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Robust Approximation and Stability of Highly Nonlinear Composite Additive–Quadratic Functional Equations in Various Banach Spaces

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Abstract:
In this groundbreaking paper, we pioneer the investigation of the generalized Ulam-Hyers-Rassias stability of a composite additive-quadratic functional equation in various Banach spaces, including Banach Space, Generalized 2 Banach space, and Random Banach Space. Leveraging the powerful classical Hyers and Radu's fixed point methods, we rigorously establish the stability results, thereby significantly advancing the frontiers of functional equation theory.

Keywords: Hybrid Functional Equation with Composite terms, generalized Ulam-Hyers-Rassias stability, Banach Space, Generalized 2 Banach space, Random Banach Space, Radu's Fixed Point Technique.

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1. INTRODUCTION

Functional equations constitute a cornerstone of mathematics, delving into the intricate study of equations wherein mappings assume the role of unknowns. These equations have far-reaching implications and applications across a broad spectrum of disciplines, including physics, engineering, economics, and computer science. In recent decades, the stability of functional equations, particularly the Hyers-Ulam stability, has emerged as a paramount research frontier, captivating the attention of mathematicians and scientists worldwide.

In 1941, the visionary mathematician Stanislaw Ulam [34] posed a profound question regarding the stability of group homomorphisms, sparking a groundbreaking inquiry. Donald Hyers [18] promptly solved this problem in the same year, laying the foundation for the Hyers-Ulam stability theory. This revolutionary concept reveals the remarkable phenomenon where a mapping that approximately aligns with a functional equation can be rigorously approximated by an exact solution of the same equation, thereby establishing a profound connection between approximate and exact solutions. In 1950 Takashi Aoki [2] generalized Hyers' result to approximately linear mappings. His work demonstrated the versatility of Hyers-Ulam stability and paved the way for further research in this area.

In 1978 Themistocles Rassias [28] generalization of Hyers-Ulam stability to Banach spaces marked a significant milestone in the development of this field. His introduction of the concept of ε-stability has had a lasting impact on the study of functional equations and stability theory. Later John Michael Rassias [25] seminal extension of Hyers-Ulam stability to normed spaces in 1982 has had a lasting influence on the field of functional equations. His introduction of super stability has revealed new insights into the stability of functional equations, opening up fresh avenues for research and exploration. In a groundbreaking achievement, Pasc Gavruta [15] revolutionized the stability landscape in 1994 by devising a pioneering generalization that subsumed all preceding stability results. Gavruta's ingenious approach entailed substituting a versatile control mapping for the unbounded Cauchy difference, astutely incorporating the summation of both the sum and the product of two $p-$ norms.

Building upon Gavruta's seminal work, Krishnan Ravi et al. [30] made a significant breakthrough in 2008, uncovering a specialized variant of Gavruta's theorem tailored to the unbounded Cauchy difference.

A plethora of renowned researchers have conducted exhaustive investigations into the stability of various functional equations and additive-quadratic mixed-type functional equations across diverse normed spaces, yielding a treasure trove of fascinating discoveries (see [4-13, 19-21, 24, 26, 29, 31]). We now highlight the influential research of Margolis, Diaz [22], and Radu [23], which has profoundly impacted the development of fixed point theory.

Theorem 1.1 [21,22] (The alternative of fixed point) Suppose that for a complete generalized metric space (W,d) and a strictly contractive mapping $T:W\to W$ with Lipschitz constant L. Then for each given $x\in W$, either $d\left(T^n x,T^{n+1}x\right)=\infty$ for all $n\geq 0$, or there exists a natural number n_0 such that

(FPC1) $d\left(T^n x,T^{n+1}x\right)<\infty$ for all $n\geq n_0$;

(FPC2) The sequence $\left(T^n x\right)$ is approaches to a fixed point y^* of T;

(FPC3) y^* is the unique fixed point of T in the set $U=\left\{y\in W : d\left(T^{n_0}x,y\right)<\infty\right\}$;

(FPC4) $d\left(y,y^*\right)\leq \frac{1}{1-L}d\left(y,Ty\right)$ for all $y\in U$.

In this pioneering paper, we make a groundbreaking introduction and rigorously establish the generalized Ulam-Hyers-Rassias stability of a composite additive-quadratic functional equation

$$\begin{aligned}
&\Omega\left(2\Omega(\omega_1)-\Omega(\omega_2)-\Omega(\omega_3)\right)+2\Omega(\omega_1)+\Omega(\omega_2)+\Omega(\omega_3)+\left[\Omega(\omega_1)+\Omega(-\omega_1)\right]\\
&\qquad+\frac{3}{2}\left[\Omega(\omega_2)+\Omega(-\omega_2)\right]+\frac{3}{2}\left[\Omega(\omega_3)+\Omega(-\omega_3)\right]+2\left[\Omega(\omega_1)-\Omega(-\omega_1)\right]\\
&=\Omega(\omega_1+\omega_2)+\Omega(\omega_1+\omega_3)+3\Omega(\omega_1-\omega_2)+3\Omega(\omega_1-\omega_3)+\Omega(\omega_2+\omega_3)\\
&\qquad+\frac{1}{2}\left[\Omega(\omega_2)-\Omega(-\omega_2)\right]+\frac{1}{2}\left[\Omega(\omega_3)-\Omega(-\omega_3)\right]
\end{aligned}\tag{1.1}$$

in various Banach spaces with the help of classical Hyers and Radus fixed point techniques.

Employing a rigorous analytical technique, we delve into the stability analysis of functional equations within the framework of complete normed spaces. Specifically, we utilize a triplet of specialized techniques tailored to three distinct spaces: Banach Space, Generalized 2 Banach space, and Random Banach Space. By leveraging the unique properties and definitions of each space, we conduct a comprehensive stability analysis.

In Section 2, we undertake a stability analysis of (1.1) within a Banach space framework, leveraging the traditional Hyers and Radu fixed point methodologies.

In Section 3, we provide an exhaustive stability examination of (1.1) in Generalized 2-Banach spaces, leveraging the classical Hyers and Radu fixed point techniques

In Section 4, we present a meticulous stability investigation of (1.1) in Random Banach spaces, facilitated by the esteemed Hyers and Radu fixed point methods.

For proving stability result, let us take

$$\begin{aligned}
\Omega_A^Q(\omega_1,\omega_2,\omega_3)&=\Omega\left(2\Omega(\omega_1)-\Omega(\omega_2)-\Omega(\omega_3)\right)+2\Omega(\omega_1)+\Omega(\omega_2)+\Omega(\omega_3)+\left[\Omega(\omega_1)+\Omega(-\omega_1)\right]\\
&\quad+\frac{3}{2}\left[\Omega(\omega_2)+\Omega(-\omega_2)\right]+\frac{3}{2}\left[\Omega(\omega_3)+\Omega(-\omega_3)\right]+2\left[\Omega(\omega_1)-\Omega(-\omega_1)\right]\\
&\quad-\Omega(\omega_1+\omega_2)-\Omega(\omega_1+\omega_3)-3\Omega(\omega_1-\omega_2)-3\Omega(\omega_1-\omega_3)+\Omega(\omega_2+\omega_3)\\
&\quad-\frac{1}{2}\left[\Omega(\omega_2)-\Omega(-\omega_2)\right]-\frac{1}{2}\left[\Omega(\omega_3)-\Omega(-\omega_3)\right].
\end{aligned}$$

2. STABILITY ANALYSIS IN BANACH SPACE

In this section, we present the stability analysis of (1.1) in Banach space with the help of classical Hyers and Radus fixed point methods.

As a precursor to proving the stability theorems, let us posit that $Q_1$is a normed space and Q_2 is a Banach space, providing the necessary framework for our investigation.

2.1 Additive Case: Stability Analysis: Direct Method

Theorem 2.1 Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\left\|\Omega_A^Q\left(\omega_1,\omega_2,\omega_3\right)\ \right\|\leq\Phi\left(\omega_1,\omega_2,\omega_3\ \right) \tag{2.1}$$

where $\Phi:Q_1^3\longrightarrow[0,\infty)$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty}\frac{1}{2^{ES}}\Phi\left(2^{ES}\omega_1,2^{ES}\omega_2,2^{ES}\omega_3\ \right)=0;\ E=\pm1 \tag{2.2}$$

for all $\omega\in Q_1$. It follows that there is a one and only one additive mapping $\Lambda_A\left(\omega\right):Q_1\longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega\left(\omega\right)-\Lambda_A\left(\omega\right)\ \right\|\leq\frac{1}{3\times2}\sum_{D=\frac{1-E}{2}}^{\infty}\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{2^{DE}} \tag{2.3}$$

where the mapping $\Lambda_A\left(\omega\right)$ is characterized by

$$\Lambda_A\left(\omega\right)=\lim_{S\longrightarrow\infty}\frac{\Omega\left(2^{ES}\omega\right)}{2^{ES}} \tag{2.4}$$

for all $\omega\in Q_1$.

Proof. Utilizing oddness of Ω in (2.1) and then changing $\omega_1=\omega_2=\omega_3=\omega$, we achieve

$$\left\|6\Omega\left(\omega\right)-\ 3\Omega\left(2\omega\right)\right\|\leq\Phi\left(\omega,\omega,\omega\ \right) \tag{2.5}$$

for all $\omega\in Q_1$. As a consequence of (2.5) that

$$\left\|2\Omega\left(\omega\right)-\ \Omega\left(2\omega\right)\right\|\leq\frac{1}{3}\Phi\left(\omega,\omega,\omega\ \right) \tag{2.6}$$

and then

$$\left\|\Omega(\omega)-\tfrac{1}{2}\Omega(2\omega)\right\|\leq\tfrac{1}{3\times2}\Phi(\omega,\omega,\omega) \tag{2.7}$$

for all $\omega\in Q_1$. For any positive integer S, the above disparity can be generalized as

$$\left\|\Omega\left(\omega\right)-\frac{1}{2^S}\Omega\left(2^S\omega\right)\right\|\leq\frac{1}{3\times2}\sum_{D=0}^{S-1}\frac{\Phi\left(2^D\omega,2^D\omega,2^D\omega\right)}{2^D} \tag{2.8}$$

for all $\omega\in Q_1$. Therefore, the sequence $\left\{\frac{1}{2^S}\Omega\left(2^S\omega\right)\right\}$ is a Cauchy sequence. Indeed, changing $\omega=2^{S_1}\omega$ in (2.8), we attain

$$\left\|\frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right)-\frac{1}{2^{S+S_1}}\Omega\left(2^{S+S_1}\omega\right)\right\|=\frac{1}{2^{S_1}}\left\|\Omega\left(\omega\right)-\frac{1}{2^S}\Omega\left(2^S\,2^{S_1}\omega\right)\right\|\leq\frac{1}{3\times2}\sum_{D=0}^{S-1}\frac{\Phi\left(2^{D+S_1}\omega,2^{D+S_1}\omega,2^{D+S_1}\omega\right)}{2^{D+S_1}} \tag{2.9}$$

for all $\omega\in Q_1$. Letting $S_1\to\infty$ in (2.9), we deduce that the sequence is Cauchy, and its limit is $\Lambda_A\left(y\right)\in Q_2$. So, we define

$$\Lambda_A\left(\omega\right)=\lim_{S\longrightarrow\infty}\frac{\Omega\left(2^S\omega\right)}{2^S} \tag{2.10}$$

for all $\omega\in Q_1$. Again letting $S\to\infty$ in (2.8) and using (2.10), we see (2.3) holds for $E=1$.

In order to show that the existence of $\Lambda_A\left(y\right)$ obey the functional equation (1.20), interchanging $\omega_1=2^S\omega_1$; $\omega_2=2^S\omega_2$; $\omega_3=2^S\omega_3$ in (2.1), we arrive

$$\left\|\frac{1}{2^S}\Omega_A^Q\left(2^S\omega_1,2^S\omega_2,2^S\omega_3\right)\right\|\leq\frac{1}{2^S}\Phi\left(2^S\omega_1,2^S\omega_2,2^S\omega_3\right) \tag{2.11}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$. Letting $S\to\infty$ in (2.11) and using (2.10), we obtain that $\Lambda_A\left(y\right)$ fulfills the functional equation (1.20).

Finally, we aim to prove that the existence of $\Lambda_A\left(y\right)$ is unique, let's assume $\Lambda_B\left(y\right)$ fulfills (1.1) and (2.10) for all $\omega\in Q_1$. So,

$$\begin{aligned}\left\|\Lambda_A(\omega)-\Lambda_B(\omega)\right\| &= \frac{1}{2^{S_1}}\left\|\Lambda_A\left(2^{S_1}\omega\right)-\Lambda_B\left(2^{S_1}\omega\right)\right\| \\ &\le \frac{1}{2^{S_1}}\left\{\left\|\Lambda_A\left(2^{S_1}\omega\right)-\Omega\left(2^{S_1}\omega\right)\right\|+\left\|\Omega\left(2^{S_1}\omega\right)-\Lambda_B\left(2^{S_1}\omega\right)\right\|\right\} \\ &\le \frac{2}{3\times 2}\sum_{D=0}^{\infty}\frac{\Phi\left(2^{D+S_1}\omega,2^{D+S_1}\omega,2^{D+S_1}\omega\right)}{2^{D+S_1}} \to 0 \;\; as \; S_1\to\infty\end{aligned}$$

for all $\omega\in Q_1$. Therefore $\Lambda_A(\omega)=\Lambda_B(\omega)$, this confirms the uniqueness of the solution. Thus, the theorem holds for $E=1$.

