

On Geodesic Semi $E - b$ –Preinvex Functions

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Abstract:

In this paper, we introduce and investigate a novel class of functions termed as geodesic semi E - b -preinvex (GSEBPI) functions on RMs (Riemannian manifolds). These functions are defined on geodesic E - b -invex (GEBI) sets with respect to a mapping η , there by extending existing generalized convexity concepts within a geometric framework. Several fundamental properties and structural characterizations are established, along with connections to related convexity notions. Further, we consider the non-linear programming problems and prove some results by using GEBPI, GSEBPI, strictly GSEBPI, GQSEBPI functions for optimality. At the end of this paper, it is shown that, under appropriate differentiability assumptions, every GEBI functions is GEBPI functions. The results contribute to the advancement of geometric optimization and nonlinear analysis on RMs.

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1. Introduction

In the area of optimization, convexity and generalized convexity plays vital role. Numerous extensions of convex functions and sets have been discussed in the literature. Among these are invex sets and preinvex functions, introduced by Weir and Mond [27] and Weir and Jeyakumar [26]. Bector and Singh [4] be established the function namely, b -vex functions and also shares the many properties with convex functions. Youness [29] established the concepts of E –convex sets and E –convex

functions, that have few crucial programs in several branches of mathematical sciences [1, 3, 14]. The preliminary results of Youness stimulated a tremendous deal of work, which has significantly extended the function of E –convexity in optimization theory; see [7, 12, 28, 8]. Syau and Lee [24] introduced the concept of E -quasiconvex functions and outlined various properties of both E -convex and E -quasiconvex functions. Additionally, Chen [5] explored a class of functions known as semi E -convex functions and provided an explanation of some of their key properties. Lately, Mishra *et al.* [16] introduced a class of functions known as semi E -b-convex functions and outlined several of their key properties. In linear topological spaces, the concept of convex sets is based on connecting any two points within the space via a line segment. However, in many real-world applications, it is not always feasible to connect points using a straight line. A manifold, unlike a linear space, presents a more complex structure, which naturally extends the concept from linear space to Riemannian manifolds (RMs). Udriste [25] and Rapcsak [15], proposed a generalization of the convexity concept by replacing the linear space with Riemannian manifolds (RMs), where the line segment between two points is substituted by geodesic segments. In this context, convex functions are characterized by the positiveness of their Hessian.

Upadhyay et al. [21] introduced a class of functions known as explicitly geodesic B -preinvex functions on Riemannian manifolds (RMs) and discussed several of their key properties. The class of Geodesic semi E -preinvex functions on RMs that's introduced by Porwal [19] and also derived some characterizations of geodesic semi E -preinvex functions in terms of E -epigraph. Motivated by the works in [2, 9, 17, 18, 20, 23] and references therein, this paper explores and derives several properties of geodesic semi $E - b -$ preinvex functions on RMs.

2. Preliminaries and definitions

In this paper, we will denote Q as RMs (Riemannian manifolds) and δ as the geodesic. This section provides some preliminary notations related to RMs and essential definitions that will be referenced throughout the paper. The foundational material in this section is adapted from [3]. For further information on differential geometry, please refer to [10, 11]. Let \mathbb{R}^n represent an $n -$ dimensional Euclidean space and \mathbb{R}_+ denote the set of nonnegative real numbers. Let Q be a C^∞ smooth manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p Q$ with the corresponding norm denoted by $\| \cdot \|_p$, which characterizes the RMs Q . The length of a piecewise differentiable curve $\delta: [a, b] \rightarrow Q$ that connects points p and q (where $\delta(a) = p$ & $\delta(b) = q$) is defined by

