=\frac{\pi}{2}\Omega$ with $\Omega\ge 0$ if one of the two conditions cited in [7] be satisfied and

$$A_n=\left(1-\frac{\upsilon_1}{\beta_1};\frac{\mu_1}{\beta_1};1\right),\ldots,\left(1-\frac{\upsilon_n}{\beta_n};\frac{\mu_n}{\beta_n};1\right);\ B_n=\left(-\sum\nolimits_{k=1}^{n}\frac{\upsilon_k}{\beta_k};\ \frac{\mu_1}{\beta_1},\ldots,\frac{\mu_n}{\beta_n};1\right) \quad (3.2)$$

Proof:

To prove the theorem 1, using (1.9), express the Pragathi-Satyanarayana's I-function in Mellin-Barnes contour integral and swap the order of integration, which is reasonable because of the absolute convergence of the integral involved in the process. Collecting the powers of $x_k\left(k=1,...,n\right)$, this gives I .

$$I=\int_{\substack{x_1\ge 0\\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+.....+\left(\frac{x_n}{a_n}\right)^{\beta_n}\le 1}}\cdots\int_{x_n\ge 0}\Pi_{k=1}^{n}\, x_k^{\upsilon_k-1}\,\psi_{p_i,q_i,r}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\ldots dx_n =$$

$$\int_{\substack{x_1\ge 0\\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+.....+\left(\frac{x_n}{a_n}\right)^{\beta_n}\le 1}}\cdots\int_{x_n\ge 0}\Pi_{k=1}^{n}\, x_k^{\upsilon_k-1}\;\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\alpha_j s\right)\prod_{j=1}^{m}\Gamma^{B_j}\left(b_j-\beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}\left(1-b_{ji}+\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}\left(a_{ji}-\alpha_{ji}s\right)\right]}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)^{s}ds\,dx_1\ldots dx_n$$

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We obtain $I = \int_{x_1\geq 0} \cdots \int_{x_n \geq 0} \Pi_{k=1}^{n} x_k^{\upsilon_k - 1} \psi_{p_i, q_i, r}^{m,n}\left(z \prod_{i=1}^{n} x_i^{\mu_1}\right) dx_1 \ldots dx_n = \Pi_{k=1}^{n} \frac{a_k}{\beta_k} \cdot$

$\left(\frac{x_1}{a_1}\right)^{\beta_1} + \ldots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1$

$$\cdot \frac{1}{2\pi\omega} \int_L \frac{\prod_{j=1}^{n} \Gamma^{A_j}\left(1 - a_j + \alpha_j s\right) \prod_{j=1}^{m} \Gamma^{B_j}\left(b_j - \beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i} \Gamma^{B_{ji}}\left(1 - b_{ji} + \beta_{ji} s\right) \prod_{j=n+1}^{p_i} \Gamma^{A_{ji}}\left(a_{ji} - \alpha_{ji} s\right)\right]} z^s \cdot \int_{x_1\geq 0} \cdots \int_{x_n\geq 0} \Pi_{k=1}^{n} x^{\upsilon_k + \mu_k s} dx_1 \ldots dx_n \, ds$$

$$\left(\frac{x_1}{a_1}\right)^{\beta_1} + \ldots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1$$

Applying the lemma, this gives:

$$I = \int_{x_1\geq 0} \cdots \int_{x_n \geq 0} \Pi_{k=1}^{n} x_k^{\upsilon_k - 1} \psi_{p_i, q_i, r}^{m,n}\left(z \prod_{i=1}^{n} x_i^{\mu_1}\right) dx_1 \ldots dx_n = \Pi_{k=1}^{n} \frac{a_k}{\beta_k}$$

$$\left(\frac{x_1}{a_1}\right)^{\beta_1} + \ldots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1$$

$$\frac{1}{2\pi\omega} \int_L \frac{\prod_{j=1}^{n} \Gamma^{A_j}\left(1 - a_j + \alpha_j s\right) \prod_{j=1}^{m} \Gamma^{B_j}\left(b_j - \beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i} \Gamma^{B_{ji}}\left(1 - b_{ji} + \beta_{ji} s\right) \prod_{j=n+1}^{p_i} \Gamma^{A_{ji}}\left(a_{ji} - \alpha_{ji} s\right)\right]} z^s \; \Gamma\left[\begin{array}{c} \frac{\upsilon_1 + \mu_1 s}{\beta_1}, \ldots, \frac{\upsilon_n + \mu_n s}{\beta_n} \\ \vdots \\ \sum_{k=1}^{n} \frac{\upsilon_k + \mu_k s}{\beta_k} + 1 \end{array}\right] \Pi_{k=1}^{n} \frac{a_k}{\beta_k} ds$$