Again, changing $\omega=\frac{\omega}{2}$ in (2.6), we reach

$$\left\|2\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)\right\|\le\frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right) \tag{2.12}$$

for all $\omega\in Q_1$. For any positive integer S, the above disparity can be generalized as

$$\left\|2^S\Omega\left(\frac{\omega}{2^S}\right)-\Omega(\omega)\right\|\le\frac{1}{3\times 2}\sum_{D=1}^{S}2^D\Phi\left(\frac{\omega}{2^D},\frac{\omega}{2^D},\frac{\omega}{2^D}\right) \tag{2.13}$$

for all $\omega\in Q_1$. The remainder of the proof follows a similar pattern to that of $E=1$ case. Hence the proof is finalized.

Corollary 2.2 Assume Δ be a positive constant and I be a real number. Suppose an odd mapping $\Omega_A^Q: Q_1\longrightarrow Q_2$ aligns with the disparity

$$\left\|\Omega_A^Q(\omega_1,\omega_2,\omega_3)\right\|\le\begin{cases}\Delta;\\ \Delta\sum_{i=1}^{3}|\omega_i|^I;\\ \Delta\sum_{i=1}^{3}|\omega_i|^{I_i};\\ \Delta\prod_{i=1}^{3}|\omega_i|^I;\\ \Delta\prod_{i=1}^{3}|\omega_i|^{I_i};\\ \Delta\left\{\prod_{i=1}^{3}|\omega_i|^I+\sum_{i=1}^{3}|\omega_i|^{3I}\right\};\end{cases} \tag{2.14}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$. It follows that there is a one and only one quadratic mapping $\Lambda_A(\omega): Q_1\longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega(\omega)-\Lambda_A(\omega)\right\|\le\begin{cases}\frac{\Delta}{|3|};\\ \frac{\Delta|\omega|^I}{|2-2^I|};I\ne 1,\\ \sum_{i=1}^{3}\frac{\Delta|\omega_i|^{I_i}}{3|2-2^{I_i}|};I_1,I_2,I_3\ne 1,\\ \frac{\Delta|\omega|^{3I}}{3|2-2^{3I}|};3I\ne 1,\\ \frac{\Delta|\omega|^{I_1+I_2+I_3}}{3|2-2^{I_1+I_2+I_3}|};I_1+I_2+I_3\ne 1,\\ \frac{2\Delta|\omega|^{3I}}{3|2-2^{3I}|};3I\ne 1,\end{cases} \tag{2.15}$$

for all $\omega\in Q_1$.

2.2 Quadratic Case: Stability Analysis: Direct Method

Theorem 2.3 Consider an even mapping $\Omega_A^Q: Q_1\longrightarrow Q_2$ aligns with the disparity

$$\left\|\Omega_A^Q(\omega_1,\omega_2,\omega_3)\right\|\le\Phi(\omega_1,\omega_2,\omega_3) \tag{2.16}$$

where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping subject to the condition

$$\lim_{S\longrightarrow\infty} \frac{1}{4^{ES}} \Phi\left(2^{ES}\omega_1, 2^{ES}\omega_2, 2^{ES}\omega_3\right) = 0; \ E = \pm 1 \tag{2.17}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$. It follows that there is a unique quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) together with the disparity

$$\left\|\Omega(\omega) - \Lambda_Q(\omega)\right\| \le \frac{1}{3\times 4} \sum_{D=\frac{1-E}{2}}^{\infty} \frac{\Phi\left(2^{DE}\omega, 2^{DE}\omega, 2^{DE}\omega\right)}{4^{DE}} \tag{2.18}$$

where the mapping $\Lambda_Q(\omega)$ is given by

$$\Lambda_Q(\omega) = \lim_{S\longrightarrow\infty} \frac{\Omega\left(2^{ES}\omega\right)}{4^{ES}} \tag{2.19}$$

for every $\omega \in Q_1$.

Proof. Utilizing evenness of Ω in (2.16) and then changing $\omega_1 = \omega_2 = \omega_3 = \omega$, we achieve

$$\left\|12\Omega(\omega) - 3\Omega(2\omega)\right\| \le \Phi(\omega,\omega,\omega) \tag{2.20}$$

for all $\omega \in Q_1$. As a consequence of (2.20) that

$$\left\|4\Omega(\omega) - \Omega(2\omega)\right\| \le \frac{1}{3}\Phi(\omega,\omega,\omega) \tag{2.21}$$

and then

$$\left\|\Omega(\omega) - \frac{1}{4}\Omega(2\omega)\right\| \le \frac{1}{3\times 4}\Phi(\omega,\omega,\omega) \tag{2.22}$$

for all $\omega \in Q_1$. For any positive integer S, the above disparity can be generalized as

$$\left\|\Omega(\omega) - \frac{1}{4^S}\Omega\left(2^S\omega\right)\right\| \le \frac{1}{3\times 4} \sum_{D=0}^{S-1} \frac{\Phi\left(2^D\omega, 2^D\omega, 2^D\omega\right)}{4^D} \tag{2.23}$$

for all $\omega \in Q_1$. Therefore, the sequence $\left\{\frac{1}{4^S}\Omega\left(2^S\omega\right)\right\}$ is a Cauchy sequence. The proof concludes in a manner analogous to that of $E = 1$ case of Theorem 2.1. Again, changing $\omega = \frac{\omega}{2}$ in (2.21), we reach

$$\left\|4\Omega\left(\frac{\omega}{2}\right) - \Omega(\omega)\right\| \le \frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right) \tag{2.24}$$

for all $\omega \in Q_1$. For any positive integer S, the above disparity can be generalized as

$$\left\|4^S\Omega\left(\frac{\omega}{2^S}\right) - \Omega(\omega)\right\| \le \frac{1}{3\times 4} \sum_{D=1}^{S} 4^D \Phi\left(\frac{\omega}{2^D},\frac{\omega}{2^D},\frac{\omega}{2^D}\right) \tag{2.25}$$

for all $\omega \in Q_1$. The rest of the proof follows the same pattern as $E = 1$ case. Therefore, the proof is now finalized.

Corollary 2.4 Assume Δ be a positive constant and I be a real number. Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\left\|\Omega_A^Q(\omega_1,\omega_2,\omega_3)\right\| \le \begin{cases} \Delta; \\ \Delta \sum_{i=1}^{3} |\omega_i|^I; \\ \Delta \sum_{i=1}^{3} |\omega_i|^{I_i}; \\ \Delta \prod_{i=1}^{3} |\omega_i|^I; \\ \Delta \prod_{i=1}^{3} |\omega_i|^{I_i}; \\ \Delta\left\{\prod_{i=1}^{3} |\omega_i|^I + \sum_{i=1}^{3} |\omega_i|^{3I}\right\}; \end{cases} \tag{2.26}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$. It follows that there is a one and only one quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega(\omega)-\Lambda_Q(\omega)\right\| \le \begin{cases} \dfrac{\Delta}{3|3|}; \\ \dfrac{\Delta\,|\,\omega\,|^{I}}{\left|4-2^{I}\right|}; I \neq 2, \\ \displaystyle\sum_{i=1}^{3}\dfrac{\Delta\,|\,\omega_i\,|^{I_i}}{3\left|4-2^{I_i}\right|}; I_1, I_2, I_3 \neq 2, \\ \dfrac{\Delta\,|\,\omega\,|^{3I}}{3\left|4-2^{3I}\right|}; 3I \neq 2, \\ \dfrac{\Delta\,|\,\omega\,|^{I_1+I_2+I_3}}{3\left|4-2^{I_1+I_2+I_3}\right|}; I_1+I_2+I_3 \neq 2, \\ \dfrac{2\Delta\,|\,\omega\,|^{3I}}{3\left|4-2^{3I}\right|}; 3I \neq 2, \end{cases} \tag{2.27}$$

for each $\omega \in Q_1$.

2.3 Additive - Quadratic Case: Stability Analysis: Direct Method

Theorem 2.5 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ fulfills the disparity

$$\left\|\Omega_A^Q(\omega_1,\omega_2,\omega_3)\right\| \le \Phi(\omega_1,\omega_2,\omega_3) \tag{2.28}$$

where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the conditions (2.2) and (2.17) for all $\omega_1,\omega_2,\omega_3 \in Q_1$. Then there exists unique additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ and a unique quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\begin{aligned}\left\|\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)\right\| \le &\frac{1}{3\times 4}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{2^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{2^{DE}}\right\} \\ &+\frac{1}{3\times 8}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{4^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{4^{DE}}\right\}\end{aligned} \tag{2.29}$$

in which the correspondence $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are explicitly stated in (2.4) and (2.19) for every $\omega \in Q_1$.

Proof. Assume a mapping

$$\Omega_{Add}(\omega) = \frac{1}{2}\left\{\Omega(\omega)-\Omega(-\omega)\right\} \text{ for all } \omega \in Q_1. \tag{2.30}$$

It is easy to prove that $\Omega_{Add}(0)=0;\ \Omega_{Add}(-\omega) = -\Omega_{Add}(\omega)$ for all $\omega \in Q_1$. By Theorem 2.1, there exists unique additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the disparity

$$\left\|\Omega_{Add}(\omega)-\Lambda_A(\omega)\right\| \le \frac{1}{3\times 2}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{2^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{2^{DE}}\right\} \tag{2.31}$$

for all $\omega \in Q_1$. Also, assume a mapping

$$\Omega_{QUD}(\omega) = \tfrac{1}{2}\{\Omega(\omega)+\Omega(-\omega)\} \text{ for all } \omega \in Q_1. \tag{2.32}$$

It is easy to prove that $\Omega_{QUD}(0)=0;\ \Omega_{QUD}(-\omega) = \Omega_{QUD}(\omega)$ for all $\omega \in Q_1$. By Theorem 2.3, there exists a one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which fulfills the disparity

$$\left\|\Omega_{QUD}(\omega)-\Lambda_Q(\omega)\right\| \le \frac{1}{3\times 2}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{2^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{2^{DE}}\right\} \tag{2.33}$$

for all $\omega \in Q_1$. Now, define a mapping

$$\Omega(\omega) = \Omega_{Add}(\omega)+\Omega_{QUD}(\omega) \tag{2.34}$$

for all $\omega \in Q_1$. As a consequence of (2.35), (2.34), (2.31), we arrive

$$\begin{aligned} \left\|\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)\right\| &= \left\|\Omega_{ADD}(\omega)+\Omega_{QUD}(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)\right\| \\ &\le \left\|\Omega_{ADD}(\omega)-\Lambda_A(\omega)\right\| + \left\|\Omega_{QUD}(\omega)-\Lambda_Q(\omega)\right\| \\ &\le \frac{1}{3\times 4}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{2^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{2^{DE}}\right\} \\ &\quad + \frac{1}{3\times 8}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(2^{DE}\omega,2^{DE}\omega,2^{DE}\omega\right)}{4^{DE}}+\frac{\Phi\left(-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega\right)}{4^{DE}}\right\} \end{aligned} \tag{2.35}$$

for all $\omega \in Q_1$. Thus, the proof is finalized.

Corollary 2.6 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ fulfills the disparity

$$\left\|\Omega_A^Q(\omega_1,\omega_2,\omega_3)\right\| \le \begin{cases} \Delta; \\ \Delta\sum_{i=1}^{3}|\omega_i|^{I}; \\ \Delta\sum_{i=1}^{3}|\omega_i|^{I_i}; \\ \Delta\prod_{i=1}^{3}|\omega_i|^{I}; \\ \Delta\prod_{i=1}^{3}|\omega_i|^{I_i}; \\ \Delta\left\{\prod_{i=1}^{3}|\omega_i|^{I}+\sum_{i=1}^{3}|\omega_i|^{3I}\right\}; \end{cases} \tag{2.36}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1 \longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which obey the functional equation (1.1) along with the disparity

$$\left\|\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)\right\| \le \begin{cases} \left\{\frac{\Delta}{|3|}+\frac{\Delta}{3|3|}\right\}; \\ \left\{\frac{\Delta|\omega|^{I}}{|2-2^{I}|}+\frac{\Delta|\omega|^{I}}{|4-2^{I}|}\right\}; I\neq 1,2, \\ \left\{\sum_{i=1}^{3}\frac{\Delta|\omega_i|^{I_i}}{3|2-2^{I_i}|}+\sum_{i=1}^{3}\frac{\Delta|\omega_i|^{I_i}}{3|4-2^{I_i}|}\right\}; I_1,I_2,I_3\neq 1,2, \\ \left\{\frac{\Delta|\omega|^{3I}}{3|2-2^{3I}|}+\frac{\Delta|\omega|^{3I}}{3|4-2^{3I}|}\right\}; 3I\neq 1,2, \\ \left\{\frac{\Delta|\omega|^{I_1+I_2+I_3}}{3|2-2^{I_1+I_2+I_3}|}+\frac{\Delta|\omega|^{I_1+I_2+I_3}}{3|4-2^{I_1+I_2+I_3}|}\right\}; I_1+I_2+I_3\neq 1,2, \\ \left\{\frac{2\Delta|\omega|^{3I}}{3|2-2^{3I}|}+\frac{2\Delta|\omega|^{3I}}{3|4-2^{3I}|}\right\}; 3I\neq 1,2, \end{cases} \tag{2.37}$$

for all $\omega \in Q_1$.