$$L(\delta) := \int_a^b \|\delta'(t)\|_{\delta(t)} dt.$$

Minimizing the length functional over the set of all piecewise differentiable curves that connect points p and q in Q , we obtain a distance function $d(p, q)$. This distance function d induces the original topology on Q . Let $\chi(Q)$ represent the space of all vector fields on Q . The Riemannian metric induces a map $f \mapsto \text{grad } f \in \chi(Q)$, which assigns the gradient of f by the rule $\langle df, X \rangle = df(X)$ for each vector field $X \in \chi(Q)$. On every Riemannian manifold, there exists a unique covariant derivative known as the Levi-Civita connection, denoted by $\nabla_X Y$ for any vector fields X and $Y \in Q$. Additionally,

a geodesic is defined as a C^∞ smooth curve δ whose tangent is parallel along the path δ . In other words, δ satisfies the following equation:

$$\nabla_{\frac{d\delta(t)}{dt}} \frac{d\delta(t)}{dt} = 0.$$

The path δ that connects points p and q in Q and satisfies $L(\delta) = d(p, q)$ is referred to as a minimal geodesic.

Definition 1. A subset $D \subseteq Q$ is considered geodesic $E - b - \text{invex}(\text{GEBI})$ w.r.t. $\eta: Q \times Q \rightarrow TQ$ if \exists mappings $E: Q \rightarrow Q, b: Q \times Q \times [0,1] \rightarrow \mathbb{R}_+$ & a unique geodesic $\delta_{E(p),E(q)}: [0,1] \rightarrow Q$ s.t. $\mu b(p, q, \mu) \in [0,1]$, where μ is a parameter along the geodesic, then

$\delta_{E(p),E(q)}(0) = E(q), \delta_{E(p),E(q)}(\mu) \in D, \delta'_{E(p),E(q)}(0) = \eta(E(p), E(q))$ for each $p, q \in D$ & $\mu \in [0,1]$.

Definition 2. A function $h: D \rightarrow \mathbb{R}$ is considered geodesic $E - b - \text{preinvex}(\text{GEBPI})$ on a geodesic $E - b - \text{invex}(\text{GEBI})$ set $D \subseteq Q$ if \exists mappings $E: Q \rightarrow Q, b: Q \times Q \times [0,1] \rightarrow \mathbb{R}_+$ s.t. $\mu b(p, q, \mu) \in [0,1]$, then

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(E(p)) + (1 - \mu b(p, q, \mu))h(E(q)),$$

for each $p, q \in D$ & $\mu \in [0,1]$. If the above inequality is strict for each $p, q \in D$ with $p \neq q$ and $\mu \in (0,1)$, then the function $h: D \rightarrow \mathbb{R}$ is called strictly GEBPI on D .

Definition 3. A function $h: D \rightarrow \mathbb{R}$ is considered geodesic semi $E - b - \text{preinvex}(\text{GSEBPI})$ on a GEBI set D if \exists mappings $E: Q \rightarrow Q, b: Q \times Q \times [0,1] \rightarrow \mathbb{R}_+$ s.t. $\mu b(p, q, \mu) \in [0,1]$, then

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q),$$

for each $p, q \in D$ & $\mu \in [0,1]$. If the above inequality is strict for each $p, q \in D$ with $p \neq q$ and $\mu \in (0,1)$, then the function $h: D \rightarrow \mathbb{R}$ is called strictly GSEBPI on D .

Definition 4. The function $h: D \rightarrow \mathbb{R}$ is considered geodesic quasi-semi $E - b - \text{preinvex}(\text{GQSEBPI})$ on a GEBI set $D \subseteq Q$, if

$$h(\delta_{E(p),E(q)}(\mu)) \leq \max\{h(p), h(q)\}, \text{ for each } p, q \in D \text{ \& } \mu \in [0,1].$$

If the above inequality is strict for each $p, q \in D, p \neq q$ & $\mu \in (0,1)$, then the function $h: D \rightarrow \mathbb{R}$ is called strictly GQSEBPI on D .

