Multiple Integrals Involving Pragathi-Satyanarayana’s I-Function, Generalized Gamma and Generalized Hypergeometric Functions

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Abstract: This paper evaluates the general multiple integrals involving Pragathi-Satyanarayana’s I-function, generalized gamma function and generalized hypergeometric function. The result is perceived as innovative and possesses the ability to generate the previous findings. Furthermore, a collection of corollaries will be revealed at the end.

Keywords: Generalized Gamma function, Pragathi-Satyanarayana’s I-function, Mellin-Barnes integral contour and generalized hyper geometric function.

1. Introduction

Saxena [6] defined the I-function as:

\[ I_{p,q}^{m,s}(z) = I_{p,q}^{m,s} \left( \left( a_j, A_j \right)_{i=1}^n, \left( b_j, B_j \right)_{i=1}^n \right) = \frac{1}{2\pi i} \int_{\gamma} \prod_{j=1}^{n} \Gamma \left( 1 - a_j + A_j s \right) \prod_{j=1}^{n} \Gamma \left( b_j - B_j s \right) \sum_{i=1}^{\rho} \prod_{j=1}^{\rho} \Gamma \left( 1 - b_j + B_j s \right) \prod_{j=1}^{\rho} \Gamma \left( a_j - A_j s \right) z^s \, ds \] (1.1)

This function is a generalization of the H-function [4]. It converges if:

\[ \Omega_j > 0, | \arg(z) | < \frac{\pi}{2} \Omega_j, \quad i = 1, \ldots, r \] (1.2)

\[ \Omega_j \geq 0, | \arg(z) | < \frac{\pi}{2} \Omega_j \quad \text{and} \quad \Re(\zeta_j) + 1 < 0 \] (1.3)

Where

\[ \Omega_j = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \max_{i \in \mathbb{D}_r} \left( \sum_{j=1}^{q_i} A_j + \sum_{j=m+1}^{r} G_j \right) \] (1.4)

and

\[ \zeta_j = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \left( \sum_{j=m+1}^{r} g_j - \sum_{j=1}^{q_i} a_j \right) + \frac{p_i - q_i}{2}, \quad i = 1, 2, \ldots, r. \] (1.5)

Earlier to Saxena, Arjun K Rathie [3] defined a generalized H-function[4], this function was defined as-

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\[ I(z) = I_{p,q}^{m,n} \left( z \begin{pmatrix} (a_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)_{1,q} \end{pmatrix} = \frac{1}{2\pi i} \int_{L} \Omega_{p,q}^{m,n}(s) z^{s} ds \] (1.6)

Where \[ \Omega_{p,q}^{m,n}(s) = \prod_{j=1}^{m} \Gamma_{B_j}^{A_j}(1-a_j + \alpha_j s) \prod_{j=1}^{n} \Gamma_{B_j}^{A_j}(1-b_j + \beta_j s) \] (1.7)

The quantities \( A_j \) and \( B_j \) are positive reals. The integral (1.6) converges when \( |\arg(z)| < \frac{1}{2} \Delta \), \( \Delta > 0 \) and \( \Delta = \sum_{j=1}^{m} B_j \beta_j - \sum_{j=m+1}^{q} B_j \beta_j + \sum_{j=1}^{p} A_j \alpha_j - \sum_{j=q+1}^{n} A_j \alpha_j \). (1.8)

Now we have a new function defined by Pragathi and Satyanarayana [7]. This function is a unification of the above mentioned I – functions. It’s called Pragathi-Satyanarayan’ I-function. We have this integral representation as:

\[ \psi_{p,q,r}^{m,n}(z) = \psi_{p,q,r}^{m,n} \left( z \begin{pmatrix} (a_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)_{1,q} \end{pmatrix} = \frac{1}{2\pi i} \int_{L} \sum_{r=1}^{r} \prod_{j=1}^{m} \Gamma_{B_j}^{A_j}(1-a_j + \alpha_j s) \prod_{j=1}^{n} \Gamma_{B_j}^{A_j}(1-b_j + \beta_j s) \right) z^{s} ds \] (1.9)

The general convergence criterion of the above integral is defined in [7].

2. Required Multiple integrals

In this section, we remark two generalized multiple integral formulæ. In lemma 1, we use the formula about the generalized Gamma-function. In lemma 2, we have a unified multiple integrals involving the generalized hypergeometric function [1, 5].