2.4 Additive Case: Stability Analysis: Radus Method

Theorem 2.7 Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (2.1) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the condition

$$\lim_{S\to\infty}\frac{1}{A_K^S}\Phi\left(A_K^S\omega_1, A_K^S\omega_2, A_K^S\omega_3\right)=0; A=\begin{cases}2 & K=1\\ \frac{1}{2} & K=-1\end{cases} \tag{2.38}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$. In the event that $L=L[K]$ be a mapping have the properties

$$\Phi(\omega,\omega,\omega)=\frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right);$$
$$\frac{1}{A_K}\Phi\left(A_K\omega, A_K\omega, A_K\omega\right)=L\Phi(\omega,\omega,\omega); \tag{2.39}$$

for all $\omega\in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega):Q_1\longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega(\omega)-\Lambda_A(\omega)\right\|\leq\frac{L^{K-1}}{1-L}\Phi(\omega,\omega,\omega) \tag{2.40}$$

where the mapping $\Lambda_A(\omega)$ is formulated by

$$\Lambda_A(\omega)=\lim_{S\longrightarrow\infty}\frac{\Omega\left(A_K^S\omega\right)}{A_K^S} \tag{2.41}$$

for all $\omega\in Q_1$.

Proof. Assume a set

$$\mathrm{M}=\left\{\Omega/\Omega:Q_1\longrightarrow Q_2,\Omega(0)=0\right\} \tag{2.42}$$

and introduce the generalized metric on the above set M as

$$d(\Omega,\Omega_1)=\inf\left\{\mathrm{K}\in(0,\infty)\big/\left\|\Omega(\omega)-\Omega_1(\omega)\right\|\leq\mathrm{K}\Phi(\omega,\omega,\omega)\right\}, \tag{2.43}$$

for all $\omega\in Q_1$. It is apparent that (M,d) is complete. Define a mapping $\mathrm{N}:\mathrm{M}\to\mathrm{M}$ by

$$\mathrm{N}\Omega(\omega)=\frac{1}{A_K}\Omega\left(A_K\omega\right);$$

(2.44)

Now $\Omega,\Omega_1\in\mathrm{M}$ and $\omega\in Q_1$, we see

$$\begin{aligned}d(\Omega,\Omega_1)\leq\mathrm{K}\ &\Rightarrow\left\|\Omega(\omega)-\Omega_1(\omega)\right\|\leq\mathrm{K}\Phi(\omega,\omega,\omega)\\&\Rightarrow\left\|\frac{1}{A_K}\Omega\left(A_K\omega\right)-\frac{1}{A_K}\Omega_1\left(A_K\omega\right)\right\|\leq\frac{\mathrm{K}}{A_K}\Phi\left(A_K\omega,A_K\omega,A_K\omega\right)\\&\Rightarrow\left\|\mathrm{N}\Omega(\omega)-\mathrm{N}\Omega_1(\omega)\right\|\leq\mathrm{K\,L}\,\Phi(\omega,\omega,\omega)\\&\Rightarrow d\left(\mathrm{N}\Omega,\mathrm{N}\Omega_1\right)\leq\mathrm{K\,L}\end{aligned}$$

i.e., N is a strictly contractive mapping on M with Lipschitz constant L.

For the case K = 0, as a consequence of (2.7) and by means of (2.39), (2.43), (2.44), we obtain

$$\left\|\Omega(\omega)-\frac{1}{2}\Omega(2\omega)\right\|\leq\frac{1}{3\times2}\Phi(\omega,\omega,\omega)\Rightarrow\left\|\Omega(\omega)-\mathrm{N}\Omega(\omega)\right\|\leq\frac{1}{2}\Phi(\omega,\omega,\omega)\Rightarrow d(\Omega,\mathrm{N}\Omega)\leq\mathrm{L}=\mathrm{L}^{1\text{-}\mathrm{K}} \tag{2.45}$$

For the case K = 1, as a consequence of (2.12) and by means of (2.39), (2.43), (2.44), we get

$$\left\|2\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)\right\|\leq\frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right)\Rightarrow\left\|\mathrm{N}\Omega(\omega)-\Omega(\omega)\right\|\leq\Phi(\omega,\omega,\omega)\Rightarrow d(\mathrm{N}\Omega,\Omega)\leq1=\mathrm{L}^{1\text{-}\mathrm{K}} \tag{2.46}$$

Combining (2.45) and (2.46), we have

$$d(\mathrm{N}\Omega,\Omega)\leq\mathrm{L}^{1\text{-}\mathrm{K}}$$

(2.47)

Therefore (FPC1) of Theorem I.1 holds.

By (FPC2) of Theorem I.1, there exists a fixed point Λ_A of N such that

$$\Lambda_A(\omega)=\lim_{S\longrightarrow\infty}\frac{\Omega\left(A_K^S\omega\right)}{A_K^S}\text{, for all }\omega\in Q_1.$$

To order to prove $\Lambda_A(\omega):Q_1\longrightarrow Q_2$ which aligns with (1.20), a similar proof can be constructed along the lines of

Theorem 2.1.
Again by (FPC3) of Theorem I.1, $\Lambda_A(\omega)$ is the unique fixed point of N in the set

$$\mathrm{M}^* = \{\Lambda_A(\omega) \in \mathrm{N} : d(\Omega(\omega), \Lambda_A(\omega)) < \infty \}.$$

Finally, by (FPC4) of Theorem 1.1, we obtain

$$d(\Omega, \Lambda_A) \le \frac{1}{1-\mathrm{L}} d(\Omega, \Lambda_A) \Rightarrow d(\Omega, \Lambda_A) \le \frac{\mathrm{L}^{1\text{-K}}}{1-\mathrm{L}} \Phi(\omega, \omega, \omega)$$

for all $\omega \in Q_1$. Thus, the proof is finished.

Corollary 2.8 Assume Δ be a positive constant and *I* be a real number. Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (2.14) for all $\omega_1, \omega_2, \omega_3 \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which obey the functional equation (1.1) and the disparity (2.15) for all $\omega \in Q_1$.
Proof. Let us take

$$\Phi(\omega_1, \omega_2, \omega_3) = \begin{cases} \Delta; \\ \Delta \sum_{i=1}^{3} |\omega_i|^I; \\ \Delta \sum_{i=1}^{3} |\omega_i|^{I_i}; \\ \Delta \prod_{i=1}^{3} |\omega_i|^I; \\ \Delta \prod_{i=1}^{3} |\omega_i|^{I_i}; \\ \Delta \left\{ \prod_{i=1}^{3} |\omega_i|^I + \sum_{i=1}^{3} |\omega_i|^{3I} \right\}; \end{cases}$$

for all $\omega_1, \omega_2, \omega_3 \in Q_1$ in Theorem 2.7. Replacing $\omega_1 = A_K^S \omega_1; \omega_2 = A_K^S \omega_2; \omega_3 = A_K^S \omega_3$ and dividing by A_K^S in above equation, we reach

$$\frac{1}{A_K^S} \Phi(A_K^S \omega_1, A_K^S \omega_2, A_K^S \omega_3) \le \begin{cases} \frac{\Delta}{A_K^S}; \\ \frac{\Delta}{A_K^S} \sum_{i=1}^{3} |A_K^S \omega_i|^I; \\ \frac{\Delta}{A_K^S} \sum_{i=1}^{3} |A_K^S \omega_i|^{I_i}; \\ \frac{\Delta}{A_K^S} \prod_{i=1}^{3} |A_K^S \omega_i|^I; \\ \frac{\Delta}{A_K^S} \prod_{i=1}^{3} |A_K^S \omega_i|^{I_i}; \\ \frac{\Delta}{A_K^S} \left\{ \prod_{i=1}^{3} |A_K^S \omega_i|^I + \sum_{i=1}^{3} |A_K^S \omega_i|^{3I} \right\}; \end{cases} = \begin{cases} \to 0 \text{ as } \mathrm{S} \to \infty; \\ \to 0 \text{ as } \mathrm{S} \to \infty; \\ \to 0 \text{ as } \mathrm{S} \to \infty; \\ \to 0 \text{ as } \mathrm{S} \to \infty; \\ \to 0 \text{ as } \mathrm{S} \to \infty; \\ \to 0 \text{ as } \mathrm{S} \to \infty. \end{cases}$$

Therefore (2.38) holds for all $\omega_1, \omega_2, \omega_3 \in Q_1$. Now, as a consequence of (2.39), we have

$$\Phi(\omega,\omega,\omega\ ) = \frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right) = \frac{1}{3}\begin{cases} \Delta; \\ \frac{3\Delta|\omega|^{I}}{|2|^{I}}; \\ \sum_{i=1}^{3}\frac{\Delta|\omega|^{I_i}}{|2|^{I_i}}; \\ \frac{\Delta|\omega|^{3I}}{|2|^{3I}}; \\ \frac{\Delta|\omega|^{I_1+I_2+I_3}}{|2|^{I_1+I_2+I_3}}; \\ \frac{4\Delta|\omega|^{3I}}{|2|^{3I}}; \end{cases}$$

$$\frac{1}{A_K}\Phi\left(A_K\omega, A_K\omega, A_K\omega\right) = \begin{cases} A_K^{-1}\ \Delta; \\ A_K^{I-1}\ 3\Delta|\omega|^{I}; \\ \sum_{i=1}^{3} A_K^{I_i-1}\Delta|\omega|^{I_i}; \\ A_K^{3I-1}\ \Delta|\omega|^{3I}; \\ A_K^{I_1+I_2+I_3-1}\ \Delta|\omega|^{I_1+I_2+I_3}; \\ A_K^{3I-1}\ 4\Delta|\omega|^{3I}; \end{cases} = \begin{cases} \mathrm{L}\Phi(\omega,\omega,\omega); \\ \mathrm{L}\Phi(\omega,\omega,\omega); \\ \mathrm{L}\Phi(\omega,\omega,\omega); \\ \mathrm{L}\Phi(\omega,\omega,\omega); \\ \mathrm{L}\Phi(\omega,\omega,\omega); \\ \mathrm{L}\Phi(\omega,\omega,\omega); \end{cases}$$

for all $\omega \in Q_1$.

For K=0, we have L= $A_0^{-1} = 2^{-1}$. From (2.40), we obtain

$$\left\|\Omega(\omega) - \Lambda_A(\omega)\ \right\| \le \frac{\left(2^{-1}\right)^{1-0}}{1-\left(2^{-1}\right)}\Phi(\omega,\omega,\omega\ ) = \frac{1}{2-1}\cdot\frac{\Delta}{3} = \frac{\Delta}{3}.$$

For K=1, we have L= $A_1^{-1} = \left(\frac{1}{2}\right)^{-1}$. From (2.40), we obtain

$$\left\|\Omega(\omega) - \Lambda_A(\omega)\ \right\| \le \frac{(2)^{1-1}}{1-2}\Phi(\omega,\omega,\omega\ ) = \frac{1}{1-2}\cdot\frac{\Delta}{3} = \frac{\Delta}{-3}.$$

For K=0, we have L= $A_0^{I-1} = 2^{I-1}$. From (2.40), we obtain

$$\left\|\Omega(\omega) - \Lambda_A(\omega)\ \right\| \le \frac{\left(2^{I-1}\right)^{1-0}}{1-\left(2^{I-1}\right)}\Phi(\omega,\omega,\omega\ ) = \frac{2^{I}}{2-2^{I}}\cdot\frac{3\Delta|\omega|^{I}}{|2|^{I}} = \frac{3\Delta|\omega|^{I}}{2-2^{I}}.$$

For K=1, we have L= $A_1^{-1} = \left(\frac{1}{2^{I-1}}\right)^{-1}$. From (2.40), we get

$$\left\|\Omega(\omega) - \Lambda_A(\omega)\ \right\| \le \frac{\left(2^{1-I}\right)^{1-1}}{1-\left(2^{1-I}\right)}\Phi(\omega,\omega,\omega\ ) = \frac{2^{I}}{2^{I}-2}\cdot\frac{3\Delta|\omega|^{I}}{|2|^{I}} = \frac{3\Delta|\omega|^{I}}{2^{I}-2}.$$

For K=0, we have L= $A_0^{3I-1} = 2^{3I-1}$. From (2.40), we arrive

$$\left\|\Omega(\omega) - \Lambda_A(\omega)\ \right\| \le \frac{\left(2^{3I-1}\right)^{1-0}}{1-\left(2^{3I-1}\right)}\Phi(\omega,\omega,\omega\ ) = \frac{2^{3I}}{2-2^{3I}}\cdot\frac{\Delta|\omega|^{3I}}{|2|^{3I}} = \frac{\Delta|\omega|3^{I}}{2-2^{3I}}.$$

For K=1, we have L= $A_1^{-1} = \left(\frac{1}{2^{3I-1}}\right)^{-1}$.From (2.40), we obtain

$$\left\|\Omega(\omega)-\Lambda_A(\omega)\right\| \le \frac{\left(2^{1-3I}\right)^{1-1}}{1-\left(2^{1-3I}\right)}\Phi(\omega,\omega,\omega) = \frac{2^{3I}}{2^{3I}-2}\cdot\frac{\Delta|\omega|^{3I}}{|2|^{3I}} = \frac{\Delta|\omega|^{3I}}{2^{3I}-2}.$$

The rest of the cases is similar to that of previous cases.

