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Topological and Coincidence Degree based on the Degree of Non-Densifiability and Applications

Djilali Benamara, Abdelghani Ouahab,
Laboratory of Mathematics, Djillali Liabes University of Sidi-Bel-Abbes,
PoBox 89, 22000 Sidi-Bel-Abbès, Algeria
Email: djillali.benamara@gmail.com, abdelghani.ouahab@univ-sba.dz

Juan J. Nieto
Departamento de Analisis Matématico, Facultad d Matematicas Universidad de
Santiago de Compostela, Santiago de Compostela, 15782, Spain}
Email: juanjose.nieto.roig@usc.es

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Abstract:

Using the concept of the degree of nondensifiability in Banach spaces, we develop a new version of the topological degree based on retraction mappings. This approach combines the degree of nondensifiability with the Leray--Schauder degree. As applications, we establish Leray--Schauder and Schaefer type fixed point theorems. Moreover, we introduce a version of Mawhin's coincidence degree via this framework for monotone operator contractions with respect to the degree of nondensifiability. Our results extend and unify several classical and recent contributions available in the literature. Finally, we apply the main theoretical results to investigate the existence of solutions for certain classes of differential equations subject to boundary value conditions.

Keywords: Degree of nondensifiability; α-dense curves; k-φ_d-contractions; topological degree; fixed points; Mawhin's coincidence degree; continuation theorem; differential equations; boundary value problems.

Introduction

Fixed point theory is a vibrant and extensively developed area of modern mathematics. It plays a central role in nonlinear analysis, providing powerful methods for proving the existence of

solutions to a wide variety of nonlinear problems arising in disciplines such as differential equations, economics, mechanics, and biology.

In many practical situations, the solution of a given problem can be formulated as a fixed point of an appropriate operator. The existence of such fixed points is generally ensured by imposing suitable assumptions on the mappings involved—whether they are single-valued or multivalued—or on the structural properties of the underlying Banach space in which the problem is posed.

The notion of measure of noncompactness (MNC) was introduced by Kuratowski and has since played a central role in functional analysis. Relying on this concept, Darbo proved a fixed point theorem formulated in terms of MNC, which generalizes both Banach's contraction principle and Schauder's topological fixed point theorem . Since then, the theory of measures of noncompactness and their applications has been extensively developed and broadened in many directions; see, for instance, for comprehensive treatments.

In 1997, Mora and Cherruault introduced the notions of α-dense curves and densifiable sets in metric spaces, extending the classical concept of space-filling curves (see). The family of densifiable sets forms a class that lies strictly between Peano continua and the class of connected precompact sets, thereby offering a more flexible framework for the study of the geometric and topological structure of metric spaces. For further developments and applications of these ideas, we refer to .

In recent years, a number of authors have developed fixed point theorems using the concept of the degree of nondensifiability, which is rooted in the theory of α-dense curves. This framework provides an alternative to the classical measure of noncompactness (MNC) for establishing fixed point results. We refer, for instance, to and the references therein for further contributions in this direction.

As indicated in , the measure of nondensifiability, or the Kuratowski measure of noncompactness, provides a direct approach to topological fixed point theorems and to existence results in many areas. However, these measures are not characterized solely by a reduction principle that allows one to shrink bounded and closed sets to compact ones.

Recently, the concept of the weak degree of nondensifiability was introduced by García and Mora .

One of the fundamental tools in fixed point theory is the concept of topological degree, which was introduced by Brouwer in 1912 for finite-dimensional spaces. This notion provided a powerful method for proving the existence of solutions to nonlinear equations in Euclidean spaces.

Later, Leray and Schauder extended this concept to infinite-dimensional Banach spaces in the case of compact perturbations of the identity operator. Their extension, known as the Leray–Schauder degree, has become a central instrument in nonlinear functional analysis and has been widely used in the study of differential and integral equations.

Gaines and Mawhin introduced the coincidence degree theory in the 1970's for the analysis of functional and differential equations . Subsequently, Mawhin further developed this theory and made significant contributions to its advancement. For this reason, it is often referred to as Mawhin's coincidence degree theory. Coincidence degree theory is a powerful tool in the study of nonlinear equations, particularly for establishing the existence of solutions. It has been widely applied to the investigation of periodic solutions of nonlinear differential equations, used in partial differential equations, bifurcation theory and fractional differential equations, and numerous authors have employed it in their research (see and the references therein).