Definition 5. A function $h: D \rightarrow \mathbb{R}$ is considered geodesic pseudo-semi $E - b - \text{preinvex}(\text{GPSEBPI})$ on a GEBI set $D \subseteq Q$, if \exists a strictly positive function $\sigma: D \times D \rightarrow \mathbb{R}$, the following condition holds:

$$h(p) < h(q) \Rightarrow h(\delta_{E(p),E(q)}(\mu)) \leq h(q) + \mu b(\mu - 1)\sigma(p, q), \forall p, q \in D \text{ \& } \mu \in (0,1).$$

Definition 6. Let $W \subseteq Q \times \mathbb{R}$ & $E: Q \rightarrow Q$, then the set W is considered geodesic $E - b - \text{invex}(\text{GEBI}) \Leftrightarrow (p, \alpha), (q, \beta) \in W \Rightarrow (\delta_{E(p),E(q)}(\mu), \mu\alpha + (1 - \mu)\beta) \in W, \forall \mu \in [0,1]$.

Definition 7. The E -epigraph of f is defined by $\{(E(p), \alpha): p \in D, \alpha \in \mathbb{R}, h(E(p)) \leq \alpha\}$

3. Properties of geodesic semi E-b-preinvex functions

In this section, we present and explore the properties on geodesic semi $E - b -$ preinvex functions on RMs.

Theorem 1. If $h: D \rightarrow \mathbb{R}$ be GEBPI function on a GEBI set D w.r.t. η , then the function h is a GSEBPI on $D \Leftrightarrow h(E(p)) \leq h(p)$ for each $p \in D$.

Proof. Let h be a GSEBPI function on a GEBI set D , then for any $p, q \in D$ & $\mu \in [0,1]$. We have

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q).$$

Therefore, for $\mu b(p, q, \mu) = 1$, we get $h(E(p)) \leq h(p)$.

Conversely, Let $h(E(p)) \leq h(p)$, for each $p \in D$, then for any $p, q \in D$ and $\mu \in [0,1]$, we have

$$\begin{aligned} h(\delta_{E(p),E(q)}(\mu)) &\leq \mu b(p, q, \mu)h(E(p)) + (1 - \mu b(p, q, \mu))h(E(q)) \\ &\leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q). \end{aligned}$$

Hence, proved. □

Theorem 2. Suppose $h: D \rightarrow \mathbb{R}$ is GSEBPI on a GEBI set $D \subseteq Q$, then for any $\alpha \in \mathbb{R}$, the $\alpha -$ level set $K_\alpha = \{p: p \in D, h(p) \leq \alpha\}$ is a GEBI set.

Proof. If for any $p, q \in K_\alpha$ and $0 \leq \mu b(p, q, \mu) \leq 1$, then $h(p) \leq \alpha, h(q) \leq \alpha$. Since h is GSEBPI on a GEBI set $D \subseteq Q$, then

$$\begin{aligned} h(\delta_{E(p),E(q)}(\mu)) &\leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) \\ &\leq \mu b(p, q, \mu)\alpha + (1 - \mu b(p, q, \mu))\alpha \\ &= \alpha. \end{aligned}$$

Hence the level set K_α is GEBI. □

Theorem 3. Suppose $D \subseteq Q$ is a GEBI set w.r.t. η . Then the function $h: D \rightarrow \mathbb{R}$ is GQSEBPI iff the level set $K_\alpha = \{p: p \in D, h(p) \leq \mu\}$ is GEBI for each $\alpha \in \mathbb{R}$.

Proof. Suppose that the function h be a GQSEBPI on D . Thus for any $p, q \in K_\alpha$ & $\mu \in [0,1], f(p) \leq \alpha, h(q) \leq \alpha$ we have

$$h(\delta_{E(p),E(q)}(\mu)) \leq \max\{h(p), h(q)\} \leq \alpha,$$

This \Rightarrow

$$h(\delta_{E(p),E(q)}(\mu)) \in K_\alpha.$$

Hence, the level set K_α is GEBI.