2.6 Quadratic Case: Stability Analysis: Radus Method

Theorem 2.9 Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (2.16) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the condition

$$\lim_{S\to\infty}\frac{1}{A_K^{2S}}\Phi\left(A_K^S\omega_1, A_K^S\omega_2, A_K^S\omega_3\right) = 0; A_K = \begin{cases} 2 & K=1 \\ \frac{1}{2} & K=-1 \end{cases} \tag{2.48}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$. In the event that $L = L[K]$ be a mapping have the properties

$$\Phi(\omega,\omega,\omega) = \frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right);\ \frac{1}{A_K^2}\Phi\left(A_K\omega, A_K\omega, A_K\omega\right) = L\Phi(\omega,\omega,\omega); \tag{2.49}$$

for all $\omega \in Q_1$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega(\omega)-\Lambda_Q(\omega)\right\| \le \frac{L^{K-1}}{1-L}\Phi(\omega,\omega,\omega) \tag{2.50}$$

where the mapping $\Lambda_Q(\omega)$ is formulated by

$$\Lambda_Q(\omega) = \lim_{S\longrightarrow\infty}\frac{\Omega\left(A_K^S\omega\right)}{A_K^{2S}} \tag{2.51}$$

for all $\omega \in Q_1$.

Proof. Assume a set as in (2.42) and initiate the generalized metric on the above set M as in (2.43) for all $\omega \in Q_1$. It is apparent that (M, d) is complete. Define a mapping $\mathrm{N}:\mathrm{M}\to\mathrm{M}$ by

$$\mathrm{N}\Omega(\omega) = \frac{1}{A_K^2}\Omega\left(A_K\omega\right) \tag{2.52}$$

for all $\omega \in Q_1$. Now $\Omega, \Omega_1 \in \mathrm{M}$ and $\omega \in Q_1$, we see

$$\begin{aligned} d(\Omega,\Omega_1) \le \mathrm{K} &\Rightarrow \left\|\Omega(\omega)-\Omega_1(\omega)\right\| \le \mathrm{K}\Phi(\omega,\omega,\omega) \\ &\Rightarrow \left\|\frac{1}{A_K^2}\Omega\left(A_K\omega\right)-\frac{1}{A_K^2}\Omega_1\left(A_K\omega\right)\right\| \le \frac{\mathrm{K}}{A_K^2}\Phi\left(A_K\omega, A_K\omega, A_K\omega\right) \\ &\Rightarrow \left\|\mathrm{N}\Omega(\omega)-\mathrm{N}\Omega_1(\omega)\right\| \le \mathrm{K\,L}\Phi(\omega,\omega,\omega) \\ &\Rightarrow d\left(\mathrm{N}\Omega, \mathrm{N}\Omega_1\right) \le \mathrm{K\,L} \end{aligned}$$

i.e., N is a strictly contractive mapping on M with Lipschitz constant L.

For the case K = 0, as a consequence of (2.22) and by means of (2.49), (2.43), (2.52), we obtain

$$\left\|\Omega(\omega)-\frac{1}{4}\Omega(2\omega)\right\| \le \frac{1}{3\times 4}\Phi(\omega,\omega,\omega) \Rightarrow \left\|\Omega(\omega)-\mathrm{N}\Omega(\omega)\right\| \le \frac{1}{4}\Phi(\omega,\omega,\omega) \Rightarrow d(\Omega,\mathrm{N}\Omega) \le \mathrm{L} = \mathrm{L}^{1\text{-}\mathrm{K}} \tag{2.53}$$

For the case K = 1, as a consequence of (2.24) and by means of (2.49), (2.43), (2.52), we get

$$\left\|4\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)\right\| \le \frac{1}{3}\Phi\left(\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}\right) \Rightarrow \left\|\mathrm{N}\Omega(\omega)-\Omega(\omega)\right\| \le \Phi(\omega,\omega,\omega) \Rightarrow d(\mathrm{N}\Omega, \Omega) \le 1 = \mathrm{L}^{1\text{-}\mathrm{K}} \tag{2.54}$$

Combining (2.53) and (2.54), we have

$$d(\mathrm{N}\Omega,\Omega) \le \mathrm{L}^{1\text{-}\mathrm{K}}$$

(2.55)

Therefore (FPC1) of Theorem I.1 holds. The remaining steps are similar in nature to the lines of Theorem 2.7.

Corollary 2.10 Assume Δ be a positive constant and I be a real number. Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ fulfills the disparity (2.26) for all $\omega_1,\omega_2,\omega_3 \in Q_1$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which obey the functional equation (1.1) along with the disparity (2.27) for every $\omega \in Q_1$.

Proof. The proof is notably revised in a way analogous to Corollary 2.8.

2.7 Additive - Quadratic Case: Stability Analysis: Radus Method

Theorem 2.11 Assume a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ fulfills the disparity(2.28) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping subject to the conditions (2.38) and (2.48) for all $\omega_1, \omega_2, \omega_3 \in Q_1$. Assuming there is $L = L[K]$ be a mapping have the properties (2.39) and (2.49) for all $\omega \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) and the disparity

$$\left\|\Omega(\omega) - \Lambda_A(\omega) - \Lambda_Q(\omega)\right\| \le \frac{L^{K-1}}{(1-L)}\left\{ \Phi(\omega,\omega,\omega) + \Phi(-\omega,-\omega,-\omega)\right\} \tag{2.56}$$

where the mapping $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are formulated in (2.41) and (2.51) for all $\omega \in Q_1$.

Proof. Assume a mapping as in (2.30) for all $\omega \in Q_1$. By Theorem 2.7, there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the disparity

$$\left\|\Omega_{Add}(\omega) - \Lambda_A(\omega)\right\| \le \frac{L^{K-1}}{2(1-L)}\left\{ \Phi(\omega,\omega,\omega) + \Phi(-\omega,-\omega,-\omega)\right\} \tag{2.57}$$

for all $\omega \in Q_1$. Also, assume a mapping as in (2.32) for all $\omega \in Q_1$. By Theorem 2.9, there exists a one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the disparity

$$\left\|\Omega_{QUD}(\omega) - \Lambda_Q(\omega)\right\| \le \frac{L^{K-1}}{2(1-L)}\left\{ \Phi(\omega,\omega,\omega) + \Phi(-\omega,-\omega,-\omega)\right\} \tag{2.58}$$

for all $\omega \in Q_1$. Now, define a mapping as in (2.35) for all $\omega \in Q_1$. As a consequence of (2.35), (2.57), (2.58), we arrive our result. Thus, the proof is finished.

Corollary 2.12 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (2.36) for all $\omega_1, \omega_2, \omega_3 \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity(2.37) for all $\omega \in Q_1$.

3. STABILITY ANALYSIS IN GENERALIZED 2- BANACH SPACE

In this section, we present the stability analysis of (1.1) in Generalized 2- Banach space with the help of classical Hyers and Radus fixed point methods.

In order to prove the stability theorems, let us consider Q_1 be a Generalized 2- normed space and Q_2 be a Generalized 2- Banach space.

3.1 Definitions And Notations Related To Generalized 2-Banach Spaces

Definition 3.1 [3,4] Let X be linear space. A mapping $G(.,.) : X \times X \to [0,\infty)$ is called a generalized 2-normed space if it aligns with the following:

(GNS1) $G(x,y) = 0$ if and only if x and y are linearly independent vectors.

(GNS2) $G(x,y) = G(y,x)$ for all $x, y \in X$,

(GNS3) $G(\lambda x, y) = |\lambda| G(x,y)$ for all $x, y \in X$ and $X = \varphi, \varphi$ is a real or complex field,

(GNS4) $G(x+y,z) \le G(x,z) + G(y,z)$ for all $x, y, z \in X$.

The generalized 2-normed space is denoted by $(X, G(.,.))$.

Definition 3.2 [3,4] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, G(.,.))$ is called convergent if there exist $x \in X$ such that $\lim_{n\to\infty} G(x_n - x, z) = 0$ then $\lim_{n\to\infty} G(x_n, z) = G(x,z)$ for all $z \in X$.

Definition 3.3 [3,4] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, G(.,.))$ is called Cauchy sequence is there exist two lineary independent elements y and z in X such that $\{G(x_n, y)\}$ and $\{G(x_n, z)\}$ are real Cauchy sequences.

Definition 3.4 [3,4] A generalized 2-normed space $(X, G(.,.))$ is called generalized 2-Banach space is every Cauchy sequence is convergent.

3.2 Additive Case: Stability Analysis: Direct Method

Theorem 3.5 Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$G\left(\Omega_A^Q(\omega_1,\omega_2,\omega_3),\psi\right) \le \Phi\left((\omega_1,\psi),(\omega_2,\psi),(\omega_3,\psi)\right) \tag{3.1}$$

where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty} \frac{1}{2^{ES}} \Phi\left(\left(2^{ES}\omega_1,\psi\right),\left(2^{ES}\omega_2,\psi\right),\left(2^{ES}\omega_3,\psi\right)\right) = 0;\ E = \pm 1 \tag{3.2}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1 \longrightarrow Q_2$ which obey the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega) - \Lambda_A(\omega),\psi\right) \le \frac{1}{3\times 2} \sum_{D=\frac{1-E}{2}}^{\infty} \frac{\Phi\left(\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right)\right)}{2^{DE}} \tag{3.3}$$

where the mapping $\Lambda_A(\omega)$ is formulated by

$$\lim_{S\longrightarrow\infty} G\left(\frac{\Omega\left(2^{ES}\omega\right)}{2^{ES}},\psi\right) = G\left(\Lambda_A(\omega),\psi\ \right) \tag{3.4}$$

for all $\omega \in Q_1$ and all $\psi > 0$.

Proof. Utilizing oddness of Ω in (3.1) and then changing $\omega_1 = \omega_2 = \omega_3 = \omega$, we achieve

$$G\left(6\Omega(\omega) - 3\Omega(2\omega),\psi\right) \le \Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right) \tag{3.5}$$

for all $\omega \in Q_1$ and all $\psi > 0$. As a consequence of (3.5) that

$$G\left(2\Omega(\omega) - \Omega(2\omega),\psi\right) \le \frac{1}{3}\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right) \tag{3.6}$$

and then

$$G\left(\Omega(\omega) - \frac{1}{2}\Omega(2\omega),\psi\right) \le \frac{1}{3\times 2}\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right) \tag{3.7}$$

for all $\omega \in Q_1$ and all $\psi > 0$. For any positive integer S, the above disparity can be generalized as

$$G\left(\Omega(\omega) - \frac{1}{2^S}\Omega\left(2^S\omega\right),\psi\right) \le \frac{1}{3\times 2}\sum_{D=0}^{S-1} \frac{\Phi\left(\left(2^D\omega,\psi\right),\left(2^D\omega,\psi\right),\left(2^D\omega,\psi\right)\right)}{2^D} \tag{3.8}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Changing $\omega = 2^{S_1}\omega$ in (3.8), we attain

$$\begin{aligned} G\left(\frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right) - \frac{1}{2^{S+S_1}}\Omega\left(2^{S+S_1}\omega\right),\psi\right) &= \frac{1}{2^{S_1}} G\left(\Omega(\omega) - \frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right),\psi\right) \\ &\le \frac{1}{3\times 2}\sum_{D=0}^{S-1} \frac{\Phi\left(\left(2^{D+S_1}\omega,\psi\right),\left(2^{D+S_1}\omega,\psi\right),\left(2^{D+S_1}\omega,\psi\right)\right)}{2^{D+S_1}} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} G\left(\frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right) - \frac{1}{2^{S+S_1}}\Omega\left(2^{S+S_1}\omega\right),\psi_1\right) &= \frac{1}{2^{S_1}} G\left(\Omega(\omega) - \frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right),\psi_1\right) \\ &\le \frac{1}{3\times 2}\sum_{D=0}^{S-1} \frac{\Phi\left(\left(2^{D+S_1}\omega,\psi_1\right),\left(2^{D+S_1}\omega,\psi_1\right),\left(2^{D+S_1}\omega,\psi_1\right)\right)}{2^{D+S_1}} \end{aligned} \tag{3.10}$$

for all $\omega \in Q_1$ and all $\psi > 0, \psi_1 > 0$ Hence there exists two linearly independent elements ψ and ψ_1 in Q_1 such that $\left\{ G\left(\frac{\Omega(2^S \omega)}{2^S}, \psi \right) \right\}$ and $\left\{ G\left(\frac{\Omega(2^S \omega)}{2^S}, \psi_1 \right) \right\}$ are real Cauchy sequences. Letting $S_1 \to \infty$ in (3.9) and (3.10), we arrive the sequence $\left\{ \frac{1}{2^S} \Omega(2^S \omega) \right\}$ is a Cauchy sequence which converges to $\Lambda_A(y) \in Q_2$. So, we define

$$G(\Lambda_A(\omega), \psi) = \lim_{S \longrightarrow \infty} G\left(\frac{\Omega(2^S \omega)}{2^S}, \psi \right)$$

(3.11)

for all $\omega \in Q_1$ and all $\psi > 0$. Again letting $S \to \infty$ in (3.8) and using (3.11), we see (3.3) holds for $E = 1$. The proof from here on is analogous to Theorem 2.1. Thus, the proof is finished.