The goal of the present paper is to develop new fixed point theorems by using the concept of the degree of nondensifiability in Banach spaces.

Our main objectives are:

To introduce necessary definitions, notation, and fundamental results concerning the degree of nondensifiability, including several propositions and theorems from the existing literature that will be used throughout the paper.

To employ the concept of the degree of nondensifiability to introduce a Leray–Schauder type topological degree, extending classical topological degree theory to a broader class of operators.

To introduce the notion of coincidence degree based on the degree of nondensifiability and establish several continuation theorems that generalize existing results in the literature.

To apply our theoretical results to certain classes of second order differential equations in Hilbert space, particularly focusing on periodic boundary value problems.

These objectives extend and unify several classical and recent contributions available in the literature, providing new tools for analyzing nonlinear problems in functional analysis and differential equations.

In this section, we introduce the fundamental mathematical framework and methodological approach used in developing our main results. We begin by establishing the necessary notation, definitions, and auxiliary results concerning the degree of nondensifiability. Subsequently, we develop the topological degree theory based on this concept and introduce the coincidence degree framework.

Preliminaries and fundamental definitions

In this subsection, we introduce the notation, definitions, and auxiliary results that will be used throughout the paper. Let X be a metric space (in particular, a normed space). We denote by

$$P(X)=\{A \subset X: A \neq \emptyset\}$$

the family of all nonempty subsets of X. Furthermore, we define

$$P_{cl}(X)=\{A \in P(X): A \text{ is closed}\},$$

$$P_b(X)=\{A \in P(X): A \text{ is bounded}\},$$

$$P_{cv}(X)=\{A \in P(X): A \text{is convex}\},$$

$$P_{cp}(X)=\{A \in P(X): A \text{is compact}\},$$

and

$$P_{arc}(X)=\{A \in P(X): A \text{is path-connected}\}.$$

The concept of the measure of noncompactness enables us to quantify and compare the degree of noncompactness of subsets of X. For further details and properties of this notion, we refer the reader to .

Definition 3.1. *Let (X, d) be a complete metric space. A map $\varphi: P_b(X) \to [0,+\infty)$ is called a measure of noncompactness (MNC) defined on X if the following properties are fulfilled:*

1. *Regularity: For any $B \in P_b(X)$, $\varphi_d(B)=0$ if and only if B is a relatively compact set.*
2. *Invariant under closure: $\varphi(B)=\varphi(\overline{B})$, for any $B \in P_b(X)$.*
3. *Semi-additivity: $\varphi(B_1 \cup B_2)=max\{\varphi(B_1), \varphi(B_2)\}$, for any $B_1, B_2 \in P_b(X)$.*

The notion of α-dense curve was introduced by G. Mora in 1997 but the notion of DND appeared in 2015 as an application of such a theory.

Definition 3.2. *Let (X, d) be a metric space, $\alpha \geq 0$ and $B \in P_b(X)$ a mapping $\gamma \in C([0,1], \mathbb{R}_+)$ is called curve α-dense in B if*

$\gamma([0,1]) \subset B$.

For any $x \in B$, there exists $t \in [0,1]$ such that

$$d(x, \gamma(t))<\alpha.$$

The bounded sets B is said to be densifiable, if for all $\alpha>0$ there exists an α-dense curve in B.

For any $\alpha \geq 0$ and $B \in P_b(X)$ we denote the sets α-dense curves by $\Gamma_{\alpha,B.}$

Definitin3.3. *The degree of nondensifiability (DND) is a mapping $\varphi_d: P_b(X) \to \mathbb{R}_+$ defined by*

$$\varphi_d(B)=inf\{\alpha \geq 0: \Gamma_{\alpha,B} \neq \emptyset\}, B \in P_b(X).$$

Remark 3.1.

From the definition 3.3, we deduce

$$\varphi_d(B) \leq \delta(B) \text{for all} B \in P_b(X),$$

where $\delta(B)=\sup\{d(x, y): x, y \in B\}$. This implies that φ_d is well defined.

By the Hahn-Mazurkiewicz theorem (see, for instance,) we know that a set $B \in P_b(X)$ is a Peano Continuum, if and only if, it is the continuous image of $[0,1]$. So, the $DND \varphi_d$ measures, in the specified sense, the distance from B to the class of Peano Continua contained in it.