**Lemma 1.** Prudnikov et al. ([2], Ch 3.3.3, 2 page 588) equality is given by

\[ \int_{x_{k,0}}^{x_{k,1}} \cdots \int_{x_{k,0}}^{x_{k,1}} \Pi_{k=1}^{n} x_{k}^{\nu_{k}} d x_{k} = \Gamma_{\nu_{1}}^{\beta_{1}} \cdots \Gamma_{\nu_{n}}^{\beta_{n}} \left[ \begin{array}{c} \nu_{1} \\ \beta_{1} \\ \vdots \\ \nu_{n} \\ \beta_{n} \end{array} \right] \left[ \begin{array}{c} 1 \\ \rho_{1} \\ \vdots \\ \rho_{n} \\ 1 \end{array} \right] \] (2.1)

Provided \( a_{k}, \beta_{k}, \nu_{k} (k = 1, \ldots, n) > 0 \).

**Lemma 2.** Prudnikov et al. ([2], Ch 3.3.5, 14 page 595) equality is given by

\[ \int_{0}^{1} \cdots \int_{0}^{1} \prod_{k=1}^{n} x_{k}^{\alpha_{k}} (1-x_{k})^{\beta-\alpha-1} \left( 1 - \prod_{k=1}^{n} x_{k} \right)^{\psi} d x_{1} \cdots d x_{n} = \]
Provided $Re(\nu), Re(\alpha_k), Re(\beta_k) > 0$, $(k=1, \ldots, n)$ and $|\arg(1-z)| < \pi$.

3. Main Multiple integrals

With reference to above lemmas, we have the generalized integrals as –

**Theorem 1.** Prove that

$$
\int_{x_k>0} \cdots \int_{x_k>0} \prod_{k=1}^{n} x_k^{\nu_{k}-1} \psi_{\nu_{k}, \alpha_{k}, \beta_{k}} \left(z \prod_{i=1}^{n} x_i^{\alpha_i} \right) dx_1 \ldots dx_n =
\prod_{k=1}^{n} \frac{\alpha_k}{\beta_k} \cdot \psi_{\nu_{k}, \alpha_{k}, \beta_{k}} \left(z \prod_{i=1}^{n} x_i^{\alpha_i} \right)
$$

Provided $\alpha_k, \beta_k, \mu_k, \nu_k > 0$, $\frac{\nu_k + \mu_k}{\beta_k}$, $\min Re(B_j) > 0$, $(k=1, \ldots, n)$ and $\Omega > 0$, $|\arg(z)| < \frac{\pi}{2}$.

where $\Omega = \sum_{j=1}^{m} \beta_j B_j + \sum_{j=1}^{m} \alpha_j A_j - \max_{1 \leq i \leq n} \left[ \sum_{j=1}^{m} \beta_j B_j + \sum_{j=1}^{m} \alpha_j A_j \right]$ (or $|\arg(z)| = \frac{\pi}{2}$) with $\Omega \geq 0$ if one of the two conditions cited in [7] be satisfied and

$$
A_n = \left(1 - \frac{\nu_1}{\beta_1}, 1\right), \ldots, \left(1 - \frac{\nu_n}{\beta_n}, 1\right); \quad B_n = \left(-\sum_{k=1}^{n} \frac{\nu_k}{\beta_k}, \frac{\mu_1}{\beta_1}, \ldots, \frac{\mu_n}{\beta_n}; 1\right)
$$

**Proof:**

To prove the theorem 1, using (1.9), express the Pragathi-Satyanarayana’s I-function in Mellin-Barnes contour integral and swap the order of integration, which is reasonable because of the absolute convergence of the integral involved in the process. Collecting the powers of $x_i$ $(k=1, \ldots, n)$, this gives

$$
I = \int_{x_k>0} \cdots \int_{x_k>0} \prod_{k=1}^{n} x_k^{\nu_{k}-1} \psi_{\nu_{k}, \alpha_{k}, \beta_{k}} \left(z \prod_{i=1}^{n} x_i^{\alpha_i} \right) dx_1 \ldots dx_n =
\prod_{k=1}^{n} \frac{\alpha_k}{\beta_k} \cdot \psi_{\nu_{k}, \alpha_{k}, \beta_{k}} \left(z \prod_{i=1}^{n} x_i^{\alpha_i} \right)
$$
We obtain

\[ I = \int_{x_{k20}} \cdots \int_{x_{k20}} \prod_{j=1}^{n} x_j^{\nu_{j-1}} \psi_{\rho, \Omega, r}^{m, n} \left( z \prod_{j=1}^{n} x_j^{\mu_j} \right) dx_1 \cdots dx_n = \prod_{k=1}^{m} \frac{a_k}{b_k}. \]

\[
\frac{1}{2\pi i} \int_{\Gamma} \sum_{j=1}^{n} \prod_{j=1}^{n} \Gamma^{n} (1-b_j - \alpha_j s) \prod_{j=1}^{n} \Gamma^{n} (a_j - \alpha_j s) \int_{\Gamma} \prod_{k=1}^{m} x_k^{\nu_k} dx_k ds d
\]