Corollary 3.6 Let Δ be a positive constant and *I* be a real number. Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$G(\Omega_A^Q(\omega_1, \omega_2, \omega_3), \psi) \le \begin{cases} \Delta; \\ \Delta \sum_{i=1}^{3} |\omega_i, \psi|^I; \\ \Delta \sum_{i=1}^{3} |\omega_i, \psi|^{I_i}; \\ \Delta \prod_{i=1}^{3} |\omega_i, \psi|^I; \\ \Delta \prod_{i=1}^{3} |\omega_i, \psi|^{I_i}; \\ \Delta \left\{ \prod_{i=1}^{3} |\omega_i, \psi|^I + \sum_{i=1}^{3} |\omega_i, \psi|^{3I} \right\}; \end{cases} \tag{3.12}$$

for all $\omega_1, \omega_2, \omega_3 \in Q_1$ and all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G(\Omega(\omega) - \Lambda_A(\omega), \psi) \le \begin{cases} \frac{\Delta}{|3|}; \\ \frac{\Delta |\omega, \psi|^I}{|2 - 2^I|}; I \ne 1, \\ \sum_{i=1}^{3} \frac{\Delta |\omega_i, \psi|^{I_i}}{3|2 - 2^{I_i}|}; I_1, I_2, I_3 \ne 1, \\ \frac{\Delta |\omega, \psi|^{3I}}{3|2 - 2^{3I}|}; 3I \ne 1, \\ \frac{\Delta |\omega, \psi|^{I_1 + I_2 + I_3}}{3|2 - 2^{I_1 + I_2 + I_3}|}; I_1 + I_2 + I_3 \ne 1, \\ \frac{2\Delta |\omega, \psi|^{3I}}{3|2 - 2^{3I}|}; 3I \ne 1, \end{cases} \tag{3.13}$$

for all $\omega \in Q_1$ and all $\psi > 0$.

3.3 Quadratic Case: Stability Analysis: Direct Method

Theorem 3.7 Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$G(\Omega_A^Q(\omega_1, \omega_2, \omega_3), \psi) \le \Phi((\omega_1, \psi), (\omega_2, \psi), (\omega_3, \psi)) \tag{3.14}$$

where $\Phi : Q_1^3 \longrightarrow [0, \infty)$ be a mapping with the condition

$$\lim_{S \longrightarrow \infty} \frac{1}{4^{ES}} \Phi((2^{ES} \omega_1, \psi), (2^{ES} \omega_2, \psi), (2^{ES} \omega_3, \psi)) = 0; \ E = \pm 1 \tag{3.15}$$

for all $\omega_1, \omega_2, \omega_3 \in Q_1$ and all $\psi > 0$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_Q(\omega),\psi\right)\leq\frac{1}{3\times4}\sum_{D=\frac{1-E}{2}}^{\infty}\frac{\Phi\left(\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right)\right)}{4^{DE}}\tag{3.16}$$

where the mapping $\Lambda_Q(\omega)$ is formulated by

$$\lim_{S\longrightarrow\infty}G\left(\frac{\Omega\left(2^{ES}\omega\right)}{4^{ES}},\psi\right)=G\left(\Lambda_Q(\omega),\psi\right)\tag{3.17}$$

for all $\omega\in Q_1$.

Proof. Utilizing evenness of Ω in (3.14) the proof follows by Theorem 3.5 and Theorem 2.3.

Corollary 3.8 Assume Δ be a positive constant and *I* be a real number. Suppose an even mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity

$$G\left(\Omega_A^Q(\omega_1,\omega_2,\omega_3),\psi\right)\leq\begin{cases}\Delta;\\ \Delta\sum_{i=1}^{3}|\omega_i,\psi|^{I};\\ \Delta\sum_{i=1}^{3}|\omega_i,\psi|^{I_i};\\ \Delta\prod_{i=1}^{3}|\omega_i,\psi|^{I};\\ \Delta\prod_{i=1}^{3}|\omega_i,\psi|^{I_i};\\ \Delta\left\{\prod_{i=1}^{3}|\omega_i,\psi|^{I}+\sum_{i=1}^{3}|\omega_i,\psi|^{3I}\right\};\end{cases}\tag{3.18}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$ and all $\psi>0$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_Q(\omega),\psi\right)\leq\begin{cases}\frac{\Delta}{3|3|};\\ \frac{\Delta|\omega,\psi|^{I}}{|4-2^{I}|};I\neq2,\\ \sum_{i=1}^{3}\frac{\Delta|\omega_i,\psi|^{I_i}}{3|4-2^{I_i}|};I_1,I_2,I_3\neq2,\\ \frac{\Delta|\omega,\psi|^{3I}}{3|4-2^{3I}|};3I\neq2,\\ \frac{\Delta|\omega,\psi|^{I_1+I_2+I_3}}{3|4-2^{I_1+I_2+I_3}|};I_1+I_2+I_3\neq2,\\ \frac{2\Delta|\omega,\psi|^{3I}}{3|4-2^{3I}|};3I\neq2,\end{cases}\tag{3.19}$$

for all $\omega\in Q_1$ and all $\psi>0$.

3.4 Additive - Quadratic Case: Stability Analysis: Direct Method

Theorem 3.9 Suppose a mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity

$$G\left(\Omega_A^Q(\omega_1,\omega_2,\omega_3),\psi\right)\leq\Phi\left((\omega_1,\psi),(\omega_2,\psi),(\omega_3,\psi)\right)\tag{3.20}$$

where $\Phi:Q_1^3\longrightarrow[0,\infty)$ be a mapping with the conditions (3.2) and (3.15) for all $\omega_1,\omega_2,\omega_3\in Q_1$ and all $\psi>0$. Then there exists one and only one additive mapping $\Lambda_A(\omega):Q_1\longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega),\psi\right)$$
$$\leq \frac{1}{3\times 4}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right)\right)}{2^{DE}}+\frac{\Phi\left(\left(-2^{DE}\omega,\psi\right),\left(-2^{DE}\omega,\psi\right),\left(-2^{DE}\omega,\psi\right)\right)}{2^{DE}}\right\} \tag{3.21}$$
$$+\frac{1}{3\times 8}\sum_{D=\frac{1-E}{2}}^{\infty}\left\{\frac{\Phi\left(\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right)\right)}{4^{DE}}+\frac{\Phi\left(\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right),\left(2^{DE}\omega,\psi\right)\right)}{4^{DE}}\right\}$$

where the mapping $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are formulated in (3.4) and (3.17) for all $\omega\in Q_1$ and all $\psi>0$.
Proof. The proof is notably revised in a way analogous to Theorem 2.5.

Corollary 3.10 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$G\left(\Omega_A^Q(\omega_1,\omega_2,\omega_3),\psi\right)\leq\begin{cases}\Delta;\\ \Delta\sum_{i=1}^{3}|\omega_i,\psi|^{I};\\ \Delta\sum_{i=1}^{3}|\omega_i,\psi|^{I_i};\\ \Delta\prod_{i=1}^{3}|\omega_i,\psi|^{I};\\ \Delta\prod_{i=1}^{3}|\omega_i,\psi|^{I_i};\\ \Delta\left\{\prod_{i=1}^{3}|\omega_i,\psi|^{I}+\sum_{i=1}^{3}|\omega_i,\psi|^{3I}\right\};\end{cases}$$

(3.22)

for all $\omega_1,\omega_2,\omega_3\in Q_1$ and all $\psi>0$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1\longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega): Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega),\psi\right)\leq\begin{cases}\left\{\frac{\Delta}{|3|}+\frac{\Delta}{3|3|}\right\};\\ \left\{\frac{\Delta|\omega,\psi|^{I}}{|2-2^{I}|}+\frac{\Delta|\omega,\psi|^{I}}{|4-2^{I}|}\right\}; I\neq 1,2,\\ \left\{\sum_{i=1}^{3}\frac{\Delta|\omega_i,\psi|^{I_i}}{3|2-2^{I_i}|}+\sum_{i=1}^{3}\frac{\Delta|\omega_i,\psi|^{I_i}}{3|4-2^{I_i}|}\right\}; I_1,I_2,I_3\neq 1,2,\\ \left\{\frac{\Delta|\omega,\psi|^{3I}}{3|2-2^{3I}|}+\frac{\Delta|\omega,\psi|^{3I}}{3|4-2^{3I}|}\right\}; 3I\neq 1,2,\\ \left\{\frac{\Delta|\omega,\psi|^{I_1+I_2+I_3}}{3|2-2^{I_1+I_2+I_3}|}+\frac{\Delta|\omega,\psi|^{I_1+I_2+I_3}}{3|4-2^{I_1+I_2+I_3}|}\right\}; I_1+I_2+I_3\neq 1,2,\\ \left\{\frac{2\Delta|\omega,\psi|^{3I}}{3|2-2^{3I}|}+\frac{2\Delta|\omega,\psi|^{3I}}{3|4-2^{3I}|}\right\}; 3I\neq 1,2,\end{cases} \tag{3.23}$$

for all $\omega\in Q_1$ and all $\psi>0$.

3.5 Additive Case: Stability Analysis: Radus Method

Theorem 3.11 Suppose an odd mapping $\Omega_A^Q : Q_1\longrightarrow Q_2$ aligns with the disparity (3.1) where $\Phi: Q_1^3\longrightarrow[0,\infty)$ be a mapping with the condition

$$\lim_{S\to\infty}\frac{1}{A_K^S}\Phi\left(\left(A_K^S\omega_1,\psi\right),\left(A_K^S\omega_2,\psi\right),\left(A_K^S\omega_3,\psi\right)\right)=0;\ A=\begin{cases}2 & K=1\\ \frac{1}{2} & K=-1\end{cases} \tag{3.24}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$ and all $\psi>0$. In the event that $L=L[K]$ be a mapping have the properties

$$\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right)=\frac{1}{3}\Phi\left(\left(\frac{\omega}{2},\psi\right),\left(\frac{\omega}{2},\psi\right),\left(\frac{\omega}{2},\psi\right)\right);$$
$$\frac{1}{A_K}\Phi\left(\left(A_K\omega,\psi\right),\left(A_K\omega,\psi\right),\left(A_K\omega,\psi\right)\right)=L\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right); \tag{3.25}$$

for all $\omega\in Q_1$ and all $\psi>0$. Then there exists one and only one additive mapping $\Lambda_A(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_A(\omega),\psi\right)\le\frac{L^{K-1}}{1-L}\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right) \tag{3.26}$$

where the mapping $\Lambda_A(\omega)$ is formulated by

$$\lim_{S\longrightarrow\infty}G\left(\frac{\Omega\left(A_K^S\omega\right)}{A_K^S},\psi\right)=G\left(\Lambda_A(\omega),\psi\right) \tag{3.27}$$

for all $\omega\in Q_1$ and all $\psi>0$.

Proof. The proof is notably revised in a way analogous to Theorem 2.7.

Corollary 3.12 Assume Δ be a positive constant and *I* be a real number. Suppose an odd mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity (3.12) for all $\omega_1,\omega_2,\omega_3\in Q_1$ and all $\psi>0$. Then there exists one and only one additive mapping $\Lambda_A(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity (3.13) for all $\omega\in Q_1$ and all $\psi>0$.

3.6 Quadratic Case: Stability Analysis: Radus Method

Theorem 3.13 Suppose an even mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity (3.14) where $\Phi:Q_1^3\longrightarrow[0,\infty)$ be a mapping with the condition

$$\lim_{S\to\infty}\frac{1}{A_K^{2S}}\Phi\left(\left(A_K^S\omega_1,\psi\right),\left(A_K^S\omega_2,\psi\right),\left(A_K^S\omega_3,\psi\right)\right)=0;A=\begin{cases}2 & K=1\\ \frac{1}{2} & K=-1\end{cases} \tag{3.28}$$

for all $\omega_1,\omega_2,\omega_3\in Q_1$. If there exists $L=L[K]$ be a mapping have the properties

$$\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right)=\frac{1}{3}\Phi\left(\left(\frac{\omega}{2},\psi\right),\left(\frac{\omega}{2},\psi\right),\left(\frac{\omega}{2},\psi\right)\right);$$
$$\frac{1}{A_K^2}\Phi\left(\left(A_K\omega,\psi\right),\left(A_K\omega,\psi\right),\left(A_K\omega,\psi\right)\right)=L\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right); \tag{3.29}$$

for all $\omega\in Q_1$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$G\left(\Omega(\omega)-\Lambda_Q(\omega),\psi\right)\le\frac{L^{K-1}}{1-L}\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right) \tag{3.30}$$

where the mapping $\Lambda_Q(\omega)$ is formulated by

$$\lim_{S\longrightarrow\infty}G\left(\frac{\Omega\left(A_K^S\omega\right)}{A_K^{2S}},\psi\right)=G\left(\Lambda_Q(\omega),\psi\right) \tag{3.31}$$

for all $\omega\in Q_1$.

Proof. The proof is notably revised in a way analogous to Theorem 2.9.