Example 1. *[42]Let X be a Banach space and $\acute{B}(0,1) \subset X$ be a closed unit ball, then*

$$\varphi_d(\overline{B}(0,1)) = \begin{cases} 1 & \text{if } X \text{has infinite dimension;} \\ 0 & \text{if } X \text{has finite dimension.} \end{cases}$$

Remark3.2. *García and Mora , prove that the degree of nondensifiability is not a measure of noncompactness.*

Now, we prove some relationships between the Kuratowski, Hausdorff $MNC\chi$ and the degree of nondensifiability φ_d.

Proposition 3.1.[18.26] *Let X be a metric space and $B \in P_{b,arc}(X)$. Then,*

$$\chi(B) \leq \varphi_d(B) \leq 2\chi(B),$$

where χ is a Hausdorff measure defined as follows

$$\chi(B) = inf\{\epsilon > 0: B \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)\},$$

and

$$\frac{1}{2}\varpi(B) \leq \varphi_d(B) \leq \varpi(B).$$

where ϖ is a Kuratowskii MNC defined by

$$\varpi(B) = inf\{\epsilon > 0: B \subset \bigcup_{i=1}^{n} B_i; Diam(B_i) \leq \epsilon\}.$$

Remark 3.3. *Notice that for every $B \in P_b(X)$ we have*

$$\chi(B) \leq \varpi(B) \leq 2\chi(B).$$

Definition 3.4. *Let X be a metric space, $C \subset X$ a nonempty subset, $N: X \to X$ and φ_d a degree of nondensifiability (DND).*

$N$is k-φ_d- contraction if

$$\varphi_d(N(B)) \leq k\varphi_d(B), B \in P_{b,cv}(X),$$

whit $k \in (0,1)$.

$N$is φ_d-condensing if

$$\varphi_d(N(B)) < \varphi_d(B), B \in P_{b,cv}(X),$$

with $\varphi_d(B) \neq 0$.

We give some neutral properties of the DND in the following result proved in [20, 21, 26, 24].

Proposition 3.2. *Let (X, d) be a complete metric space and φ_d be DND then*

1. *Regularity: For any $B \in P_{b,arc}(X)$, $\varphi_d(B)=0$ if and only if B is a relatively compact set.*

2. *Invariant under closure:* $\varphi_d(B)=\varphi_d(\overline{B})$, *for any* $B \in P_{b,arc}(X)$.

If X is a Banach space

1. $\varphi_d(x_0+B)=\varphi_d(B)$, *for any* $B \in P_b(X)$.
2. *Semi-homogeneity:* $\varphi_d(\lambda B)=|\lambda|\varphi_d(B)$, *for all* $B \in P_b(X)$ *and* $\lambda \in \mathbb{R}$.
3. $\varphi_d(co(B)) \leq \varphi_d(B)$ *for all* $B \in P_b(X)$, co *is convex hull.*
4. $\varphi_d(co(B \cup C)) \leq max(\varphi_d(coB), \varphi_d(coC))$ *for all* $B, C \in P_b(X)$.
5. $\varphi_d(B+C) \leq \varphi_d(B)+\varphi_d(C)$, $B, C \in P_b(X)$.
6. *Generalized Cantor's intersection theorem: If* $(B_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of nonempty, convex closed, and bounded subsets of* X, *and* $\lim_{n\to\infty} \varphi_d(B_n)=0$, *then the intersection* $\bigcap_{n=1}^{\infty} B_n$ *is nonempty and compact.*

Topological degree based on the degree of nondensifiability

Before introducing the concept of topological degree via the degree of nondensifiability, we present some propositions and corollaries that will be used in the definition of our degree and to provide extended versions of Darbo's theorem, Schaefer's theorem, and the Leray–Schauder nonlinear alternative.

Proposition 3.4. *Let* X *and* Y *be Banach spaces, let* $D \subset X$ *be a closed and bounded set, and let* $f: D \to Y$ *be a continuous compact mapping. Then there exists a continuous compact extension* $\hat{f}: X \to Y$ *such that*

$$\hat{f}|_D=f \text{and} \hat{f}(X) \subset convf(D).$$

Proof. By the Dugundji Extension Theorem , there exists a continuous extension $\hat{f}$ of f to the whole space X such that

$$\hat{f}(X) \subset convf(D).$$

Since f is compact, the set $f(D)$ is relatively compact in Y. Moreover, the convex hull of a relatively compact set is also relatively compact. Consequently, $convf(D)$ is relatively compact, and hence $\hat{f}: X \to Y$ is a continuous compact mapping.