Applying the lemma, this gives:

\[
I = \int_{x_{k20}} \cdots \int_{x_{k20}} \prod_{j=1}^{n} x_j^{\nu_{j-1}} \psi_{\rho, \Omega, r}^{m, n} \left( z \prod_{j=1}^{n} x_j^{\mu_j} \right) dx_1 \cdots dx_n = \prod_{k=1}^{m} \frac{a_k}{b_k}.
\]

By the definition of the Pragathi-Satyanarayana’s I-function (1.9) and the generalized Gamma-function, after a few simplifications, we get -

\[
\prod_{j=1}^{n} x_j^{\nu_{j-1}} \psi_{\rho, \Omega, r}^{m, n} \left( z \prod_{j=1}^{n} x_j^{\mu_j} \right) dx_1 \cdots dx_n = \prod_{k=1}^{m} \frac{a_k}{b_k} \cdot \psi_{\rho, \Omega, r}^{m, n+1} \left( z \prod_{j=1}^{n} x_j^{\mu_j} \right) dx_1 \cdots dx_n
\]

**Theorem 2.** Prove that

\[
\prod_{j=1}^{n} x_j^{\nu_{j-1}} \prod_{j=1}^{n} (1-x_j)^{\beta_j - a_j} \left( 1 - \prod_{j=1}^{n} x_j \right)^{\mu_j} \psi_{\rho, \Omega, r}^{m, n} \left( \prod_{j=1}^{n} x_j^{\nu_j} \right) dx_1 \cdots dx_n
\]

\[= \sum_{n=0}^{\infty} \frac{(\nu)^{n}}{n!} z^n \psi_{\rho, \Omega, r}^{m, n+2x} \left( z \prod_{j=1}^{n} x_j^{\mu_j} \right) \left( b_j, \beta_j; B_j \right)_{n} \left( b_j, \beta_j; B_j \right)_{n} \]

Provided

\[\text{Re}(\nu), \text{Re}(\beta_j), \text{Re}(\alpha_k), \text{Re}(\mu_k), \text{Re}(\nu_k) > 0 \quad (k = 1, \ldots, n), \quad |\arg(1-z)| < \pi, \quad 0 < \beta_k + \nu_k \min \text{Re} \left( \frac{B_j b_j}{\beta_j} \right) \quad \text{and}
\]

\[0 < \alpha_k + \mu_k \min \text{Re} \left( \frac{B_j b_j}{\beta_j} \right) \quad (k = 1, \ldots, n). \]

Also, \( \Omega > 0, |\arg(z)| < \frac{\pi}{2} \Omega \), where

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\[
\Omega = \sum_{j=1}^{m} \beta_j B_j + \sum_{j=1}^{n} \alpha_j A_j - \max_{k \in \mathbb{S}^2} \left[ \sum_{j=1}^{m} \beta_j B_j + \sum_{j=1}^{n} \alpha_j A_j \right] \quad \text{(or)} \quad |\arg(z)| = \frac{\pi}{2} \Omega, \quad \Omega \geq 0 \quad \text{if one of the two conditions cited by (ii) be satisfied and } \text{Re}(\nu_k) > \text{Re}(\mu_i) \text{ where}
\]

\[A_n = (1 - \alpha_n - n'\mu_i; 1), \ldots, (1 - \alpha_n - n'\mu_i; 1); (1 - \beta_i + \alpha_i; \nu_i - \mu_i; 1), \ldots, (1 - \beta_n + \alpha_n; \nu_n - \mu_n; 1) \quad \text{(3.4)}
\]

\[B_n = (1 - \beta_n - n'\nu_i; 1), \ldots, (1 - \beta_n - n'\nu_i; 1) \quad \text{(3.5)}
\]

**Proof:** To prove the theorem 2, using (1.9), express the Pragathi-Satyanarayana’s I-function in Mellin-Barnes contour integral and swap the order of integration, which is reasonable because of the absolute convergence of the integral involved in the process. Collecting the powers of \( x_i \) and \( 1 - x_k \ (k = 1, \ldots, n) \), this gives \( I \) as -