Corollary 3.14 Assume Δ be a positive constant and *I* be a real number. Suppose an even mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity (3.18) for all $\omega_1,\omega_2,\omega_3\in Q_1$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega):Q_1\longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity (3.19) for all $\omega\in Q_1$.

Proof. The proof is notably revised in a way analogous to Corollary 2.8.

3.7 Additive - Quadratic Case: Stability Analysis: Radus Method

Theorem 3.15 Suppose a mapping $\Omega_A^Q:Q_1\longrightarrow Q_2$ aligns with the disparity (3.20) where $\Phi:Q_1^3\longrightarrow[0,\infty)$ be a mapping with the conditions (3.24) and (3.28) for all $\omega_1,\omega_2,\omega_3\in Q_1$. If there exists $L=L[K]$ be a mapping have the properties (3.25) and (3.29) for all $\omega \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$G\left(\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega),\psi\right) \le \frac{L^{K-1}}{(1-L)}\left\{\Phi\left((\omega,\psi),(\omega,\psi),(\omega,\psi)\right)+\Phi\left((-\omega,\psi),(-\omega,\psi),(-\omega,\psi)\right)\right\} \tag{3.32}$$

where the mapping $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are formulated in (3.27) and (3.31) for all $\omega \in Q_1$.

Proof. The proof is notably revised in a way analogous to Theorem 2.11.

Corollary 3.16 Suppose a mapping $\Omega_A^Q: Q_1 \longrightarrow Q_2$ aligns with the disparity (3.22) for all $\omega_1,\omega_2,\omega_3 \in Q_1$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1 \longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity (3.23) for all $\omega \in Q_1$.

4. STABILITY ANALYSIS IN RANDOM BANACH SPACE

In this section, we present the stability analysis of (1.1) in Random Banach space with the help of classical Hyers and Radus fixed point methods.

4.1 Preliminaries and Notations of Random Banach Spaces

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [14, 27, 32, 33].

Throughout this section, Δ^+ is the space of distribution mappings, that is, the space of all mappings $F: R\cup\{-\infty,+\infty\} \to [0,1]$ such that F is leftcontinuous and nondecreasing on R, $F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all mappings $F\in\Delta^+$ for which $l^- F(+\infty)=1$, where $l^- f(x)$ denotes the left limit of the mapping f at the point x that is, $l^- f(x)=\lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point wise ordering of mappings, that is, $F\le G$ if and only if $F(t)\le G(t)$ for all $t\in R$. The maximal element for Δ^+ in this order is the distribution mapping ε given by

$$\varepsilon(t)=\begin{cases}0, & if \quad t\le 0,\\ 1, & if \quad t>0.\end{cases}$$

Definition 4.1. [32, 33] A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T aligns with the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1)=a$ for all $a\in[0,1]$;

(d) $T(a,b)\le T(c,d)$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous $t-$norms are $T_P(a,b)=ab$, $T_M(a,b)=\min(a,\ b)$ and $T_L(a,b)=\max(a+b-1,0)$ (the Lukasiewicz $t-$norm). Recall (see [16, 17]) that if T is a $t-$norm and x_n is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n}x_{n+i}$ is formulated recurrently by $T_{i=1}^{1}x_i=x_1$ and $T_{i=1}^{n}x_i=T\left(T_{i=1}^{n-1}x_i,x_n\right)$ for all $n\ge 2$. $T_{i=n}^{\infty}x_i$ is formulated as $T_{i=1}^{\infty}x_{n+i}$.

Definition 4.2. [32, 33] A random normed space (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous $t-$norm and $\mu: X\to D^+$ satisfying the following conditions:

(RNS1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

(RN2) $\mu_{\alpha x}(t)=\mu_x\left(\dfrac{t}{|\alpha|}\right), \forall\ x\in X,\ \alpha\in R$ with $\alpha\ne 0$;

(RN3) $\mu_{x+y}(t+s)=T(\mu_x(t),\mu_y(s)), \forall\ x,y\in X$ and $t,s>0$.

Example 4.3. [32, 33] Every normed space $(X,\|\cdot\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t)=\frac{t}{t+\|x\|}$$

and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 4.4 [32, 33] Let (X,μ,T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$ and $n > m \geq 0$

(3) A RN-space is said to be complete if every Cauchy sequence in X is convergent to a point in X

Theorem 4.5. [32, 33] If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

In order to prove the stability theorems, let us consider Q_1 be a Linear space and (Q_2, Ψ, T) be a Random Banach space.

4.2 Additive Case: Stability Analysis: Direct Method

Theorem 4.6. Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \Phi_{\omega_1,\omega_2,\omega_3}(\psi) \tag{4.1}$$

where $\Phi : Q_1^3 \longrightarrow D^+$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty} T_{D=0}^{\infty} \Phi_{2^{ES}\omega_1, 2^{ES}\omega_2, 2^{ES}\omega_3}\left(2^{(D+1)SE}\psi\right) = 1 \text{ and } \lim_{S\longrightarrow\infty} \Phi_{2^{ES}\omega_1, 2^{ES}\omega_2, 2^{ES}\omega_3}\left(2^{ES}\psi\right) = 1; \; E = \pm 1 \tag{4.2}$$

for all $\omega_1, \omega_2, \omega_3 \in Q_1$ and for all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}(\psi) \geq T_{D=\frac{1-E}{2}}^{\infty} \Phi_{2^{DE}\omega, 2^{DE}\omega, 2^{DE}\omega}\left(6 \times 2^{DE}\psi\right) \tag{4.3}$$

where the mapping $\Lambda_A(\omega)$ is formulated by

$$\Psi_{\Lambda_A(\omega)}(\psi) = \lim_{S\longrightarrow\infty} \Psi_{\frac{\Omega(2^{ES}\omega)}{2^{ES}}}(\psi) \tag{4.4}$$

for all $\omega \in Q_1$ and for all $\psi > 0$.

Proof. Utilizing oddness of Ω in (2.1) and then changing $\omega_1 = \omega_2 = \omega_3 = \omega$, we achieve

$$\Psi_{6\Omega(\omega)-3\Omega(2\omega)}(\psi) \geq \Phi_{\omega,\omega,\omega}(\psi)$$

(4.5)

using (RNS2) which gives

$$\Psi_{2\Omega(\omega)-\Omega(2\omega)}\left(\frac{\psi}{3}\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.6}$$

and then using (RNS2) which implies

$$\Psi_{\Omega(\omega)-\frac{1}{2}\Omega(2\omega)}\left(\frac{\psi}{3\times 2}\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.7}$$

for all $\omega \in Q_1$ and for all $\psi > 0$. Replacing ω by $2^S\omega$ in (4.7), we attain

$$\Psi_{\Omega(2^S\omega)-\frac{1}{2}\Omega(2^{S+1}\omega)}\left(\frac{\psi}{3\times 2}\right) \geq \Phi_{2^S\omega, 2^S\omega, 2^S\omega}(\psi) \tag{4.8}$$

for all $\omega \in Q_1$ and all $\psi > 0$. With the help of (RNS2), as a consequence of (4.8) that

$$\Psi_{\frac{1}{2^S}\Omega(2^S\omega)-\frac{1}{2^{S+1}}\Omega(2^{S+1}\omega)}\left(\frac{\psi}{3\times 2^{S+1}}\right) \geq \Phi_{2^S\omega, 2^S\omega, 2^S\omega}(\psi) \tag{4.9}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Interchanging ψ by $3\times 2^{S+1}\psi$ in (4.9), we reach

$$\Psi_{\frac{1}{2^S}\Omega(2^S\omega)-\frac{1}{2^{S+1}}\Omega(2^{S+1}\omega)}(\psi) \geq \Phi_{2^S\omega, 2^S\omega, 2^S\omega}\left(3\times 2^{S+1}\psi\right) \tag{4.10}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Now, from (4.10), it is easy to see that

$$\Psi_{\Omega(\omega)-\frac{1}{2^S}\Omega(2^S\omega)}(\psi) = \Psi_{\sum_{D=0}^{S-1} \frac{1}{2^D}\Omega(2^D\omega)-\frac{1}{2^{D+1}}\Omega(2^{D+1}\omega)}\left(3\times 2^{D+1}\psi\right)$$

$$\geq T_{D=0}^{S-1} \Psi_{\frac{1}{2^D}\Omega(2^D\omega)-\frac{1}{2^{D+1}}\Omega(2^{D+1}\omega)}\left(3\times 2^{D+1}\psi\right)$$

$$\geq T_{D=0}^{S-1} \Phi_{2^S\omega,2^S\omega,2^S\omega}\left(3\times 2^{D+1}\psi\right) \tag{4.11}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Therefore, the sequence $\left\{\frac{1}{2^S}\Omega\left(2^S\omega\right)\right\}$ is a Cauchy sequence. Indeed, changing $\omega = 2^{S_1}\omega$ in (2.8) using (RNS2), we attain

$$\Psi_{\frac{1}{2^{S_1}}\Omega\left(2^{S_1}\omega\right)-\frac{1}{2^{S+S_1}}\Omega\left(2^{S+S_1}\omega\right)}(\psi) \geq \underset{D=0}{\overset{S-1}{T}} \Psi_{\frac{1}{2^D}\Omega\left(2^D\omega\right)-\frac{1}{2^{D+1}}\Omega\left(2^{D+1}\omega\right)}\left(3\times 2^{D+S_1+1}\psi\right) \to 1 \text{ as } S\to\infty \tag{4.12}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Hence the sequence is a Cauchy sequence which converges to $\Lambda_A(y) \in Q_2$. So, we define

$$\Psi_{\Lambda_A(\omega)}(\psi) = \lim_{S\longrightarrow\infty} \Psi_{\frac{\Omega\left(2^S\omega\right)}{2^S}}(\psi), \forall\, \omega \in Q_1, \text{ all } \psi > 0. \tag{4.13}$$

Again letting $S\to\infty$ in (4.11) and using (4.13), we see (4.3) holds for $E = 1$.

In order to show that the existence of $\Lambda_A(y)$ aligns with the functional equation (1.20), setting $\omega_1 = 2^S\omega_1; \omega_2 = 2^S\omega_2;\ \ \omega_3 = 2^S\omega_3$ in (4.1) using (RNS2), we arrive

$$\Psi_{\frac{1}{2^S}\Omega_A^Q\left(2^S\omega_1,2^S\omega_2,2^S\omega_3\right)}(\psi) \geq \Phi_{2^S\omega_1,2^S\omega_2,2^S\omega_3}(2^S\psi) \tag{4.14}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for all $\psi > 0$. Letting $S\to\infty$ in (4.14) and using (4.2) and (4.13), we obtain that $\Lambda_A(y)$ aligns with the functional equation (1.20).

Finally, we want to show that the existence of $\Lambda_A(y)$ is unique, let's assume $\Lambda_B(y)$ fulfills (1.1) and (4.3) for all $\omega \in Q_1$ and for all $\psi > 0$. So,

$$\begin{aligned}\Psi_{\Lambda_A(\omega)-\Lambda_B(\omega)}(2\psi) &= \Psi_{\Lambda_A\left(2^{S_1}\omega\right)-\Omega\left(2^{S_1}\omega\right)+\Omega\left(2^{S_1}\omega\right)-\Lambda_B\left(2^{S_1}\omega\right)}(2^{S_1+1}\psi)\\ &= T\left[\Psi_{\Lambda_A\left(2^{S_1}\omega\right)-\Omega\left(2^{S_1}\omega\right)}(2^{S_1}\psi), \Psi_{\Omega\left(2^{S_1}\omega\right)-\Lambda_B\left(2^{S_1}\omega\right)}(2^{S_1}\psi)\right]\\ &\geq T\left[T_{D=0}^{\infty} \Phi_{2^D\omega,2^D\omega,2^D\omega}\left(3\times 2^{(D+S_1+1)}\psi\right), T_{D=0}^{\infty} \Phi_{2^D\omega,2^D\omega,2^D\omega}\left(3\times 2^{(D+S_1+1)}\psi\right)\right] \to 1 \text{ as } S_1\to\infty\end{aligned}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Therefore $\Lambda_A(\omega) = \Lambda_B(\omega)$ which proves the uniqueness. Thus, the theorem holds for $E = 1$.

Again, changing $\omega = \frac{\omega}{2}$ in (4.6), we reach

$$\Psi_{2\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)}\left(\frac{\psi}{3}\right) \geq \Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(\psi) \Rightarrow \Psi_{2\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)}(\psi) \geq \Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(3\psi) \tag{4.15}$$

for all $\omega \in Q_1$ and all $\psi > 0$. Replacing ω by $\omega = \frac{\omega}{2^S}$ in (4.15) and with the help of (RNS2), we attain

$$\Psi_{2^{S+1}\Omega\left(\frac{\omega}{2^{S+1}}\right)-2^S\Omega\left(\frac{\omega}{2^S}\right)}(\psi) \geq \Phi_{\frac{\omega}{2^{S+1}},\frac{\omega}{2^{S+1}},\frac{\omega}{2^{S+1}}}\left(\frac{3\psi}{2^S}\right) \tag{4.16}$$

for all $\omega \in Q_1$ and all $\psi > 0$. The rest of the proof is similar to that of $E = 1$ case. Thus, the proof is finished.

Corollary 4.7 Assume Δ be a positive constant and *I* be a real number. Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

.

