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On Fixed Point for Nonexpansive Mappings in Partial Metric Spaces

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Abstract:
In this paper, we establish some fixed points theorem for nonexpansive mappings in partial metric spaces. Our result generalizes Vetro's results (2015) in the setting of partial metric spaces. This work proves and generalizes some results of Aydi (2017). Suitable example is provided to illustrate the usability of our results.

Keywords: Fixed points, nonexpansive mapping, partial metric spaces.

2020 Mathematics Subject Classification: 47H10, 47H09

1. Introduction

The Banach Contraction Principle is a fundamental topic in mathematics, especially in the focus of fixed point theory. S. Banach introduced the Banach Contraction Principle in 1922 [2]. The Banach Contraction Principle guarantees the existence of fixed points from a contraction mapping. In its development, studies regarding the existence of fixed points from contraction mapping have attracted much interest from researchers. Various studies were carried out in an effort to develop fixed point theory, including by providing a new definition of contraction mapping in various applications [11, 12,14] as a generalization of the Banach Contraction Principle [3].

Contraction mapping is a special case of Lipschitz-λ mapping [16]. Note that if we are given a metric space (X, d) and given a mapping $f: X \to X$, then the mapping f is said to be a Lipschitz-λ mapping if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for every $x, y \in X$ with $\lambda \geq 0$. In this case, if $\lambda \in [0,1)$, then the mapping f is said to be a contraction mapping, whereas if $\lambda = 1$, the mapping f is said to be a non-expansive mapping. Like contraction mapping widely studied in relation to fixed points, non-expansive mapping also has an equally important role in fixed point studies [4,8,13,15]. One of them was carried out by Vetro [16], and the existence of fixed points from non-expansive mapping in metric space was successfully demonstrated. Furthermore, Aydi [1] extend a fixed point theorem for α-nonexpansive mappings on partial metric spaces. Motivated by Vetro [16] and Aydi [1], we will prove the fixed point theorem for nonexpansive mapping in partial metric spaces. And we also prove a more general theorem.

2. Preliminaries

In 1992, Matthews introduced a new concept as a generalization of standard metrics, namely partial metrics. Let we consider the following definitions.

Definition 2.1 [9,10] Let X be any nonempty set. A partial metric on X is a mapping $p: X \times X \to [0, \infty)$ which satisfies following conditions:
(P1) $p(x, y) = p(y, x)$,
(P2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$,
(P3) $p(x, x) \leq p(x, y)$,
(P4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$,
for all $x, y, z \in X$. Then pair (X, p) is called a partial metric space.

Partial metric p will be a metric if $p(x, x) = 0$. Several properties of partial metric spaces are given as follow [5,6,7,9,10].

Definition 2.2. Let (X, p) be a partial metric space. A sequence (x_n) is said to converges to a point $x \in X$ if and only if

$$lim_{n\to\infty}\, p(x_n, x) = p(x, x).$$

Definition 2.3. Let (X, p) be a partial metric space. A sequence (x_n) is called Cauchy sequence if and only if

$$lim_{n\to\infty}\, p(x_n, x_m)$$

is finite.

Definition 2.4. Let (X, p) be a partial metric space. If every Cauchy sequence (x_n) converges to a point $x \in X$ such that

$$lim_{n\to\infty}\, p(x_n, x_m) = p(x, x),$$

then (X, p) is known as complete partial metric space.