Conversely, Suppose $D \subseteq Q$ be a GEBI set & K_α is GEBI set for each $\alpha \in \mathbb{R}$, we have to show that the function h is GQSEBPI. By definition of GEBI set for each $p, q \in D$ and $\mu \in [0,1]$ s.t.

$$h(\delta_{E(p),E(q)}(\mu)) \in D.$$

Let $\alpha = \max\{h(p), h(q)\}$, thus $p, q \in K_\alpha$, since K_α is GEBI set, then $\mu_{E(p),E(q)}(\lambda) \leq \alpha = \max\{h(p), h(q)\}$. Hence, the function h is GQSEBPI.

□

Theorem 4. Suppose the functions $h_i: S \rightarrow \mathbb{R}; i = 1, \dots, k$ are GQSEBPI on D . Then the set $Q = \{p \in D: h_i(p) \leq 0, i = 1, \dots, k\}$ is GEBI.

Proof. The proof directly follows from the previous Theorem 3.

□

Now, the map $E: Q \rightarrow Q$. & consider another map $E \times I: Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}$ is defined by

$$(E \times I)(p, t) = (E(p), t), \forall (p, t) \in Q \times \mathbb{R}.$$

It's a lot easier to show that the subset $D \subseteq Q$ is GEBI, $\Leftrightarrow Q \times \mathbb{R}$ is geodesic $E \times I - b -$ invex. Now, we are going to describe the GSEBPI function in terms of its epigraph.

Theorem 5. Assume S be a GEBI set, then h is GSEBPI function on $D \Leftrightarrow \text{epi}(h)$ is geodesic $E \times I - b -$ invex on $D \times \mathbb{R}$.

Proof. Suppose that h be a GSEBPI function on D , let $(p, \alpha), (y, \beta) \in \text{epi}(h), \mu \in [0,1]$, we have

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) \leq \mu b(p, q, \mu)\alpha + (1 - \mu b(p, q, \mu))\beta.$$

Thus

$$(\delta_{E(p),E(q)}(\mu), \mu b(p, q, \mu)\alpha + (1 - \mu b(p, q, \mu))\beta) \in \text{epi}(h),$$

this $\Rightarrow \text{epi}(h)$ is geodesic $E \times I - b -$ invex on $D \times \mathbb{R}$. Conversely, let $\text{epi}(h)$ is geodesic $E \times I - b -$ invex on $D \times \mathbb{R}$. Let $p, q \in D, \mu \in [0,1]$ then $(p, h(p)), (q, h(q)) \in \text{epi}(h)$. Since $\text{epi}(h)$ is geodesic $E \times I - b -$ invex on $D \times \mathbb{R}$, we have

$$(\delta_{E(p),E(q)}(\mu), \mu b(p, q, \mu)\alpha + (1 - \mu b(p, q, \mu))\beta) \in \text{epi}(h),$$

i. e.

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q).$$

Hence, the function h is GSEBPI on D .

Theorem 6. If $h: D \rightarrow \mathbb{R}$ be a GSEBPI function on a GEBI set D w.r.t. η then the function h is GPSEBPI.

Proof. Since $h(p) < h(q)$ & h is GSEBPI function on a GEBI set D then $\forall p, q \in D$ & $\mu \in (0,1)$, we have

$$\begin{aligned}
 h(\delta_{E(p),E(q)}(\mu)) &\leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) \\
 &= h(q) + \mu b(p, q, \mu)(h(p) - h(q)) \\
 &< h(q) + \mu b(p, q, \mu)(1 - \mu)(h(p) - h(q)) \\
 &= h(q) + \mu b(p, q, \mu)(\mu - 1)(h(q) - h(p)) \\
 &= h(q) + \mu b(p, q, \mu)(\mu - 1)\sigma(p, q),
 \end{aligned}$$

where $\sigma(p, q) = h(q) - h(p) > 0$, hence the function h is GPSEBPI. □

Theorem 7. Suppose $h: D \rightarrow \mathbb{R}$ be a GQSEBPI function on a GEBI set D , if $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function, then the composition $\psi \circ h$ is a GQSEBPI function on D .