Proposition 3.4. *Let* X *be a Banach space,* $U \subseteq X$ *be open bounded, and let* $f: \overline{U} \to X$ *be a continous* φ_D*-condensing mapping. Define the set of fixed points by*

$$Fix(f)=\{x \in \overline{U}: f(x)=x\}.$$

Then there exists a compact, convex set $S_* \subseteq X$ *such that:*

Fix$(f) \subseteq S_*$;

if $x_0 \in \overline{conv}(S_* \cup \{f(x_0)\})$*, then* $x_0 \in S_*$*;*

$S_* = \overline{conv} f(S_* \cap \overline{U})$.

Movere S_* *is retraction of* X.

Proof.

$$S = \{K : \text{Fix}(f) \subseteq K, K \text{closed and convex}, f(\overline{U} \cap K) \subseteq K, \text{and (b) holds for} K\}.$$

Note that $\overline{conv} f(\overline{U}) \in S$; hence $S \neq \emptyset$. We set

$$S_* = \bigcap_{K \in S} K.$$

Evidently, S_* satisfies (a), (b), and (c), and it is closed and convex. It remains to show that it is compact.

Suppose, by contradiction, that S_* is not compact. Then there exists a sequence $C_1 = \{x_n\}_{n\geq 1} \subseteq S_*$ with no convergent subsequence. By (c), we have

$$S_* = \overline{conv} f(\overline{U} \cap S_*).$$

Hence, there exists a countable set $E_1 \subseteq \overline{U} \cap S_*$ such that $C_1 \subseteq \overline{conv} f(E_1)$. Define

$$S_1 = \overline{conv} f(\overline{U} \cup C_1).$$

Then S_1 is separable, and so is $\overline{U} \cap S_1$. It follows that we can find countable sets $D_1 \subseteq S_1$ and $D_1^* \subseteq \overline{U} \cap S_1$ such that $D_1 = S_1$ and $D_1^* = \overline{U} \cap S_1$. Let

$$C_2 = C_1, \cup E_1 \cup D_1 \cup D_1^*.$$

Then

$$C_1 \subseteq C_2, conv f(\overline{U} \cap C_1) \subseteq C_2, \overline{conv} f(\overline{U} \cap C_1) \cap \overline{U} \subseteq \overline{U} \cap \overline{C_2}.$$

Using these relations and induction, we can construct a sequence $\{C_n\}_{n\geq 1}$ of subsets of S_* such that for all $n \geq 1$, the same properties hold.

$$C_n \subseteq C_{n+1}, \overline{conv} f(\overline{U} \cap C_n) \subseteq \overline{C_{n+1}}, \overline{conv} f(\overline{U} \cap C_n) \cap \overline{U} \subseteq \overline{U} \cap \overline{C_{n+1}}.$$

Set

$$C = \bigcup_{n\geq 1} C_n.$$

We show that

$$\overline{C} = \overline{conv} f(\overline{U} \cap C).$$

Indeed, $E_1 \subset U \cap C$ and $E_1 \subset C_2$ this implies that

$$C_1 \subset conv f(\overline{U} \cap C_2) \subset conv f(\overline{U} \cap C).$$

Thus

$$\overline{C_1} \subset \overline{conv} f(\overline{U} \cap C_2) \subset \overline{conv} f(\overline{U} \cap C).$$

Therefore

$$C_1 \subset \overline{conv} f(\overline{U} \cap C)$$

and

$$\overline{conv} f(\overline{U} \cap C_1) \subset \overline{conv} f(\overline{U} \cap C_2) \subset \overline{C_3} \subset \bigcup_{n\geq 1} \overline{C_n}.$$

By induction for every $n \in \mathbb{N}$, we obtain

$$C_n \subset \overline{conv} f(\overline{U} \cap C)$$

And

$$\overline{conv} \varphi(\overline{U} \cap C_n) \subset \bigcup_{n\geq 1} \overline{C_n}.$$

Then

$C \subset \overline{conv} f(\overline{U} \cap C)$(3.1)

and

$$\bigcup_{n\geq 1} \overline{conv} f(\overline{U} \cap C_n) \subset \bigcup_{n\geq 1} \overline{C_n}.$$