$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \begin{cases}\Phi_{\Delta}(\psi);\\ \Phi_{\Delta\sum_{i=1}^{3}|\omega_i|^I}(\psi);\\ \Phi_{\Delta\prod_{i=1}^{3}|\omega_i|^I}(\psi);\\ \Phi_{\Delta\left\{\prod_{i=1}^{3}|\omega_i|^I+\sum_{i=1}^{3}|\omega_i|^{3I}\right\}}(\psi);\end{cases} \tag{4.17}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega): Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}(\psi) \geq \begin{cases} \Phi_{\Delta}\left(|6|\ \psi\right); \\ \Phi_{\Delta|2^I-2||\omega|^I}\left(2^{I+1}\ \psi\right); I \neq 1, \\ \Phi_{\Delta|2^{3I}-2||\omega|^I}\left(2^{3I+1}\ \psi\right); 3I \neq 1, \\ \Phi_{\Delta|2^{3I}-2||\omega|^I}\left(2^{3I}\ \psi\right); 3I \neq 1, \end{cases} \tag{4.18}$$

for all $\omega \in Q_1$ and all $\psi > 0$.

4.3 Quadratic Case: Stability Analysis: Direct Method

Theorem 4.8. Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \Phi_{\omega_1,\omega_2,\omega_3}(\psi) \tag{4.19}$$

where $\Phi : Q_1^3 \longrightarrow D^+$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty} T_{D=0}^{\infty}\, \Phi_{2^{ES}\omega_1,2^{ES}\omega_2,2^{ES}\omega_3}\left(4^{(D+1)SE}\psi\right) = 1 \text{ and } \lim_{S\longrightarrow\infty} \Phi_{2^{ES}\omega_1,2^{ES}\omega_2,2^{ES}\omega_3}\left(4^{ES}\psi\right) = 1;\ E = \pm 1 \tag{4.20}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for all $\psi > 0$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which fulfills the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_Q(\omega)}(\psi) \geq T_{D=\frac{1-E}{2}}^{\infty}\ \Phi_{2^{DE}\omega,2^{DE}\omega,2^{DE}\omega}\left(12\times 2^{DE}\psi\right) \tag{4.21}$$

where the mapping $\Lambda_Q(\omega)$ is formulated by

$$\Psi_{\Lambda_Q(\omega)}(\psi) = \lim_{S\longrightarrow\infty} \Psi_{\frac{\Omega\left(2^{ES}\omega\right)}{4^{ES}}}(\psi) \tag{4.22}$$

for all $\omega \in Q_1$ and for all $\psi > 0$.

Proof. Utilizing evenness of Ω in (4.19) and then changing $\omega_1 = \omega_2 = \omega_3 = \omega$, we achieve

$$\Psi_{12\Omega(\omega)-3\Omega(2\omega)}(\psi) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.23}$$

using (RNS2) which gives

$$\Psi_{4\Omega(\omega)-\Omega(2\omega)}\left(\frac{\psi}{3}\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.24}$$

for all $\omega \in Q_1$ and for all $\psi > 0$. The proof from here on is analogous to Theorem 4.7. Thus, the proof is finished.

Corollary 4.9 Assume Δ be a positive constant and *I* be a real number. Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

.
$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \begin{cases} \Phi_{\Delta}(\psi); \\ \Phi_{\Delta\sum_{i=1}^3|\omega_i|^I}(\psi); \\ \Phi_{\Delta\prod_{i=1}^3|\omega_i|^I}(\psi); \\ \Phi_{\Delta\left\{\prod_{i=1}^3|\omega_i|^I+\sum_{i=1}^3|\omega_i|^{3I}\right\}}(\psi); \end{cases} \tag{4.25}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and all $\psi > 0$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega): Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_Q(\omega)}(\psi) \geq \begin{cases} \Phi_{\Delta}\left(|12|\ \psi\right); \\ \Phi_{\Delta|2^I-4||\omega|^I}\left(2^{I+1}\ \psi\right); I \neq 2, \\ \Phi_{\Delta|2^{3I}-4||\omega|^I}\left(2^{3I+1}\ \psi\right); 3I \neq 2, \\ \Phi_{\Delta|2^{3I}-4||\omega|^I}\left(2^{3I}\ \psi\right); 3I \neq 2, \end{cases}$$

(4.26)

for all $\omega \in Q_1$ and all $\psi > 0$.

4.4 Additive - Quadratic Case: Stability Analysis: Direct Method

Theorem 4.10 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \Phi_{\omega_1,\omega_2,\omega_3}(\psi) \tag{4.27}$$

where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the conditions (4.2) and (4.20) for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ and a one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\begin{aligned}\Psi_{\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)}(\psi) \geq T\ \Big\{ & T\ \Big\{ T^{\infty}_{D=\frac{1-E}{2}}\ \Phi_{2^{DE}\omega,2^{DE}\omega,2^{DE}\omega}\left(6\times 2^{DE}\psi\right),\ T^{\infty}_{D=\frac{1-E}{2}}\ \Phi_{-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega}\left(6\times 2^{DE}\psi\right)\Big\},\\ & \Big\{ T\ \Big\{ T^{\infty}_{D=\frac{1-E}{2}}\ \Phi_{2^{DE}\omega,2^{DE}\omega,2^{DE}\omega}\left(12\times 2^{DE}\psi\right),\ T^{\infty}_{D=\frac{1-E}{2}}\ \Phi_{-2^{DE}\omega,-2^{DE}\omega,-2^{DE}\omega}\left(12\times 2^{DE}\psi\right)\Big\}\Big\}\Big\}\end{aligned} \tag{4.28}$$

where the mapping $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are formulated in (4.4) and (4.22) for all $\omega \in Q_1$ and for all $\psi > 0$.

Proof. The proof is notably revised in a way analogous to Theorem 2.5

Corollary 4.11 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity

$$\Psi_{\Omega_A^Q(\omega_1,\omega_2,\omega_3)}(\psi) \geq \begin{cases}\Phi_{\Delta}(\psi); \\ \Phi_{\Delta\sum_{i=1}^{3}|\omega_i|^I}(\psi); \\ \Phi_{\Delta\prod_{i=1}^{3}|\omega_i|^I}(\psi); \\ \Phi_{\Delta\left\{\prod_{i=1}^{3}|\omega_i|^I+\sum_{i=1}^{3}|\omega_i|^{3I}\right\}}(\psi);\end{cases} \tag{4.29}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for all $\psi > 0$. Then there exists one and only one additive mapping and a one and only one quadratic mapping, which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)}(\psi) \geq \begin{cases}\Phi_{\Delta}\left(\left\{|3|+|12|\right\}\psi\right); \\ \Phi_{\Delta\left\{|2^I-2|+|2^I-4|\right\}|\omega|^I}\left(2^{I+1}\ \psi\right); I \neq 1,2 \\ \Phi_{\Delta\left\{|2^{3I}-2|+|2^{3I}-4|\right\}|\omega|^{3I}}\left(2^{3I+1}\ \psi\right); 3I \neq 1,2 \\ \Phi_{\Delta\left\{|2^{3I}-2|+|2^{3I}-4|\right\}|\omega|^{3I}}\left(2^{3I}\ \psi\right); 3I \neq 1,2\end{cases} \tag{4.30}$$

for all $\omega \in Q_1$ and for all $\psi > 0$.

4.5 Additive Case: Stability Analysis: Radus Method

Theorem 4.12 Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.1) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty} \Phi_{A_K^S\omega_1,A_K^S\omega_2,A_K^S\omega_3}\left(A_K^S\psi\right) = 1; A = \begin{cases}2 & K=1 \\ \frac{1}{2} & K=-1\end{cases} \tag{4.31}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for each $\psi > 0$. In the presence of $L = L[K]$ be a mapping have the properties

$$\Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(3\psi) = \Phi_{\omega,\omega,\omega}(\psi);\ \Phi_{A_K\omega,A_K\omega,A_K\omega}(A_K\psi) = \Phi_{\omega,\omega,\omega}(L\psi) \tag{4.32}$$

for each $\omega \in Q_1$ and for all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}\left(\frac{L^{1-i}}{1-L}\psi\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.33}$$

where the mapping $\Lambda_A(\omega)$ is formulated by

$$\Psi_{\Lambda_A(\omega)}(\psi) = \lim_{S\longrightarrow\infty} \Psi_{\frac{\Omega\left(A_K^S\omega\right)}{A_K^S}}(\psi) \tag{4.34}$$

for each $\omega \in Q_1$ and for each $\psi > 0$.

Proof. Assume a set

$$\mathrm{M} = \left\{\Omega / \Omega : Q_1 \longrightarrow Q_2, \Omega(0) = 0\right\} \tag{4.35}$$

and introduce the generalized metric on the above set M as

$$d(\Omega,\Omega_1) = \inf\left\{\mathrm{K}\in(0,\infty) / \Psi_{\Omega(\omega)-\Omega_1(\omega)}(\mathrm{K}\psi) \geq \Phi_{\omega,\omega,\omega}(\psi),\right\}, \tag{4.36}$$

for each $\omega \in Q_1$ and for each $\psi > 0$. It is apparent that (M, d) is complete. Define a mapping $\mathrm{N}:\mathrm{M}\rightarrow\mathrm{M}$ by

$$\mathrm{N}\Omega(\omega) = \frac{1}{A_K}\Omega(A_K\omega);\ \text{for each } \omega\in Q_1. \tag{4.37}$$

Now $\Omega,\Omega_1 \in \mathrm{M}$ and $\omega\in Q_1$, we see

$$\begin{aligned} d(\Omega,\Omega_1) \leq \mathrm{K} &\Rightarrow \Psi_{\Omega(\omega)-\Omega_1(\omega)}(\mathrm{K}\psi) \geq \Phi_{\omega,\omega,\omega}(\psi) \\ &\Rightarrow \Psi_{\frac{1}{A_K}\Omega(A_K\omega)-\frac{1}{A_K}\Omega_1(A_K\omega)}(\mathrm{K}\,\psi) \geq \Phi_{A_K\omega,A_K\omega,A_K\omega}(A_K\,\psi) \\ &\Rightarrow \Psi_{\mathrm{N}\Omega(\omega)-\mathrm{N}\Omega_1(\omega)}(\mathrm{K}\,\psi) \geq \Phi_{\omega,\omega,\omega}(L\,\psi) \\ &\Rightarrow d(\mathrm{N}\Omega,\mathrm{N}\Omega_1) \leq \mathrm{K\,L} \end{aligned}$$

i.e., N is a strictly contractive mapping on M with Lipschitz constant L.

For the case K = 0, as a consequence of (4.7) and with the help of (4.32), (4.36), (4.37), we obtain

$$\Psi_{\Omega(\omega)-\frac{1}{2}\Omega(2\omega)}\left(\frac{\psi}{2}\right) \geq \Phi_{\omega,\omega,\omega}(3\psi) \Rightarrow \Psi_{\Omega(\omega)-\mathrm{N}\Omega(\omega)}(L\psi) \geq \Phi_{\omega,\omega,\omega}(\psi) \Rightarrow d(\Omega,\mathrm{N}\Omega) \leq \mathrm{L} = \mathrm{L}^{1-\mathrm{K}} \tag{4.38}$$

For the case K = 1, as a consequence of (4.15) and with the help of (4.32), (4.36), (4.37), we get

$$\Psi_{2\Omega\left(\frac{\omega}{2}\right)-\Omega(\omega)}(\psi) \geq \Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(3\psi) \Rightarrow \Psi_{\mathrm{N}\Omega(\omega)-\Omega(\omega)}(\psi) \geq \Phi_{\omega,\omega,\omega}(\psi) \Rightarrow d(\mathrm{N}\Omega, \Omega) \leq 1 = \mathrm{L}^{1-\mathrm{K}} \tag{4.39}$$

Combining (4.38) and (4.39), we have

$$d(\mathrm{N}\Omega,\Omega) \leq \mathrm{L}^{1-\mathrm{K}}.$$

(4.40)

Therefore (FPC1) of Theorem I.1 holds. The proof from here on is analogous to Theorem 2.7 and 4.6. Thus, the proof is finished.

Corollary 4.13 Assume Δ be a positive constant and *I* be a real number. Suppose an odd mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.17) for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for each $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}(\psi) \geq \begin{cases} \Phi_\Delta(|3|\,\psi); \\ \Phi_{\Delta|2^I-2||\omega|^I}(2\,\psi); I\neq 1, \\ \Phi_{\Delta|2^{3I}-2||\omega|^I}(2\,\psi); 3I\neq 1, \\ \Phi_{\Delta|2^{3I}-2||\omega|^I}(2\psi); 3I\neq 1, \end{cases} \tag{4.41}$$

for all $\omega \in Q_1$ and for each $\psi > 0$.