Proof. Since h is a GQSEBPI function, then for $\forall p, q \in D$ and $\mu \in (0,1)$, we have

$$h(\delta_{E(p),E(q)}(\mu)) \leq \max\{h(p), h(q)\}.$$

Thus,

$$\begin{aligned}
 \psi[h(\delta_{E(p),E(q)}(\mu))] &\leq \psi[\max\{h(p), h(q)\}], \\
 \psi \circ h(\delta_{E(p),E(q)}(\mu)) &\leq \max\{\psi \circ h(p), \psi \circ h(q)\},
 \end{aligned}$$

From this, it follows that the composite function $\psi \circ h$ is GQSEBPI function on D . □

Next, we examine the issue:

$$\text{Min } h(p), \text{ s. t. } p \in D = \{p \in Q: h_i(p) \leq 0, i = 1, 2, \dots, m\},$$

where $h: Q \rightarrow \mathbb{R}$ & $h_i: Q \rightarrow \mathbb{R}, i = 1, 2, \dots, m$, are real valued functions defined on Q & D is a GEBI set.

$$(P_1)$$

We also consider the problem:

$$\begin{aligned}
 &\text{Minimum } h(E(p)), \\
 &\text{s. t. } p \in D
 \end{aligned}$$

Theorem 8. Suppose D be a GEBI set w.r.t. η & $h(E(p)) \leq h(p)$ for each $p \in D$. If \bar{p} is a solution of the problem (P_2) , then $E(\bar{p})$ is a solution of the problem (P_1) .

Proof. Suppose $E(\bar{p})$ is not a solution of the problem (P_1) . Then, $\exists q \in D$ s.t. $h(q) < h(E(\bar{p})) \Rightarrow h(E(q)) \leq h(q) < h(E(\bar{p}))$, which contradicts the assumption that \bar{p} is the optimal solution to problem (P_2) . Hence, $E(\bar{p})$ must be a solution to problem (P_1) . □

Theorem 9. Let $h: Q \rightarrow \mathbb{R}$ be a GSEBPI function defined on a GEBI set $D \subseteq Q$, & suppose that \bar{p} is a solution of the problem (P_2) . Then, $E(\bar{p})$ is a solution to problem (P_1) .

Proof. The proof follows a similar approach to that of Theorem 8.

□

Theorem 10. If $p^\circ = E(z^\circ) \in E(A)$ is the minimum of problem (P_1) on a GEBI set D , & if $h: Q \rightarrow \mathbb{R}$ is a GEBPI function on D , with the condition that $h(E(p)) \leq h(p) \forall p \in D$, then p° is the global minimum of the problem (P_1) on D .

Proof. Suppose $p^\circ = E(z^\circ) \in E(A)$ is a nonglobal minimum of the problem (P_1) on D . Then, $\exists q \in D$ s.t. $h(q) < h(p^\circ) = h(E(z^\circ))$. Since the function $h: Q \rightarrow \mathbb{R}$ is a GEBPI function & $h(E(p)) \leq h(p)$ for each $p \in D$, this \implies

$$\begin{aligned} h(\delta_{E(z^\circ), E(q)}(\mu)) &\leq \mu b(p, q, \mu)h(E(z^\circ)) + (1 - \mu b(p, q, \mu))h(E(q)) \\ &\leq \mu b(p, q, \mu)h(p^\circ) + (1 - \mu b(p, q, \mu))h(q) \\ &< h(p^\circ), \end{aligned}$$

for any small $\mu \in (0,1)$, which leads to a contradiction with the assumption that p° is the optimal solution for problem (P_1) . Therefore, p° must be the global minimum of problem (P_1) on D .