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k} (1 - x_k)^{\beta_k - \alpha_k - 1} \left( 1 - \prod_{k=1}^{n} x_k z \right)^{-\nu} \prod_{k=1}^{n} x_k^{\alpha_k} (1 - x_k)^{\nu_k - \mu_k} dx_1 \cdots dx_n
\]

\[
= \frac{1}{2\pi i} \int_{L} \sum_{r=1}^{\infty} \prod_{p=1}^{r} \Gamma^{\alpha_p} \left( 1 - a_p + \alpha_p, s \right) \prod_{j=1}^{m} \Gamma^{\beta_j} \left( b_j - \beta_j, s \right) \prod_{j=1}^{m} \Gamma^{\beta_j} \left( a_p - \alpha_p, s \right) z^{-s} \prod_{k=1}^{n} x_k^{\mu_k} \left( 1 - x_k \right)^{\nu_k - \mu_k} ds dx_1 \cdots dx_n
\]

We can transform the above formula and leads to

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k + \mu_k s} (1 - x_k)^{\beta_k - \alpha_k + \nu_k s - \mu_k s - 1} \left( 1 - \prod_{k=1}^{n} x_k \right)^{-\nu} ds \ dx_1 \cdots dx_n
\]

Evaluating the inner \((x_1, \ldots, x_n)-\) integrals using the lemma, this gives by applying the generalized Gamma- function, we obtain

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} x_k^{\alpha_k} (1 - x_k)^{\beta_k - \alpha_k - 1} \left( 1 - \prod_{k=1}^{n} x_k \right)^{\nu} \prod_{k=1}^{n} x_k^{\alpha_k} (1 - x_k)^{\nu_k - \mu_k} dx_1 \cdots dx_n
\]

\[
= \frac{1}{2\pi i} \int_{L} \sum_{r=1}^{\infty} \prod_{p=1}^{r} \Gamma^{\alpha_p} \left( 1 - a_p + \alpha_p, s \right) \prod_{j=1}^{m} \Gamma^{\beta_j} \left( b_j - \beta_j, s \right) \prod_{j=1}^{m} \Gamma^{\beta_j} \left( a_p - \alpha_p, s \right) z^{-s} \prod_{k=1}^{n} x_k^{\mu_k} \left( 1 - x_k \right)^{\nu_k - \mu_k} ds dx_1 \cdots dx_n
\]
Using the expression of the generalized hypergeometric function [5] in terms of series $\sum_{n=0}^{\infty}$, under the hypothesis, by interchanging this series and the integrals, we obtain:

\[
I = \int_{0}^{1} \ldots \int_{0}^{1} \prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}} \left(1 - \prod_{k=1}^{n} x_{k} z\right)^{-\nu} \psi_{n, n}^{m, n} \left(\prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}}\right) d x_{1} \ldots d x_{n}
\]

\[
= \sum_{n=0}^{\infty} \left(\nu\right)_{n} z^{-n} \frac{1}{2\pi i} \int_{C} \prod_{j=1}^{r} \frac{\Gamma \left(1 - a_{j} + \alpha_{j} s\right)^{m} \Gamma \left(b_{j} - \beta_{j} s\right)}{\Gamma \left(1 - b_{j} + \beta_{j} s\right)^{m} \Gamma \left(a_{j} - \beta_{j} s\right)}\frac{d z}{z^{n+1}}
\]

We can change the expression by using the generalized Gamma-function:

\[
I = \int_{0}^{1} \ldots \int_{0}^{1} \prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}} \left(1 - \prod_{k=1}^{n} x_{k} z\right)^{-\nu} \psi_{n, n}^{m, n} \left(\prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}}\right) d x_{1} \ldots d x_{n}
\]

\[
= \sum_{n=0}^{\infty} \left(\nu\right)_{n} z^{-n} \frac{1}{2\pi i} \int_{C} \prod_{j=1}^{r} \frac{\Gamma \left(1 - a_{j} + \alpha_{j} s\right)^{m} \Gamma \left(b_{j} - \beta_{j} s\right)}{\Gamma \left(1 - b_{j} + \beta_{j} s\right)^{m} \Gamma \left(a_{j} - \beta_{j} s\right)}\frac{d z}{z^{n+1}}
\]