For p metric spaces on X, the mapping $p^s: X \times X \to [0, \infty)$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for each $x, y \in X$ is a metric on X. A sequence (x_n) is Cauchy in partial metric spaces (X, p) if and only if (x_n) is Cauchy sequence in metric space (X, p^s). It implies, a partial metric space (X, p) is complete if and only if metric spaces (X, p^s) is complete. Therefore, for (x_n) is sequence in partial metric spaces (X, p) and $x \in X$, we have $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$$

Definition 2.5. Let (X, p) be a partial metric space. Mapping $f: X \to X$ is (sequentially) continuous if $p(x_n, x) \to p(x, x)$ then for $n \to \infty$ we have

$$p\big(f(x_n), f(x)\big) \to p\big(f(x), f(x)\big).$$

Lemma 2.6. Suppose that (X, p) be a partial metric space, then

1. If $p(x, y) = 0$ then $x = y$
2. If $x \neq y$ then $p(x, y) > 0$

Lemma 2.7. [16] If (a_n) is nonincreasing sequences of nonnegative real numbers, then the sequence

$$\left(\frac{a_n + a_{n+1}}{a_n + a_{n+1} + 1}\right)$$

is nonincreasing too.

We extend Corollary 2.5 in [16] to partial metric spaces.

Corollary 2.8. Let (X, p) be a partial metric space. Suppose that $f: X \to X$ be a nonexpansive mapping and $x_0 \in X$. If (x_n) is a Picard sequence of initial point x_0, then the sequence

$$\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1}\right) \tag{2.1}$$

is nonincreasing.

Proof:

Since (x_n) is a Picard sequence of initial point x_0 then $x_n = f(x_{n-1}) = f^n(x_0)$ for all $n \in \mathbb{N}$. Furthermore, since f is nonexpansive mapping then

$$p(x_n, x_{n+1}) = p(f(x_{n-1}), f(x_n)) \leq p(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Hence by Lemma 2.7, the statement (2.1) holds. This completes the proof.

3. Main Results

Theorem 3.1. *Let (X, p) be a complete partial metric spaces endowed with a binary relation $\Re$ on X. Suppose that $f: X \to X$ be a nonexpansive mappings such that*

$$p\big(f(x), f(y)\big) \leq \left(\frac{p\big(x, f(y)\big) + p(y, f(x))}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) p(x, y) \tag{3.1}$$

for each $x, y \in \Re$, where $k \in [0,1)$. Assume that:

1. *f is preserving mapping*
2. *f is continuous mapping*
3. *$Fix\ (f)$ is well ordered with respect to $\Re$.*

If there exist $x_0 \in X$ such that $\big(x_0, f(x_0)\big) \in \Re$ and

$$\frac{p\big(x_0, f(x_0)\big) + p(f(x_0), f^2(x_0))}{p\big(x_0, f(x_0)\big) + p\big(f(x_0), f^2(x_0)\big) + 1} + k < 1 \tag{3.2}$$

then there exist $z \in X$ such that $p(z, z) = 0$. Furthermore

a. *There exists $z \in X$ fixed point of f*
b. *The Picard sequences of initial point $x_0 \in X$ converges to fixed point of f*
c. *If z and w are fixed point of f where $z \neq w$ then*

$$\frac{p(z, w)}{p(z, z) + p(w, w) + 1} \geq \frac{1 - k}{2}$$

Proof:

Let $x_0 \in X$ be such that $\left(x_0, f(x_0)\right) \in \mathfrak{R}$ and (3.2) holds. Suppose that (x_n) be a Picard sequence of initial point x_0 such that

$$x_n = f(x_{n-1}) = f^n(x_0)$$

for all $n \in \mathbb{N}$. Let we consider the following cases:

Case 1. Suppose that $x_{n-1} = x_n$ for some $n \in \mathbb{N}$ then

$$x_{n-1} = f(x_{n-1})$$

It means x_{n-1} is fixed point of f. Hence, the existence of a fixed point of f is proved.