Since $(C_n)_{n\geq 1}$ is an increasing sequence of convex subsets, both $\bigcup_{n\geq 1} C_n, \bigcup_{n\geq 1} \overline{C_n}$ are convex. So,

$$\overline{conv}\left(\bigcup_{n\geq 1} \overline{conv} f(\overline{U} \cap C_n)\right) \subset \bigcup_{n\geq 1} \overline{C_n} \subset \overline{\bigcup_{n\geq 1} C_n}.$$

From [cov-00], we get

$$\overline{C} \subset \overline{conv} f(\overline{U} \cap C) \subset \overline{C}.$$

It follows that

$$\overline{C} = \overline{conv} f(\overline{U} \cap C).$$

By propositions [3.1], [3.2], we have

$$\begin{aligned} \chi(\overline{C}) &= \chi(\overline{conv} f(\overline{U} \cap C)) \\ &\leq \chi(\overline{conv} f(\overline{U} \cap \overline{C})) \\ &\leq \chi(\overline{conv} f(conv(\overline{U} \cap \overline{C}))) \\ &\leq \varphi_d(\overline{conv} f(\overline{conv}(\overline{U} \cap \overline{C}))) \\ &\leq \varphi_d(f(\overline{conv}(\overline{U} \cap \overline{C}))) \\ &\leq k\varphi_d(\overline{conv}(\overline{U} \cap \overline{C})). \end{aligned}$$

Thus,

$$\chi(\overline{C})) \leq 2k\chi(\overline{U} \cap \overline{C}) < \chi(\overline{U} \cap \overline{C}),$$

which is contradiction. This prove that $\overline{S}_*$ is compact.

Corollary 3.1. *Let $f: \overline{U} \cap S_* \to S_*$ be a continuous mapping, where S_* is defined in Proposition [3.4.]. If*

$$0 \notin (I\text{-}f)(\partial U),$$

where I denotes the identity operator on X. Then

$$0 \notin (I\text{-}f \circ r)(\partial(U \cap r^{-1}(U))).$$

where r is retraction of the X into S_*.

Proof. Assume, by contradiction, that

$$0 \in (I\text{-}f \circ r)(\partial(U \cap r^{-1}(U))).$$

Then there exists $x \in \partial(U \cap r^{-1}(U))$ such that

$$x\text{-}f(r(x))=0,$$

that is,

$$x=f(r(x)) \in S_*.$$

Since $x \in \partial(U \cap r^{-1}(U))$, we have either $x \in \partial U$ or $x \in \partial r^{-1}(U)$.

Case 1: $x \in \partial U$. Then, $x \in \overline{U} \cap S_*$, thus $r(x)=x$ and hence

$$x=f(x),$$

which implies $0 \in (I\text{-}f)(\partial U)$, a contradiction.

Case 2: $x \in \partial r^{-1}(U)$. Then $x \in \overline{r^{-1}(U)}\{r^{-1}(U)$, which implies, that there exists sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{r^{-1}(U)}$ such that

$$\lim_{n\to\infty} x_n = x \text{and} r(x) \in X \setminus U$$

By continuity of r, we get $r(x) \in \overline{U} \cap S_*$, and $r(x) \in \partial U$, then $f(r(x)) \in S_*$. Hence

$$r(x)=r(f(r(x)))=f(r(x)),$$

which implies that

$$0 \in (I\text{-}f)(\partial U),$$

again a contradiction.

In both cases we arrive at a contradiction. Therefore,

$$0 \notin (I\text{-}f \circ r)(\partial(U \cap r^{-1}(U))).$$

By using Propositions [3.4] and [3.3], together with Corollary [3.1], we introduce the definition of the topological degree via the degree of non-densifiability.

Definition 3.5. *Let X be a Banach space, $U \subset X$ be a bounded open set, and $f: U \to X$ be k-φ_d-contraction with $k \in (0, \frac{1}{2}]$. Suppose that $0 \notin (I\text{-}f)(\partial U)$, and let $S_* \subset X$ be a compact convex set as in Proposition [3.4], and $r: X \to S_*$ a retraction onto S_*.*

If $U \cap S_* = \emptyset$, we define the degree

$$d_{DT}(I\text{-}f, U, 0) = 0.$$

If $U \cap S_* \neq \emptyset$, we define

$$d_{DT}(I\text{-}f, U, 0) = d_{LS}(I\text{-}f \circ r, U \cap r^{-1}(U), 0).$$

where d_{LS} is Leary-Schauder degree.