then we can use the known relation $\Gamma(a)(a) = \Gamma(a+n)$ where $\text{Re}(a) > 0$, we get:

\[
I = \int_{0}^{1} \ldots \int_{0}^{1} \prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}} \left(1 - \prod_{k=1}^{n} x_{k} z\right)^{-\nu} \psi_{n, n}^{m, n} \left(\prod_{k=1}^{n} x_{k}^{\alpha_{k} - \alpha_{k-1}}\right) d x_{1} \ldots d x_{n}
\]

\[
= \sum_{n=0}^{\infty} \left(\nu\right)_{n} z^{-n} \frac{1}{2\pi i} \int_{C} \prod_{j=1}^{r} \frac{\Gamma \left(1 - a_{j} + \alpha_{j} s\right)^{m} \Gamma \left(b_{j} - \beta_{j} s\right)}{\Gamma \left(1 - b_{j} + \beta_{j} s\right)^{m} \Gamma \left(a_{j} - \beta_{j} s\right)}\frac{d z}{z^{n+1}}
\]

\[
\cdot \prod_{k=1}^{n} \frac{\Gamma \left(\alpha_{k} + \mu_{k} s + n\right)}{\Gamma \left(\beta_{k} + \alpha_{k} s + n\right)} d s
\]
Interpreting the above Mellin-Barnes integral contour with the help of the definition (1.9), we have the desired relation.

Now, we observe the special cases.

IV. Special Cases

Corollary 1.

From Theorem 1, The Pragathi-Satyanarayana’s I-function reduces to I-function defined by Rathie [3], in this situation, we have \( r = 1 \) then:

\[
\int_{k=1}^{b} \cdots \int_{k=1}^{b} \Pi_{k=1}^{n} x_{k}^{-1} I_{p,q}^{n,n} \left( z \Pi_{i=1}^{n} x_{i} \right) dx_{1} \cdots dx_{n} = \Pi_{k=1}^{n} \frac{a_{i}}{\beta_{k}} I_{p,q+1}^{n,m} \left( z \right) \left( b_{i}, \beta_{p} ; A_{i} \right) \left( B_{i}^{n} \right) \tag{4.1}
\]

Under the conditions and notations verified by the theorem 1 of the above section. \( A_{i} \) and \( B_{i} \) are defined by the equation (3.2).

Corollary 2.

From Theorem 1, The Pragathi-Satyanarayana’s I-function reduces to I-function defined by Saxena [6] when \( A_{j} = B_{j} = A_{ji} = B_{ji} = 1 \) and we have the result:

\[
\int_{k=1}^{b} \cdots \int_{k=1}^{b} \Pi_{k=1}^{n} x_{k}^{-1} I_{p,q}^{n,n} \left( z \Pi_{i=1}^{n} x_{i} \right) dx_{1} \cdots dx_{n} = \Pi_{k=1}^{n} \frac{a_{i}}{\beta_{k}} I_{p,q+1}^{n,m} \left( z \right) \left( a_{i}, \alpha_{i} A_{j} \right) \left( B_{i}^{n} \right) \tag{4.2}
\]

Where \( A'_{i} = \left( 1 - \frac{V_{i}}{\beta_{i}} \right) \cdots \left( 1 - \frac{V_{i}}{\beta_{i}} \right) \); \( B'_{i} = \left( - \sum_{k=1}^{n} \frac{V_{k}}{\beta_{k}} \right) \cdots \left( - \sum_{k=1}^{n} \frac{V_{k}}{\beta_{k}} \right) \)

Corollary 3.

Taking \( r = 1 \), (4.2) reduces to H-function [4] and we have the result:

\[
\int_{k=1}^{b} \cdots \int_{k=1}^{b} \Pi_{k=1}^{n} x_{k}^{-1} H_{p,q}^{n,n} \left( z \Pi_{i=1}^{n} x_{i} \right) dx_{1} \cdots dx_{n} = \Pi_{k=1}^{n} \frac{a_{i}}{\beta_{k}} H_{p,q+1}^{n,m} \left( z \right) \left( A'_{i} \left( a_{j} \right) ; B'_{i} \right) \tag{4.3}
\]

under the same conditions verified by the corollary 2.

Corollary 4.

We suppose that \( A = \left( \alpha_{j} \right)_{1,p} = \left( \beta_{j} \right)_{1,q} = 1 \), then (4.3) leads to Meijer G-function [1], this gives:
Corollary 5.