Case 2. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Let we consider that $(x_0, x_1) = \left(x_0, f(x_0)\right) \in \mathfrak{R}$. Since f is preserving mapping then we have

$$\left(f(x_0), f^2(x_0)\right) \in \mathfrak{R}$$

By induction, we have

$$(x_{n-1}, x_n) = \left(f^{n-1}(x_0), f^n(x_0)\right) \in \mathfrak{R}$$

for all $n \in \mathbb{N}$. Since $(x_{n-1}, x_n) \in \mathfrak{R}$ then by (3.1) we obtain

$$\begin{aligned}
p(x_n, x_{n+1}) &= p\left(f(x_{n-1}), f(x_n)\right) \\
&\leq \left(\frac{p\left(x_{n-1}, f(x_n)\right) + p\left(x_n, f(x_{n-1})\right)}{p\left(x_{n-1}, f(x_{n-1})\right) + p\left(x_n, f(x_n)\right) + 1} + k\right) p(x_{n-1}, x_n) \\
&= \left(\frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) p(x_{n-1}, x_n) \\
&\leq \left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) p(x_{n-1}, x_n) \\
&= \left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) p(x_{n-1}, x_n)
\end{aligned}$$

Since f in nonexpansive mapping, then by using Corollary 2.8 we obtain

$$\begin{aligned}
p(x_n, x_{n+1}) &\leq \left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) p(x_{n-1}, x_n) \\
&\leq \left(\frac{p(x_0, x_1) + p(x_1, x_2)}{p(x_0, x_1) + p(x_1, x_2) + 1} + k\right) p(x_{n-1}, x_n)
\end{aligned}$$

By (3.2) then we have

$$\begin{aligned}
p(x_n, x_{n+1}) &\leq \left(\frac{p(x_0, x_1) + p(x_1, x_2)}{p(x_0, x_1) + p(x_1, x_2) + 1} + k\right) p(x_{n-1}, x_n) \\
&\leq \left(\frac{p(x_0, f(x_0)) + p(f(x_0), f^2(x_0))}{p(x_0, f(x_0)) + p(f(x_0), f^2(x_0)) + 1} + k\right) p(x_{n-1}, x_n) \\
&= \alpha p(x_{n-1}, x_n)
\end{aligned}$$

where $\alpha < 1$. Hence (x_n) is Cauchy sequences in X since $p(x_n, x_{n+1}) \leq \alpha p(x_{n-1}, x_n), \alpha < 1$. Therefor we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0$$

Since X is complete partial metric spaces, then sequences (x_n) is converges, namely $x_n \to z \in X$. It implies

$$\lim_{n\to\infty} p(x_n, z) = p(z, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Therefor, $p(z, z) = 0$. Furthermore, we will prove that z is fixed point of f. Since f is nonexpansive mapping then

$$p\big(f(z), f(z)\big) \leq p(z, z) = 0$$

Hence $p\big(f(z), f(z)\big) = 0$. By hypothesis (2), the continuity of f then by (3.1)

$$\begin{aligned} p(z, f(z)) &= \lim_{n\to\infty} p\big(x_{n+1}, f(z)\big) \\ &= \lim_{n\to\infty} p\big(f(x_n), f(z)\big) \\ &= p(z, z) \end{aligned}$$

Since $p(z, z) = 0$, it implies $p\big(z, f(z)\big) = 0$, i.e. $z \in f(z)$. It means z is fixed point of f. Suppose that w is another fixed point of f where $z \neq w$. By hypothesis (3) we obtain $(z, w) \in \Re$, then (3.1) we have

$$\begin{aligned} p(z, w) &= p\big(f(z), f(w)\big) \\ &\leq \left(\frac{p\big(z, f(w)\big) + p\big(w, f(z)\big)}{p\big(z, f(z)\big) + p\big(w, f(w)\big) + 1} + k\right) p(z, w) \\ &= \left(\frac{p(z, w) + p(w, z)}{p(z, z) + p(w, w) + 1} + k\right) p(z, w) \\ &= \left(\frac{2p(z, w)}{p(z, z) + p(w, w) + 1} + k\right) p(z, w) \end{aligned}$$

Furthermore,

$$\frac{p(z, w)}{p(z, z) + p(w, w) + 1} \geq \frac{1-k}{2}$$

This completes the proof.

We can replace the continuity hypothesis in Theorem 3.1 with the convergence hypothesis as the following theorem.