□

Theorem 11. Let $h: Q \rightarrow \mathbb{R}$ be strictly GSEBPI function defined on a GEBI set $D \subseteq Q$. Then, the global optimal solutions of problem (P_1) is unique.

Proof. Let $p, q \in D$ be two distinct global optimum solutions of problem (P_1) . Then, we have $h(p) = h(q)$. Since h is a strictly GSEBPI function on D , it follows that

$$h(\delta_{E(p), E(q)}(\mu)) < \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) = h(p),$$

for each $\mu \in (0,1)$. This contradicts the optimality of p in problem (P_1) . Therefore, the global optimal solution to problem (P_1) is unique.

□

Theorem 12. Let $h: Q \rightarrow \mathbb{R}$ be a GQSEBPI function defined on a GEBI set $D \subseteq Q$, & let $\alpha = \min_{p \in D} h(p)$. Then, the set $P = \{p \in D: h(p) = \alpha\}$ consists of the optimal solutions to problem (P_1) & is GEBI. If h is strictly GQSEBPI on D , then the set P is a singleton.

Proof. Let $p, q \in D$ be two distinct global solutions of problem (P_1) , & let $\mu b(p, q, \mu) \in [0,1]$. Then, we have $h(p) = \alpha$ & $h(q) = \alpha$. Since $h: Q \rightarrow \mathbb{R}$ is a GQSEBPI on D , it follows that

$$h(\delta_{E(p), E(q)}(\mu)) \leq \max\{h(p), h(q)\} = \alpha,$$

which $\implies \delta_{E(p), E(q)}(\mu) \in P$. Thus, it follows that P is GEBI.

Now we need to demonstrate that P is singleton. To the contrary, suppose that $p, q \in P$ with $p \neq q$. For $\mu \in (0,1)$, we have $\delta_{E(p),E(q)}(\mu) \in D$. Furthermore, since h is strictly GQSEBPI on D , we obtain

$$h(\delta_{E(p),E(q)}(\mu)) < \max\{h(p), h(q)\} = \alpha.$$

This contradicts the fact that $\alpha = \min_{x \in S} h(x)$. Therefore, P must be a singleton.

□

Theorem 13. If $h: Q \rightarrow \mathbb{R}$ is a GSEBPI function defined on a GEBI set $D \subseteq Q$, then the set of optimal solutions to problem (P_1) is GEBI.

Proof. Let p^* be the optimal solution to problem (P_1) , and let $\alpha = h(p^*)$. Let P be the set of optimal solutions for problem (P_1) defined as follows:

$$P = \{p \in D: h(p) \leq \alpha\},$$

for any $p, q \in P$ with $p \neq q$, & $\mu \in (0,1)$. Since $h: Q \rightarrow \mathbb{R}$ is a GSEBPI on the GEBI set $D \subseteq Q$, we have

$$h(\delta_{E(p),E(q)}(\mu)) \leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) \leq \alpha.$$

Thus, $\delta_{E(p),E(q)}(\mu) \in P$, it follows that P is GEBI.

□

Theorem 14. If $h: Q \rightarrow \mathbb{R}$ & $h_i: Q \rightarrow \mathbb{R}, i = 1, 2, \dots, m$ are GQSEBPI functions on Q , then the set of optimal solutions to problem (P_1) is GEBI.

Proof. From Theorem 4, it follows that $D \subseteq Q$ is a GEBI set. According to Theorem 12, the set $P = \{p \in D: h(p) = \alpha\}$ of optimal solutions to problem (P_1) is GEBI.

□

Theorem 15. Let $h: Q \rightarrow \mathbb{R}$ be a GSEBPI function defined on a GEBI set D . If $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is a positively homogeneous and non-decreasing function, then the composite function $\psi \circ h$ is also GSEBPI on D .