From Theorem 2, The Pragathi-Satyanarayana’s I-function reduces to I-function defined by Rathie [3], in this situation, we have \( r = 1 \) then:

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^n x_k^{a_k} (1 - x_k)^{\beta_k - a_k - 1} \left( 1 - \prod_{k=1}^n x_k z \right)^{-\nu} I_{p, q, r}^{m, n} \left( \prod_{k=1}^n x_k^{\mu_k} (1 - x_k)^{\nu_k - \mu_k} \right) dx_1 \cdots dx_n
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\nu}{n!} \right) \left( \prod_{k=1}^n x_k \right)^{-\nu} \left( \prod_{k=1}^n x_k \right)^{-\nu} \frac{A_n(a_j, A_j)}{l, p} \left( b_j, G_j \right)_{l, q, \beta}'
\]

(4.5)

Under the conditions and notations verified by the theorem 2 of the above section. \( A_n \) and \( B_n \) are defined by the equations (3.4) and (3.5).

Corollary 6.

From Theorem 2, The Pragathi-Satyanarayana’s I-function reduces to I-function defined by Saxena [6] when \( A_j = B_j = A_j = B_j = 1 \) and we have the result:

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^n x_k^{a_k} (1 - x_k)^{\beta_k - a_k - 1} \left( 1 - \prod_{k=1}^n x_k z \right)^{-\nu} I_{p, q, r}^{m, n} \left( \prod_{k=1}^n x_k^{\mu_k} (1 - x_k)^{\nu_k - \mu_k} \right) dx_1 \cdots dx_n
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\nu}{n!} \right) \left( \prod_{k=1}^n x_k \right)^{-\nu} \left( \prod_{k=1}^n x_k \right)^{-\nu} \frac{A_n(a_j, A_j)}{l, p} \left( b_j, G_j \right)_{l, q, \beta}'
\]

(4.6)

Where, \( A_n' = (1 - \alpha_1 - n', \mu_1), \ldots, (1 - \alpha_n - n', \mu_n), (1 - \beta_1 + \alpha_1 - \nu_1 - \mu_1), \ldots, (1 - \beta_n + \alpha_n - \nu_n - \mu_n) \) and \( B_n' = (1 - \beta_1 - n'; \nu_1), \ldots, (1 - \beta_n - n'; \nu_n) \).

Corollary 7.

Taking \( r = 1 \), (4.6) reduces to H-function [4] and we have the result:

\[
I = \int_0^1 \cdots \int_0^1 \prod_{k=1}^n x_k^{a_k} (1 - x_k)^{\beta_k - a_k - 1} \left( 1 - \prod_{k=1}^n x_k z \right)^{-\nu} H_{p, q, r}^{m, n} \left( \prod_{k=1}^n x_k^{\mu_k} (1 - x_k)^{\nu_k - \mu_k} \right) dx_1 \cdots dx_n
\]
under the same conditions verified by the corollary 6.

**Corollary 8.**

Taking $A = (\alpha_j)_{i,p} = (\beta_j)_{i,q} = 1$ in (4.7), then Meijer G-function [1] replace the H-function and we have:

$$I = \int_0^1 \cdots \int_0^1 \frac{x^n}{\prod_{k=1}^{n} (1-x_k)} \beta_{l-1} \left(1 - \prod_{k=1}^{n} x_k z \right)^{-\alpha_0} G_{\alpha_0,p,q}^{m,n} \left(1 \prod_{k=1}^{n} x_k^{\alpha_k} (1-x_k)^{\beta_k} \right) dx_1 \cdots dx_n$$

$$= \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} z^n H_m^{n+2n}_{p+2n,q+2n} \left(\frac{A_n}{(a_j, \alpha_j)_{i,p}}, \prod_{j=1}^{m} (b_j, \beta_j)_{i,q}, B_n' \right)$$

(4.8)

5. **Conclusion**

In the study of Pragathi-Satyanarayana’s I-function [7] by specializing several parameters as well as variables, obtained like [4], lead to a large number of results concerning remarkably wide variety of useful special functions (or product of such special functions) expressible in terms of I-function defined by Saxena [6], defined by Rathie [3], H-function [4], Meijer’s G-function [1] and hypergeometric function of one variable [1, 5]. The nature of the multiple integrals involving Pragathi-Satyanarayana’s I-function, generalized Gamma function and generalized Hypergeometric function was studied. The theorems developed in this study are quite broad in nature and may be helpful in a number of interesting examples that emerge in literature relating to pure and applied mathematics as well as mathematical physics.

**References**


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