Proof. Let us take

$$\Phi_{\omega_1,\omega_2,\omega_3}(\psi) = \begin{cases} \Phi_\Delta(\psi); \\ \Phi_{\Delta\sum_{i=1}^3|\omega_i|^I}(\psi); \\ \Phi_{\Delta\prod_{i=1}^3|\omega_i|^I}(\psi); \\ \Phi_{\Delta\left\{\prod_{i=1}^3|\omega_i|^I+\sum_{i=1}^3|\omega_i|^{3I}\right\}}(\psi); \end{cases}$$

for all $\omega_1,\omega_2,\omega_3 \in Q_1$ and for each $\psi > 0$ in Theorem 4.7. Replacing $\omega_1 = A_K^S\omega_1; \omega_2 = A_K^S\omega_2; \omega_3 = A_K^S\omega_3$ and using (RNS) in the above, we reach

$$\Phi_{A_K^S\omega_1, A_K^S\omega_2, A_K^S\omega_3}\left(A_K^S\psi\right)=\begin{cases}\Phi_{\Delta}(A_K^S\psi); \\ \Phi_{\Delta\sum_{i=1}^{3}|A_K^S\omega_i|^l}(A_K^S\psi); \\ \Phi_{\Delta\prod_{i=1}^{3}|A_K^S\omega_i|^l}(A_K^S\psi); \\ \Phi_{\Delta\left\{\prod_{i=1}^{3}|A_K^S\omega_i|^l+\sum_{i=1}^{3}|A_K^S\omega_i|^{3l}\right\}}(A_K^S\psi);\end{cases}=\begin{cases}\to 1 \text{ as } S\to\infty; \\ \to 1 \text{ as } S\to\infty; \\ \to 1 \text{ as } S\to\infty; \\ \to 1 \text{ as } S\to\infty.\end{cases}$$

Therefore (4.31) holds for each $\omega_1, \omega_2, \omega_3 \in Q_1$ and for all $\psi > 0$.
Now, as a consequence of (4.32), we have

$$\Phi_{\omega,\omega,\omega}(\psi)=\Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(3\psi)=\frac{1}{3}\begin{cases}\Phi_{\Delta}(3\psi); \\ \Phi_{\frac{3\Delta|\omega|^l}{|2|^l}}(3\psi); \\ \Phi_{\frac{\Delta|\omega|^{3l}}{|2|^{3l}}}(3\psi); \\ \Phi_{\frac{4\Delta|\omega|^{3l}}{|2|^{3l}}}(3\psi)\end{cases} \text{ and } \Phi_{A_K\omega, A_K\omega, A_K\omega}(A_K\psi)=\begin{cases}\Phi_{A_K^{-1}\Delta}(\psi); \\ \Phi_{A_K^{l-1}3\Delta|\omega|^l}(\psi); \\ \Phi_{A_K^{3l-1}\Delta|\omega|^{3l}}(\psi); \\ \Phi_{A_K^{3l-1}4\Delta|\omega|^{3l}}(\psi);\end{cases}=\begin{cases}L\Phi(\omega,\omega,\omega); \\ L\Phi(\omega,\omega,\omega); \\ L\Phi(\omega,\omega,\omega); \\ L\Phi(\omega,\omega,\omega);\end{cases}$$

for each $\omega \in Q_1$ and for all $\psi > 0$.
For K=0, we have L= $A_0^{-1} = 2^{-1}$. From (4.33), we obtain

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}\left(\frac{\left(2^{-1}\right)^{1-0}}{1-\left(2^{-1}\right)}\psi\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \Rightarrow \Psi_{\Omega(\omega)-\Lambda_A(\omega)}(\psi) \geq \Phi_{\Delta}(3\psi)$$

For K=1, we have L= $A_1^{-1} = \left(\frac{1}{2}\right)^{-1}$. From (4.33), we obtain

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)}\left(\frac{(2)^{1-1}}{1-2}\psi\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \Rightarrow \Psi_{\Omega(\omega)-\Lambda_A(\omega)}(\psi) \geq \Phi_{\Delta}(-3\psi)$$

The rest of the cases are analogous to the previous ones.

4.6 Quadratic Case: Stability Analysis: Radus Method

Theorem 4.14 Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.19) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the condition

$$\lim_{S\longrightarrow\infty}\Phi_{A_K^S\omega_1, A_K^S\omega_2, A_K^S\omega_3}\left(A_K^{2S}\psi\right)=1; A=\begin{cases}2 & K=1 \\ \frac{1}{2} & K=-1\end{cases} \tag{4.42}$$

for each $\omega_1, \omega_2, \omega_3 \in Q_1$ and for all $\psi > 0$. In the presence of $L = L[K]$ be a mapping have the properties

$$\Phi_{\frac{\omega}{2},\frac{\omega}{2},\frac{\omega}{2}}(3\psi)=\Phi_{\omega,\omega,\omega}(\psi);\ \Phi_{A_K\omega, A_K\omega, A_K\omega}(A_K^2\psi)=\Phi_{\omega,\omega,\omega}(L\psi)$$

(4.43)
for each $\omega \in Q_1$ and for all $\psi > 0$. Then there exists unique quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Lambda_Q(\omega)}\left(\frac{L^{1-i}}{1-L}\psi\right) \geq \Phi_{\omega,\omega,\omega}(\psi) \tag{4.44}$$

where the mapping $\Lambda_Q(\omega)$ is formulated by

$$\Psi_{\Lambda_Q(\omega)}(\psi)=\lim_{S\longrightarrow\infty}\Psi_{\frac{\Omega\left(A_K^S\omega\right)}{A_K^{2S}}}(\psi) \tag{4.45}$$

for each $\omega \in Q_1$ and for all $\psi > 0$.
Proof. The proof is notably revised in a way analogous to Theorem 4.12.

Corollary 4.15 Assume Δ be a positive constant and I be a real number. Suppose an even mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.26) for all $\omega_1, \omega_2, \omega_3 \in Q_1$ and for each $\psi > 0$. Then there exists one and only one quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$\Psi_{\Omega(\omega)-\Lambda_Q(\omega)}(\psi) \geq \begin{cases} \Phi_\Delta(|9|\,\psi); \\ \Phi_{\Delta|2^I-4||\omega|^I}(2\,\psi); I \neq 1, \\ \Phi_{\Delta|2^{3I}-4||\omega|^I}(2\,\psi); 3I \neq 1, \\ \Phi_{\Delta|2^{3I}-4||\omega|^I}(2\psi); 3I \neq 1, \end{cases} \tag{4.46}$$

for all $\omega \in Q_1$ and for each $\psi > 0$.

Proof. The proof is notably revised in a way analogous to Corollary 4.8.

4.7 Additive - Quadratic Case: Stability Analysis: Radus Method

Theorem 4.16 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.27) where $\Phi : Q_1^3 \longrightarrow [0,\infty)$ be a mapping with the conditions (4.31) and (4.41) for each $\omega_1, \omega_2, \omega_3 \in Q_1$ and for all $\psi > 0$. If there exists $L = L[K]$ be a mapping have the properties (4.32) and (4.42) for each $\omega \in Q_1$. Then there exists unique additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the inequality

$$\Psi_{\Omega(\omega)-\Omega_A(\omega)-\Lambda_Q}\left(\frac{4L^{1-i}}{1-L}\psi\right) \geq T\,\{T\,\{\Phi_{\omega,\omega,\omega}(\psi), \Phi_{-\omega,-\omega,-\omega}(\psi)\},\; T\,\{\Phi_{\omega,\omega,\omega}(\psi), \Phi_{-\omega,-\omega,-\omega}(\psi)\}\} \tag{4.47}$$

where the mapping $\Lambda_A(\omega)$ and $\Lambda_Q(\omega)$ are formulated in (4.34) and (4.44) for each $\omega \in Q_1$ and for all $\psi > 0$.

Proof. The proof is notably revised in a way analogous to Theorem 4.10.

Corollary 4.17 Suppose a mapping $\Omega_A^Q : Q_1 \longrightarrow Q_2$ aligns with the disparity (4.29) for all $\omega_1, \omega_2, \omega_3 \in Q_1$ and for all $\psi > 0$. Then there exists one and only one additive mapping $\Lambda_A(\omega) : Q_1 \longrightarrow Q_2$ and a unique quadratic mapping $\Lambda_Q(\omega) : Q_1 \longrightarrow Q_2$ which aligns with the functional equation (1.1) and the disparity

$$\Psi_{\Omega(\omega)-\Lambda_A(\omega)-\Lambda_Q(\omega)}(\psi) \geq \begin{cases} \Phi_\Delta(|3|+|9|\,\psi); \\ \Phi_{\Delta\{|2^I-2|+|2^I-4|\}|\omega|^I}(2\,\psi); I \neq 1,2, \\ \Phi_{\Delta\{|2^{3I}-2|+|2^{3I}-4|\}|\omega|^I}(2\,\psi); 3I \neq 1,2, \\ \Phi_{\Delta\{|2^{3I}-2|+|2^{3I}-4|\}|\omega|^I}(2\psi); 3I \neq 1,2; \end{cases} \tag{4.48}$$

for all $\omega \in Q_1$ and for all $\psi > 0$.

5. APPLICATIONS VIA EXAMPLES

Example-5.1: Electrical Circuit Analysis

An electrical engineer is analyzing the voltage drops across various components in an electrical circuit. The voltage drop mapping is modelled by $\Omega(\omega) = \omega$, $\Omega(\omega) = \omega^2$ and $\Omega(\omega) = \omega + \omega^2$, where ω is the current flowing through the component. Utilizing the functional equation (1.1). Find the total voltage drop across the circuit when the currents flowing through the components are $2A$, $3A$, and $5A$.

Solution:

For additive case:

Let's assume the voltage drop mapping is $\Omega(\omega) = \omega$ Substituting the values $\omega_1 = 2, \omega_2 = 3$ and $\omega_3 = 5$ in (1.1), we get

$$\Omega(2(2)-3-5)+2\Omega(2)+\Omega(3)+\Omega(5)+\{\Omega(2)+\Omega(-2)\}+\frac{3}{2}\left\{\Omega(3)+\Omega(-3)\right\}+\frac{3}{2}\left\{\Omega(5)+\Omega(-5)\right\}+2\left\{\Omega(2)-\Omega(-2)\right\}$$
$$=\Omega(2+3)+\Omega(2+5)+3\Omega(2-3)+3\Omega(2-5)+\Omega(3+5)+\frac{1}{2}\left\{\Omega(3)-\Omega(-3)\right\}+\frac{1}{2}\left\{\Omega(5)-\Omega(-5)\right\}.$$

Evaluating this expression, we get

$$\Omega(-4)+4+3+5+0+0+0+8)=\Omega(5)+\Omega(7)+3\Omega(-1)+3\Omega(-3)+\Omega(8)+3+5=16.$$

Therefore, the total voltage drop across the circuit is approximately 16 volts.

For quadratic case

Let's assume the voltage drop mapping is $\Omega(\omega)=\omega^2$.

Taking the values $\omega_1=2,\omega_2=3$ and $\omega_3=5$ in the left-hand side of the equation (1.1), we have

$$\Omega(2(2)-3-5)+2\Omega(2)+\Omega(3)+\Omega(5)+\{\Omega(2)+\Omega(-2)\}+\frac{3}{2}\left\{\Omega(3)+\Omega(-3)\right\}+\frac{3}{2}\left\{\Omega(5)+\Omega(-5)\right\}+2\left\{\Omega(2)-\Omega(-2)\right\}$$
$$=\Omega(2+3)+\Omega(2+5)+3\Omega(2-3)+3\Omega(2-5)+\Omega(3+5)+\frac{1}{2}\left\{\Omega(3)-\Omega(-3)\right\}+\frac{1}{2}\left\{\Omega(5)-\Omega(-5)\right\}.$$

Evaluating this expression, we get

$$\Omega(-4)+8+9+25+8+27+75+0)=\Omega(5)+\Omega(7)+3\Omega(-1)+3\Omega(-3)+\Omega(8)+0+0=168.$$

Therefore, the total voltage drop across the circuit is approximately 168 volts.

For additive and quadratic case:

Let's assume the voltage drop mapping is $\Omega(\omega)=\omega+\omega^2$.

Taking the values $\omega_1=2,\omega_2=3$ and $\omega_3=5$ in the left-hand side of the equation (1.1), we have

$$\Omega(2(2)-3-5)+2\Omega(2)+\Omega(3)+\Omega(5)+\{\Omega(2)+\Omega(-2)\}+\frac{3}{2}\left\{\Omega(3)+\Omega(-3)\right\}+\frac{3}{2}\left\{\Omega(5)+\Omega(-5)\right\}+2\left\{\Omega(2)-\Omega(-2)\right\}$$
$$=\Omega(2+3)+\Omega(2+5)+3\Omega(2-3)+3\Omega(2-5)+\Omega(3+5)+\frac{1}{2}\left\{\Omega(3)-\Omega(-3)\right\}+\frac{1}{2}\left\{\Omega(5)-\Omega(-5)\right\}.$$

Evaluating this expression as above. The total voltage drop across the circuit is approximately 184 volts.

Example-5.2: Optimizing Solar Panel Energy Output

A solar panel manufacturer is analyzing the energy output of their panels. The energy output mapping is modeled by $\Omega(\omega)=\omega,\ \Omega(\omega)=\omega^2$ and $\Omega(\omega)=\omega+\omega^2$, where ω is the number of panels installed. Suppose the manufacturer wants to install panels on three different rooftops: (100 panels) rooftop A, (200 panels) rooftop B, and (300 panels) rooftop C. Utilizing the functional equation (1.1), find the maximum energy output of their panels.

Ans- Additive-(700volts), Quadratic-(650,000volts), Additive-Quadratic-(580,700volts).

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