Theorem 3.2. *Let (X, p) be a complete partial metric spaces endowed with a binary relation $\Re$ on X. Suppose that $f: X \to X$ be a nonexpansive mappings such that*

$$p\big(f(x), f(y)\big) \leq \left(\frac{p\big(x, f(y)\big) + p\big(y, f(x)\big)}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) p(x, y) \qquad (3.3)$$

for each $x, y \in \Re$, *where* $k \in [0,1)$. *Assume that:*

1. f *is preserving mapping*
2. *If* x_n *sequences in* X *such that* $(x_{n-1}, x_n) \in \Re$ *for each* $n \in \mathbb{N}$ *and* $x_n \to z \in X$ *as* $n \to \infty$, *then* $(x_{n-1}, z) \in \Re$ *for all* $n \in \mathbb{N}$
3. $Fix\ (f)$ *is well ordered with respect to* $\Re$.

If there exist $x_0 \in X$ *such that* $\big(x_0, f(x_0)\big) \in \Re$ *and*

$$\frac{p\big(x_0, f(x_0)\big) + p(f(x_0), f^2(x_0))}{p\big(x_0, f(x_0)\big) + p\big(f(x_0), f^2(x_0)\big) + 1} + k < 1 \tag{3.4}$$

then there exist $z \in X$ *such that* $p(z,z) = 0$. *Furthermore*

a. *There exists* $z \in X$ *fixed point of* f
b. *The Picard sequences of initial point* $x_0 \in X$ *converges to fixed point of* f
c. *If* z *and* w *are fixed point of* f *where* $z \neq w$ *then*

$$\frac{p(z,w)}{p(z,z) + p(w,w) + 1} \geq \frac{1-k}{2}$$

Proof:

The proof is following Theorem 3.1's proof. We only have to check that z is fixed point of f. By hypothesis (2) we have $(x_n, z) \in \Re$ for all $n \in \mathbb{N}$. Therefor by (3.3) we obtain

$$\begin{aligned} p(x_{n+1}, f(z)) &= p\big(f(x_n), f(z)\big) \\ &\leq \left(\frac{p\big(x_n, f(z)\big) + p\big(z, f(x_n)\big)}{p\big(x_n, f(x_n)\big) + p\big(z, f(z)\big) + 1} + k\right) p(x_n, z) \\ &= \left(\frac{p\big(x_n, f(z)\big) + p(z, x_{n+1})}{p(x_n, x_{n+1}) + p\big(z, f(z)\big) + 1} + k\right) p(x_n, z) \end{aligned}$$

Taking $n \to \infty$ then we obtain

$$p(z, f(z)) \leq \left(\frac{p\big(z, f(z)\big) + p(z,z)}{p(z,z) + p\big(z, f(z)\big) + 1} + k\right) p(z,z)$$

Since $p(z,z) = 0$, it implies $p\big(z, f(z)\big) \leq 0$. Hence $p\big(z, f(z)\big) = 0$, i.e. $z \in f(z)$. It means z is fixed point of f. For the following proof on the similar line with Theorem 3.1's proof. This completes the proof.

Let we consider, if $p(z,z)$ and $p(w,w)$ in Theorem 3.1 and 3.2 are zero, then we have

$$p(z,w) \geq \frac{1-k}{2}$$

Furthermore, we provide the following example motivated by Aydi [1].

Example 3.3. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. Define $f: X \to X$ by $f(x) = x^2$ for all $x \in X$. Clearly, (X,p) is complete partial metric spaces and f is continuous mappings. Suppose that partial metric spaces (X,p) is endowed with binary relations $\Re$ where $(x,y) \in \Re$ it means $x \geq y$.