By using the definition of the Pragathi-Satyanarayana's I-function (1.9) and the generalized Gamma-function, after a few simplifications, we get -

$$\int_{x_1\geq 0} \cdots \int_{x_n \geq 0} \Pi_{k=1}^{n} x_k^{\upsilon_k - 1} \psi_{p_i, q_i, r}^{m,n}\left(z \prod_{i=1}^{n} x_i^{\mu_1}\right) dx_1 \ldots dx_n = \Pi_{k=1}^{n} \frac{a_k}{\beta_k} \cdot \psi_{p_i+n, q_i+1, r}^{m, n+n}\left(z \left| \begin{array}{l} A_n, \left(a_j, \alpha_j; A_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}; A_{ji}\right)_{n+1, p_i} \\ \left(b_j, \beta_j; B_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}; B_{ji}\right)_{m+1, q_i}, B_n \end{array} \right.\right)$$

$$\left(\frac{x_1}{a_1}\right)^{\beta_1} + \ldots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1$$

Theorem 2. Prove that

$$\int_0^1 \ldots \int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k} \left(1 - x_k\right)^{\beta_k - \alpha_k - 1} \left(1 - \prod_{k=1}^{n} x_k z\right)^{-\upsilon} \psi_{p_i, q_i, r}^{m,n}\left(\prod_{k=1}^{n} x_k^{\mu_k} \left(1 - x_k\right)^{\upsilon_k - \mu_k}\right) dx_1 \ldots dx_n$$

$$= \sum_{n'=0}^{\infty} \frac{(\upsilon)_{n'}}{n'!} z^{n'} \psi_{p_i+2n, q_i+n, \tau_i, r}^{m, n+2n}\left(z \left| \begin{array}{l} A_n, \left(a_j, \alpha_j; A_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}; A_{ji}\right)_{n+1, p_i} \\ \left(b_j, \beta_j; B_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}; B_{ji}\right)_{m+1, q_i}, B_n \end{array} \right.\right) \quad (3.3)$$

Provided

$\mathrm{Re}(\upsilon), \mathrm{Re}(\beta_k), \mathrm{Re}(\alpha_k), \mathrm{Re}(\mu_k), \mathrm{Re}(\upsilon_k) > 0 \; (k = 1, \ldots, n)$, $|\arg(1 - z)| < \pi$, $0 < \beta_k + \upsilon_k \min_{1\leq j\leq m} \mathrm{Re}\left(B_j \frac{b_j}{\beta_j}\right)$ and

$0 < \alpha_k + \mu_k \min_{1\leq j\leq m} \mathrm{Re}\left(B_j \frac{b_j}{\beta_j}\right), (k = 1, \ldots, n)$. Also, $\Omega > 0, |\arg(z)| < \frac{\pi}{2}\Omega$, where

$\Omega=\sum_{j=1}^{m}\beta_j B_j+\sum_{j=1}^{n}\alpha_j A_j-\max_{1\le i\le r}\left[\sum_{j=1+m}^{q_i}\beta_{ji}B_{ji}+\sum_{j=1+n}^{p_i}\alpha_{ji}A_{ji}\right]$ (or) $|\arg(z)|=\frac{\pi}{2}\Omega$, $\Omega\ge 0$ if one of the two conditions cited by (ii) be satisfied and $\mathrm{Re}(\upsilon_k)>\mathrm{Re}(\mu_k)$ where

$$A_n=(1-\alpha_1-n';\mu_1;1),...,(1-\alpha_n-n';\mu_n;1),(1-\beta_1+\alpha_1;\upsilon_1-\mu_1;1),...,(1-\beta_n+\alpha_n;\upsilon_n-\mu_n;1) \quad (3.4)$$

$$B_n=(1-\beta_1-n';\upsilon_1;1),...,(1-\beta_n-n';\upsilon_n;1) \quad (3.5)$$

Proof: To prove the theorem 2, using (1.9), express the Pragathi-Satyanarayana's I-function in Mellin-Barnes contour integral and swap the order of integration, which is reasonable because of the absolute convergence of the integral involved in the process. Collecting the powers of x_k and $1-x_k\,(k=1,...,n)$, this gives I as -

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}(1-x_k)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}(1-x_k)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}(1-x_k)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}.$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma^{B_j}(b_j-\beta_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}(a_{ji}-\alpha_{ji}s)\right]}\;z^s\;\prod_{k=1}^{n}x_k^{\mu_k s}(1-x_k)^{\upsilon_k s-\mu_k s}\,ds\,dx_1...dx_n$$