Proof. Since h is a GSEBPI function on the GEBI set D , we have, $\forall p, q \in D$ and $\mu \in [0,1]$,

$$\begin{aligned} h(\delta_{E(p),E(q)}(\mu)) &\leq \mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q) \\ \psi \circ h(\delta_{E(p),E(q)}(\mu)) &\leq \psi \circ [\mu b(p, q, \mu)h(p) + (1 - \mu b(p, q, \mu))h(q)] \\ &\leq \mu b(p, q, \mu)\psi \circ h(p) + (1 - \mu b(p, q, \mu))\psi \circ h(q), \end{aligned}$$

from which it follows that $\psi \circ h$ is GSEBPI on D .

□

Theorem 16. Let $h: Q \rightarrow \mathbb{R}$ be differentiable GEBPI on the GEBI set $D \subseteq Q$, & let $p \in D$ be a fixed point of the map E , meaning $E(p) = p$. Then, p is the minimum of h on $D \Leftrightarrow \langle h'(E(p)), \eta(E(p), E(q)) \rangle \geq 0, \forall q \in D$, where h' is the differential of h at $E(p)$.

Proof. Let $p \in D$ be a minimum of h on D . Then

$$h(E(p)) \leq h(E(q)), \forall q \in D.$$

Since D is GEBI set, we have

$$\delta_{E(q), E(p)}(0) = E(p), \delta_{E(q), E(p)}(\mu) \in D, \delta'_{E(q), E(p)}(0) = \eta(E(p), E(q)), \forall p, q \in D.$$

Since h is differentiable on D , we have

$$\begin{aligned} h(E(p)) &\leq h(\delta_{E(q), E(p)}(\mu)) \\ &= h(E(p)) + \mu b(x, y, \mu)(h(E(q)) - h(E(p))), \end{aligned}$$

This implies that

$$\begin{aligned} 0 &\leq \mu b(x, y, \mu)(h(E(q)) - h(E(p))) \\ &= b(x, y, \mu)(h(E(q)) - h(E(p))). \end{aligned}$$

Taking $\mu \rightarrow 0$, and by example 10, in we obtain

$$\langle h'(E(p)), \eta(E(p), E(q)) \rangle \geq 0, \forall q \in D,$$

This concludes the required result. The converse can be established with relative ease. □

Theorem 17. Let $D \subseteq Q$ be a GEBI set w.r.t. map η . If $h: D \rightarrow \mathbb{R}$ be a differentiable and GEBPI on D , then h is a GEBI function.

Proof. Since D is GEBI set, we have

$$\gamma_{E(p), E(q)}(0) = E(q), \gamma'_{E(p), E(q)}(0) = \eta(E(p), E(q)), \gamma_{E(p), E(q)}(\mu) \in S.$$

Next, by definition of GEBPI on D , we have

$$\begin{aligned} h(\gamma_{E(p), E(q)}(\mu)) &\leq \mu b h(E(p)) + (1 - \mu b) h(E(q)). \\ h(\gamma_{E(p), E(q)}(\mu)) - h(E(q)) &\leq \mu b [h(E(p)) - h(E(q))]. \\ \frac{h(\gamma(\mu)) - h(\gamma(0))}{\mu} &\leq b(h(E(p)) - h(E(q))) \end{aligned}$$

Taking limit as $\mu \rightarrow 0$, we get

$$h'(\gamma(0)) \leq b[h(E(p)) - h(E(q))].$$

Equivalently,

$$dh_{\gamma(0)}(\gamma'(0)) \leq b[h(E(p)) - h(E(q))].$$

That is,

$$dh_{E(q)}(\eta(E(p), E(q))) \leq b[h(E(p)) - h(E(q))].$$

Hence, By definition [6], h is GEPI function.

4. Conclusion

This study presents a new class of functions called GSEBPI functions on RMs, marking a significant advancement in the field of optimization on RMs. In section 3, our findings reveal crucial relationships among different types of GEBPI functions, including GSEBPI, GQSEBPI and GPSEBPI functions. This work builds foundational research by Porwal [22], providing new insights and paving the way for future exploration to this area.

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