Without loss of generality, take $x \geq y > 0$. Let we consider that f is nonexpasive mapping since

$$p\big(f(x), f(y)\big) = \max\{x^2, y^2\} = x^2 \leq x = \max\{x, y\} = p(x, y).$$

Take $k = \frac{1}{2}$ and let we consider the following cases:

Case 1: For $x^2 \geq y$ we have

$p\big(f(x), f(y)\big) = x^2$, and

$$\begin{aligned}
\left(\frac{p\big(x, f(y)\big) + p\big(y, f(x)\big)}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) p(x, y) &= \left(\frac{x + x^2}{x + y + 1} + \frac{1}{2}\right) x \\
&= \frac{2x^2 + 2x^3 + (1 + x + y)x}{2(x + y + 1)} \\
&= \frac{2x^2(1 + x) + 2x^2 y - 2x^2 y + (1 + x + y)x}{2(x + y + 1)} \\
&= x^2 + \frac{(1 + x + y - 2xy)x}{2(x + y + 1)}
\end{aligned}$$

Since $2xy \leq x^2 + y^2 \leq x + y$ then

$$\frac{(1 + x + y - 2xy)x}{2(x + y + 1)} > 0$$

It implies

$$\left(\frac{p\big(x, f(y)\big) + p\big(y, f(x)\big)}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) p(x, y) \geq x^2 = p\big(f(x), f(y)\big) = x^2$$

Case 2: For $x^2 < y$ we have

$p\big(f(x), f(y)\big) = x^2$, and

$$\begin{aligned}
\left(\frac{p\big(x, f(y)\big) + p\big(y, f(x)\big)}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) p(x, y) &= \left(\frac{x + y}{x + y + 1} + \frac{1}{2}\right) x \\
&= \frac{2x^2 + 2xy + (1 + x + y)x}{2(x + y + 1)} \\
&= \frac{2x^2(1 + y) + 2x^3 - 2x^3 + (1 + x + y)x}{2(x + y + 1)} \\
&= x^2 + \frac{(1 + x + y - 2x^2)x}{2(x + y + 1)}
\end{aligned}$$

Since $x + y \leq x^2 + y^2 \leq 2x^2$ then

$$\frac{(1 + x + y - 2x^2)x}{2(x + y + 1)} > 0$$

Hence

$$\left(\frac{p\big(x,f(y)\big)+p\big(y,f(x)\big)}{p\big(x,f(x)\big)+p\big(y,f(y)\big)+1}+k\right)p(x,y)\geq x^2=p\big(f(x),f(y)\big)$$

This condition shows that (3.1) in Theorem 3.1 (resp. (3.3) in Theorem 3.2) is verified for all $x,y\in X$. Furthermore, f is preserving mappings since $\big(x,f(x)\big)\in\mathfrak{R}$, i.e. $x\geq x^2$ then we have $x^2\geq x^3$ or $\big(f(x),f^2(x)\big)\in\mathfrak{R}$. For $x_0=\frac{1}{2}$ we have $\big(x_0,f(x_0)\big)\in\mathfrak{R}$ since $\frac{1}{2}>\frac{1}{4}=\left(\frac{1}{2}\right)^2$ and

$$\frac{p\big(x_0,f(x_0)\big)+p(f(x_0),f^2(x_0))}{p\big(x_0,f(x_0)\big)+p\big(f(x_0),f^2(x_0)\big)+1}+k=\frac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}+1}+\frac{1}{2}=\frac{13}{14}<1.$$

Therefore, all hypotheses of Theorem 3.1 are satisfied. In this case f has two fixed points which are $z=0$ and $w=1$, and

$$\frac{p(z,w)}{p(z,z)+p(w,w)+1}=\frac{p(0,1)}{p(0,0)+p(1,1)+1}=\frac{1}{2}>\frac{1}{4}=\frac{1-k}{2}.$$

In the next results, we replace $p(x,y)$ in (3.1) by $\mathcal{N}(x,y)$ as on the following theorem. This result generalizes Theorem 3.1.