We can transform the above formula and leads to

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}(1-x_k)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}(1-x_k)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma^{B_j}(b_j-\beta_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}(a_{ji}-\alpha_{ji}s)\right]}\;z^s\;.$$

$$\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k+\mu_k s}(1-x_k)^{\beta_k-\alpha_k+\upsilon_k s-\mu_k s-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}ds\,dx_1...dx_n$$

Evaluating the inner $(x_1,...,x_n)-$ integrals using the lemma, this gives by applying the generalized Gamma- function, we obtain

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}(1-x_k)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}(1-x_k)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}(1-a_j+\alpha_j s)\prod_{j=1}^{m}\Gamma^{B_j}(b_j-\beta_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}(a_{ji}-\alpha_{ji}s)\right]}\;z^s\;.$$

$$\Gamma\begin{bmatrix}\alpha_1+\mu_1 s,...,\alpha_n+\mu_n s,\beta_1-\alpha_1+\upsilon_1 s-\mu_1 s,...,\beta_k-\alpha_k+\upsilon_n s-\mu_n s\\ \vdots\\ \beta_1+\upsilon_1 s,...,\beta_n+\upsilon_n s\end{bmatrix}\cdot\ {}_{n+1}F_n\left[\begin{matrix}\upsilon,\alpha_1+\mu_1 s,...,\alpha_n+\mu_n s\\ \beta_1+\upsilon_1 s,...,\beta_n+\upsilon_n s\end{matrix}\middle| z\right]ds$$

Using the expression of the generalized hypergeometric function [5] in terms of series $\sum_{n'=0}^{\infty}$, under the hypothesis, by interchanging this series and the integrals, we obtain:

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\sum_{n'=0}^{\infty}\frac{(\upsilon)_{n'}}{n'!}z^{n'}.\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\alpha_j s\right)\prod_{j=1}^{m}\Gamma^{B_j}\left(b_j-\beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}\left(1-b_{ji}+\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}\left(a_{ji}-\alpha_{ji}s\right)\right]}z^s\ .$$

$$\Gamma\begin{bmatrix}\alpha_1+\mu_1 s,...,\alpha_n+\mu_n s,\beta_1-\alpha_1+\upsilon_1 s-\mu_1 s,...,\beta_k-\alpha_k+\upsilon_n s-\mu_n s\\ \vdots\\ \beta_1+\upsilon_1 s,...,\beta_n+\upsilon_n s\end{bmatrix}\prod_{k=1}^{n}\frac{\left(\alpha_k+\mu_k s\right)_{n'}}{\left(\beta_k+\upsilon_k s\right)_{n'}}ds$$

We can change the expression by using the generalized Gamma-function:

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\sum_{n'=0}^{\infty}\frac{(\upsilon)_{n'}}{n'!}z^{n'}.\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\alpha_j s\right)\prod_{j=1}^{m}\Gamma^{B_j}\left(b_j-\beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}\left(1-b_{ji}+\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}\left(a_{ji}-\alpha_{ji}s\right)\right]}z^s\ .$$

$$.\prod_{k=1}^{n}\frac{\Gamma\left(\alpha_k+\mu_k s\right)}{\Gamma\left(\beta_k+\upsilon_k s\right)}\prod_{k=1}^{n}\frac{\left(\alpha_k+\mu_k s\right)_{n'}}{\left(\beta_k+\upsilon_k s\right)_{n'}}\prod_{k=1}^{n}\Gamma\left(\beta_k-\alpha_k+\upsilon_k s-\mu_k s\right)ds$$

then we can use the known relation $\Gamma(a)(a)_n=\Gamma(a+n)$ where $\operatorname{Re}(a)>0$, we get:

$$I=\int_0^1....\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon}\psi_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1...dx_n$$

$$=\sum_{n'=0}^{\infty}\frac{(\upsilon)_{n'}}{n'!}z^{n'}.\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\alpha_j s\right)\prod_{j=1}^{m}\Gamma^{B_j}\left(b_j-\beta_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}\left(1-b_{ji}+\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}\left(a_{ji}-\alpha_{ji}s\right)\right]}z^s\ .$$

$$.\prod_{k=1}^{n}\frac{\Gamma\left(\alpha_k+\mu_k s+n'\right)}{\Gamma\left(\beta_k+\upsilon_k s+n'\right)}\prod_{k=1}^{n}\Gamma\left(\beta_k-\alpha_k+\upsilon_k s-\mu_k s\right)ds$$

Interpreting the above Mellin-Barnes integral contour with the help of the definition (1.9), we have the desired relation.