Remark 3.4. *From Corollary [3.1], it is clear that if*

$$0 \notin (I\text{-}f)(\partial U),$$

then

$$0 \notin (I\text{-}f \circ r)(\partial(U \cap r^{-1}(U))).$$

Therefore, the Leray–Schauder degree given in Definition [3.5] is well defined.

Using the homotopy invariance property of the Leray–Schauder degree, it follows that Definition [3.5] does not depend on the choice of the retraction.

Lemma 3.2. *Definition [3.5] is independent of the choice of the retraction.*

Proof. Let $r_1, r_2: X \to S_*$ be two retractions of X onto S_*. Define the homotopy $H: [0,1] \times X \to S_*$ by

$$H(t, x) = f(h_t(x)), (t, x) \in [0,1] \times X,$$

where

$$H_t(x) = tr_1(x) + (1\text{-}t)r_2(x), (t, x) \in [0,1] \times X.$$

For every $t \in [0,1]$, the mapping h_t is a retraction of X onto S_*. Moreover, we have

$$x \neq f(h_t(x)), (t, x) \in \partial(U \cap r_1^{-1}(U) \cap r_2^{-1}(U)).$$

Hence, by Definition [3.5], the topological degree is well defined and satisfies

$$d_{DT}(I\text{-}f, U, 0) = d_{LS}(I\text{-}f \circ h_t, U \cap r_1^{-1}(U) \cap r_2^{-1}(U), 0).$$

By the homotopy invariance property of the Leray–Schauder degree, we obtain

$$d_{LS}(I\text{-}f \circ h_0, U \cap r_1^{-1}(U) \cap r_2^{-1}(U),0)=d_{LS}(I\text{-}f \circ h_1, U \cap r_1^{-1}(U) \cap r_2^{-1}(U),0).$$

We easily verify that

$$0 \notin U \cap r_1^{-1}(U) \setminus \left(U \cap r_1^{-1}(U) \cap r_2^{-1}(U)\right);$$

and

$$0 \notin U \cap r_2^{-1}(U) \setminus \left(U \cap r_1^{-1}(U) \cap r_2^{-1}(U)\right);$$

Therefore, by the excision property of the Leray–Schauder degree, we have

$d_{LS}(I\text{-}f \circ r_1, U \cap r_1^{-1}(U),0)=d_{LS}(I\text{-}f \circ r_1, U \cap r_1^{-1}(U) \cap r_2^{-1}(U),0)$,(3.2)

and

$$d_{LS}(I\text{-}f \circ r_2, U \cap r_2^{-1}(U),0)=d_{LS}(I\text{-}f \circ r_2, U \cap r_1^{-1}(U) \cap r_2^{-1}(U),0).$$

Combining these equalities with (3.2), we conclude that

$$d_{LS}(I\text{-}f \circ r_1, U \cap r_1^{-1}(U),0)=d_{LS}(I\text{-}f \circ r_2, U \cap r_2^{-1}(U),0).$$

The topological degree vis a degree of nondensifiability in normed space conserves the basic features of the degree Leray-Schauder.

Theorem 3.5.*Let X be a Banach space, $U \subset X$ be an open bounded subset and $f: U \to X$ be a continuous $\frac{k}{2}$-φ_d-contractions map,* $0<\frac{k}{2}<1$. *If* $0 \notin (I\text{-}f)(\partial U)$*, then there exists an integer* $d_{DT}(I\text{-}f, U, 0)$ *satisfying the following properties:*

(Solvability) $d_{DT}(f, U, 0) \neq 0$ *then* $x\text{-}f(x)=0$ *has a solution in* U*;*

(Homotopy invariance) If $H: [01] \times \overline{U} \to X$ *is continuous,* $0 \notin \cup_{t\in[0,1]}(I\text{-}H(t, \partial U))$ *and*

$$\varphi_d(H(t,B)) \leq \frac{k}{2}\varphi_d(B) \text{for all} B \in P_{b,cv}(\overline{U}),$$

then $d_{DT}(I\text{-}H(t,\cdot), U, 0)$ *does not depend on* $t \in [0,1]$*;*

$d_{DT}(I\text{-}f+y_0, U, y_0)=d_{DT}(I\text{-}f, U, 0)$*for all* $y_0 \in X$.