Theorem 3.4. *Let (X,p) be a complete partial metric spaces endowed with a binary relation $\mathfrak{R}$ on X. Suppose that $f\colon X\to X$ be a nonexpansive mappings such that*

$$p\big(f(x),f(y)\big)\leq\frac{1}{2}\left(\frac{p\big(x,f(y)\big)+p\big(y,f(x)\big)}{p\big(x,f(x)\big)+p\big(y,f(y)\big)+1}+k\right)\mathcal{N}(x,y)\tag{3.5}$$

for each $x,y\in\mathfrak{R}$, where $k\in[0,1)$ and

$$\mathcal{N}(x,y)=\max\{p(x,y),p\big(x,f(x)\big),p\big(y,f(y)\big),p\big(x,f(y)\big),p(y,f(x))\}\tag{3.6}$$

Assume that:

1. *f is preserving mapping*
2. *f is continuous mapping*
3. *$Fix\ (f)$ is well ordered with respect to $\mathfrak{R}$.*

If there exist $x_0\in X$ such that $\big(x_0,f(x_0)\big)\in\mathfrak{R}$ and

$$\frac{p\big(x_0,f(x_0)\big)+p(f(x_0),f^2(x_0))}{p\big(x_0,f(x_0)\big)+p\big(f(x_0),f^2(x_0)\big)+1}+k<1\tag{3.7}$$

then there exist $z\in X$ such that $p(z,z)=0$. Furthermore

a. *There exists $z\in X$ fixed point of f*
b. *The Picard sequences of initial point $x_0\in X$ converges to fixed point of f*

c. *If z and w are fixed point of f where $z \neq w$ then*

$$\frac{p(z,w)}{p(z,z)+p(w,w)+1} \geq \frac{1-k}{2}$$

Proof:

Suppose that $x_0 \in X$ such that $(x_0, f(x_0)) \in \Re$ and (3.5) holds. Let (x_n) be a Picard sequence of initial point x_0 such that

$$x_n = f(x_{n-1}) = f^n(x_0)$$

for all $n \in \mathbb{N}$. Let we consider, if $x_{n-1} = x_n$ for some $n \in \mathbb{N}$ then $x_{n-1} = f(x_{n-1})$. It means x_{n-1} is fixed point of f. Hence, the existence of a fixed point of f is proved. Another condition, suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Let we consider that $(x_0, x_1) = (x_0, f(x_0)) \in \Re$. Since f is preserving mapping then we have

$$(f(x_0), f^2(x_0)) \in \Re$$

By induction, we have

$$(x_{n-1}, x_n) = (f^{n-1}(x_0), f^n(x_0)) \in \Re$$

for all $n \in \mathbb{N}$. Since $(x_{n-1}, x_n) \in \Re$ then by (3.5) we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &= p(f(x_{n-1}), f(x_n)) \\ &\leq \frac{1}{2}\left(\frac{p(x_{n-1}, f(x_n)) + p(x_n, f(x_{n-1}))}{p(x_{n-1}, f(x_{n-1})) + p(x_n, f(x_n)) + 1} + k\right)\mathcal{N}(x_{n-1}, x_n) \end{aligned}$$

Let we consider

$$\begin{aligned} \mathcal{N}(x_{n-1}, x_n) &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, f(x_{n-1})), p(x_n, f(x_n)), p(x_{n-1}, f(x_n)), p(x_n, f(x_{n-1}))\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1})\} \end{aligned}$$

Since f is nonexpansive mapping then

$$p(x_{n-1}, x_n) = p(f(x_n), f(x_{n+1})) \leq p(x_n, x_{n+1})$$

Therefore

$$\mathcal{N}(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_{n+1})\}$$

Let we consider

$$\begin{aligned} p(x_{n-1}, x_{n+1}) &\leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) \\ &\leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) \\ &\leq p(x_{n-1}, x_n) + p(x_{n-1}, x_n) \\ &= 2p(x_{n-1}, x_n) \end{aligned}$$

Thus

$$\mathcal{N}(x_{n-1}, x_n) \leq \max\{p(x_{n-1}, x_n), 2p(x_{n-1}, x_n)\} = 2p(x_{n-1}, x_n)$$