Now, we observe the special cases.

IV. Special Cases

Corollary 1.

From **Theorem 1**, The Pragathi-Satyanarayana's I-function reduces to I-function defined by Rathie [3], in this situation, we have r = 1 then:

$$\int_{\substack{x_1\geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+\dots+\left(\frac{x_n}{a_n}\right)^{\beta_n}\leq 1}}\cdots\int_{x_n\geq 0} \Pi_{k=1}^{n}\ x_k^{\upsilon_k-1}\ I_{p,q}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\dots dx_n = \Pi_{k=1}^{n}\frac{a_k}{\beta_k}.I_{p+n,q+1}^{m,\mathbf{n}+n}\left(z\left|\begin{matrix}A_n,\left(a_j,\alpha_j;A_j\right)_{1,p}\\ \vdots \\ \left(b_j,\beta_j;B_j\right)_{1,q},B_n\end{matrix}\right.\right) \quad (4.1)$$

Under the conditions and notations verified by the theorem 1 of the above section. A_n and B_n are defined by the equation (3.2).

Corollary 2.

From **Theorem 1**,The Pragathi-Satyanarayana's I-function reduces to I-function defined by Saxena [6] when $A_j = B_j = A_{ji} = B_{ji} = 1$ and we have the result:

$$\int_{\substack{x_1\geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+\dots+\left(\frac{x_n}{a_n}\right)^{\beta_n}\leq 1}}\cdots\int_{x_n\geq 0} \Pi_{k=1}^{n}\ x_k^{\upsilon_k-1}\ I_{p_i,q_i,r}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\dots dx_n = \Pi_{k=1}^{n}\frac{a_k}{\beta_k}.I_{p_i+n,q_i+1,r}^{m,\mathbf{n}+2n}\left(z\left|\begin{matrix}A'_n,\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,p_i}\\ \vdots \\ \left(b_j,\beta_j\right)_{1,m};\left(b_{ji},\beta_{ji}\right)_{m+1,q_i},B'_n\end{matrix}\right.\right)$$

(4.2)

Where $A'_n = \left(1-\frac{\upsilon_1}{\beta_1};\frac{\mu_1}{\beta_1}\right),\dots,\left(1-\frac{\upsilon_n}{\beta_n};\frac{\mu_n}{\beta_n}\right);\quad B'_n = \left(-\sum_{k=1}^{n}\frac{\upsilon_k}{\beta_k};\frac{\mu_1}{\beta_1},\dots,\frac{\mu_n}{\beta_n}\right)$

Corollary 3.

Taking r = 1, (4.2) reduces to H-function [4] and we have the result:

$$\int_{\substack{x_1\geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+\dots+\left(\frac{x_n}{a_n}\right)^{\beta_n}\leq 1}}\cdots\int_{x_n\geq 0} \Pi_{k=1}^{n}\ x_k^{\upsilon_k-1}\ H_{p,q}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\dots dx_n = \Pi_{k=1}^{n}\frac{a_k}{\beta_k}.H_{p+n,q+1}^{m,\mathbf{n}+n}\left(z\left|\begin{matrix}A'_n,\left(a_j,\alpha_j\right)_{1,p}\\ \vdots \\ \left(b_j,\beta_j\right)_{1,q},B'_n\end{matrix}\right.\right) \quad (4.3)$$

under the same conditions verified by the corollary 2.

Corollary 4.

We suppose that $A = \left(\alpha_j\right)_{1,p} = \left(\beta_j\right)_{1,q} = 1$, then (4.3) leads to Meijer G-function [1], this gives:

$$\int_{\substack{x_1\geq 0\\ \left(\frac{x_1}{a_1}\right)^{\beta_1}+\dots+\left(\frac{x_n}{a_n}\right)^{\beta_n}\leq 1}}\cdots\int_{x_n\geq 0} \Pi_{k=1}^{n} x_k^{\upsilon_k-1} G_{p,q}^{m,n}\left(z\prod_{i=1}^{n}x_i^{\mu_1}\right)dx_1\dots dx_n = \Pi_{k=1}^{n}\frac{a_k}{\beta_k}.G_{p+n,q+1}^{m,\mathbf{n}+n}\left(z\left|\begin{matrix}A_n',(a_j)_{1,p}\\ \vdots\\ (b_j)_{1,q},B_n'\end{matrix}\right.\right) \tag{4.4}$$

Corollary 5.

From **Theorem 2**, The Pragathi-Satyanarayana's I-function reduces to I-function defined by Rathie [3], in this situation, we have r = 1 then:

$$I=\int_0^1\dots\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon} I_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1\dots dx_n$$

$$=\sum_{n'=0}^{\infty}\frac{(\upsilon)_{n'}}{n'!}z^{n'}.I_{p+2n,q+n}^{m,\mathbf{n}+2n}\left(z\left|\begin{matrix}A_n,(a_j,A_j)_{1,p}\\ \vdots\\ (g_j,G_j)_{1,q},B_n\end{matrix}\right.\right) \tag{4.5}$$

Under the conditions and notations verified by the theorem 2 of the above section. A_n and B_n are defined by the equations (3.4) and (3.5).

Corollary 6.

From **Theorem 2**, The Pragathi-Satyanarayana's I-function reduces to I-function defined by Saxena [6] when $A_j = B_j = A_{ji} = B_{ji} = 1$ and we have the result:

$$I=\int_0^1\dots\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon} I_{p_i,q_i,r}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1\dots dx_n$$

$$=\sum_{n'=0}^{\infty}\frac{(\upsilon)_{n'}}{n'!}z^{n'}.I_{p_i+2n,q_i+n,r}^{m,\mathbf{n}+2n}\left(z\left|\begin{matrix}A_n',(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \vdots\\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},B_n'\end{matrix}\right.\right) \tag{4.6}$$

Where, $A_n' = (1-\alpha_1-n';\mu_1),\dots,(1-\alpha_n-n';\mu_n),(1-\beta_1+\alpha_1;\upsilon_1-\mu_1),\dots,(1-\beta_n+\alpha_n;\upsilon_n-\mu_n)$ and $B_n' = (1-\beta_1-n';\upsilon_1),\dots,(1-\beta_n-n';\upsilon_n)$.

Corollary 7.

Taking r = 1, (4.6) reduces to H-function [4] and we have the result:

$$I=\int_0^1\dots\int_0^1\prod_{k=1}^{n}x_k^{\alpha_k}\left(1-x_k\right)^{\beta_k-\alpha_k-1}\left(1-\prod_{k=1}^{n}x_k z\right)^{-\upsilon} H_{p,q}^{m,n}\left(\prod_{k=1}^{n}x_k^{\mu_k}\left(1-x_k\right)^{\upsilon_k-\mu_k}\right)dx_1\dots dx_n$$

$$= \sum_{n'=0}^{\infty} \frac{(\upsilon)_{n'}}{n'!} z^{n'} . H_{p+2n,q+n}^{m,\mathbf{n}+2n} \left(z \left| \begin{matrix} A'_n, (a_j, \alpha_j)_{1,p} \\ . \\ . \\ (b_j, \beta_j)_{1,q}, B'_n \end{matrix} \right. \right) \tag{4.7}$$

under the same conditions verified by the corollary 6.

Corollary 8.

Taking $A = (\alpha_j)_{1,p} = (\beta_j)_{1,q} = 1$ in (4.7), then Meijer G-function [1] replace the H-function and we have:

$$I = \int_0^1 ....\int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k} (1-x_k)^{\beta_k - \alpha_k - 1} \left(1 - \prod_{k=1}^{n} x_k z\right)^{-\upsilon} G_{p,q}^{m,n} \left(\prod_{k=1}^{n} x_k^{\mu_k} (1-x_k)^{\upsilon_k - \mu_k} \right) dx_1 ... dx_n$$

$$= \sum_{n'=0}^{\infty} \frac{(\upsilon)_{n'}}{n'!} z^{n'} . G_{p+2n,q+n}^{m,\mathbf{n}+2n} \left(z \left| \begin{matrix} A'_n, (a_j)_{1,p} \\ . \\ . \\ (b_j)_{1,q}, B'_n \end{matrix} \right. \right) \tag{4.8}$$

5. Conclusion

In the study of Pragathi-Satyanarayana's I-function [7] by specializing several parameters as well as variables, obtained like [4], lead to a large number of results concerning remarkably wide variety of useful special functions (or product of such special functions) expressible in terms of I-function defined by Saxena [6], defined by Rathie [3], H-function [4], Meijer's G-function [1] and hypergeometric function of one variable [1, 5]. The nature of the multiple integrals involving Pragathi-Satyanarayana's I-function, generalized Gamma function and generalized Hypergeometric function was studied. The theorems developed in this study are quite broad in nature and may be helpful in a number of interesting examples that emerge in literature relating to pure and applied mathematics as well as mathematical physics.

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