Hence

$$\begin{aligned} p(x_n, x_{n+1}) &= p\big(f(x_{n-1}), f(x_n)\big) \\ &\le \frac{1}{2}\left(\frac{p\big(x_{n-1}, f(x_n)\big) + p\big(x_n, f(x_{n-1})\big)}{p\big(x_{n-1}, f(x_{n-1})\big) + p\big(x_n, f(x_n)\big) + 1} + k\right) 2p(x_{n-1}, x_n) \\ &= \frac{1}{2}\left(\frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) 2p(x_{n-1}, x_n) \\ &\le \frac{1}{2}\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) 2p(x_{n-1}, x_n) \\ &= \frac{1}{2}\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + 1} + k\right) 2p(x_{n-1}, x_n) \\ &\le \frac{1}{2}\left(\frac{p(x_0, x_1) + p(x_1, x_2)}{p(x_0, x_1) + p(x_1, x_2) + 1} + k\right) 2p(x_{n-1}, x_n) \\ &= \left(\frac{p(x_0, x_1) + p(x_1, x_2)}{p(x_0, x_1) + p(x_1, x_2) + 1} + k\right) p(x_{n-1}, x_n) \\ &\le \alpha p(x_{n-1}, x_n) \end{aligned}$$

where $\alpha < 1$. Hence (x_n) is Cauchy sequences in X since $p(x_n, x_{n+1}) \le \alpha p(x_{n-1}, x_n), \alpha < 1$. The next proof is the same as the proof of Theorem 3.1. This completes the proof.

Analogous to Theorem 3.4, we can generalize theorem 3.2 as follows.

Theorem 3.5. *Let (X, p) be a complete partial metric spaces endowed with a binary relation $\mathfrak{R}$ on X. Suppose that $f: X \to X$ be a nonexpansive mappings such that*

$$p\big(f(x), f(y)\big) \le \frac{1}{2}\left(\frac{p\big(x, f(y)\big) + p\big(y, f(x)\big)}{p\big(x, f(x)\big) + p\big(y, f(y)\big) + 1} + k\right) \mathcal{N}(x, y)$$

for each $x, y \in \mathfrak{R}$, where $k \in [0,1)$ and

$$\mathcal{N}(x, y) = \max\{p(x, y), p\big(x, f(x)\big), p\big(y, f(y)\big), p\big(x, f(y)\big), p(y, f(x))\}$$

Assume that:

1. *f is preserving mapping*
2. *If x_n sequences in X such that $(x_{n-1}, x_n) \in \mathfrak{R}$ for each $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in \mathfrak{R}$ for all $n \in \mathbb{N}$*
3. *$Fix\ (f)$ is well ordered with respect to $\mathfrak{R}$.*

If there exist $x_0 \in X$ such that $\big(x_0, f(x_0)\big) \in \mathfrak{R}$ and

$$\frac{p\big(x_0, f(x_0)\big) + p(f(x_0), f^2(x_0))}{p\big(x_0, f(x_0)\big) + p\big(f(x_0), f^2(x_0)\big) + 1} + k < 1$$

then there exist $z \in X$ such that $p(z, z) = 0$. Furthermore

a. *There exists $z \in X$ fixed point of f*
b. *The Picard sequences of initial point $x_0 \in X$ converges to fixed point of f*

c. *If z and w are fixed point of f where $z \neq w$ then*

$$\frac{p(z,w)}{p(z,z)+p(w,w)+1} \geq \frac{1-k}{2}$$

Proof:

The proof of this theorem is following Theorem 3.4 and Theorem 3.2's proof.

4. Conclusions

In this article, we have extended the fixed point theorem to non-expansive mappings resulting from Vetro in the setting of partial metric spaces. These results also provide proof of one of Aydi's results. Furthermore, we generalize our previous result by using the $\mathcal{N}$ function. We also give examples to illustrate our results.

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