Let $g \in C(\overline{U}, X)$ *be a compact application. If* $f_{|_{\partial U}}=g_{|_{\partial U}}$*, then*

$$d_{DT}(I\text{-}f, U, 0)=d_{SL}(I\text{-}g, U, 0).$$

We present several fixed point results based on this notion of topological degree. In particular, we establish an extension of Darbo's fixed point theorem.

Theorem3.6.*Let $U \subset X$ be a bounded, open set, and let $f: \overline{U} \to X$ be a continuous $\frac{k}{2}$-φ_d-contraction mapping such that*

$$f(\partial U) \subset U.$$

Then f has at least one fixed point in U.

If f has a fixed point on ∂U, there is nothing to prove. Thus, we assume that

$$0 \notin (I\text{-}f)(\partial U),$$

Proof. Fix an element $x_0 \in U$ and define the homotopy $H: [0,1] \times U \to X$ by

$$H(t,x)\text{=}tf(x)\text{+}(1\text{-}t)x_0, (t,u) \in [0,1] \times U.$$

We claim that

$$H(t,x) \neq u \text{for all} t \in [0,1] \text{and all} u \in \partial U.$$

Indeed, suppose that there exist $t \in [0,1]$ and $x \in \partial U$ such that $H(t,x)$=x. Then

$$tf(x)\text{=}tx_0\text{+}(x\text{-}x_0).$$

Since $x \neq x_0$, we must have $t \neq 0$, and hence

$$f(x)\text{=}x_0\text{+}\frac{1}{t}(x\text{-}x_0).$$

Because $t \in (0,1]$, it follows that $\frac{1}{t}$>1, and thus $f(x) \notin U$, which contradicts the assumption $f(\partial U) \subset U$. Thus,

$$0 \notin \bigcup_{t\in[0,1]}(I\text{-}H(t,\partial U)).$$

For every $B \in P_{cv}(\overline{U})$,

$$\varphi_d(H(t,B))\text{=}\varphi_d(tf(B)\text{+}(1\text{-}t)\{x_0\})$$
$$\leq \varphi_d(tf(B)) \leq t\varphi_d(f(B)) \leq \frac{k}{2}\varphi_d(B).$$

Hence

$$\varphi_d(H(t,B)) \leq \frac{k}{2}\varphi(B) \text{for all} B \in P_{cv}(\overline{U}).$$

Therefore, $\{H(t,\cdot)\}_{t\in[0,1]}$ is a homotopy of $\frac{k}{2}$-φ_d- contraction mappings, the homotopy invariance property of the topological degree defined in Definition [3.5] (see Theorem [3.53]) yields

$$d_{DT}(I\text{-}f, U, 0)\text{=}d_{DT}(I\text{-}x_0, U, 0)\text{=}d_{LS}(I\text{-}x_0, U \cap r^{-1}(U),0)\text{=}1.$$

where r is a retraction of X onto the compact convex set S_* given in Proposition [3.4]. Therefore, there exists $\hat{x} \in U$ such that

$$\hat{x}\text{=}f(\hat{x}),$$

which completes the proof.

Now, for the second result, we establish a version of Schaefer's fixed point theorem. **Theorem 3.7.** *Let $f: X \to X$ be a continuous $\frac{k}{2}$-φ_d-contraction mapping and define*

$$M_f{:=}\{u \in X: \exists\lambda \in (0,1) \text{such that} x{=}\lambda f(x)\}.$$

Then the following alternative holds:

1. *M_f is unbounded; or*
2. *f has at least one fixed point in X.*

Proof. Assume that M_f is bounded. Then there exists $\rho{>}0$ such that

$$M_f \subset B_\rho{:=}\{x \in X: \| x \| {<}\rho\}.$$

For each $t \in (0,1)$, consider the following continuous mapping

$$x \mapsto tf(x), x \in \overline{B}_\rho.$$

By definition of M_f, the equation

$$x{=}tf(x)$$

has no solution on ∂B_ρ for any $t \in (0,1)$. Hence,

$$0 \notin (I\text{-}tf)(\partial B_\rho){=}H(t, \partial B_\rho), \forall t \in (0,1).$$

Thus, the application H: $[0,1] \times X \to X$ given by

$$H(t,x){=}x\text{-}tf(x), (t,x) \in [0,1] \times \overline{B}_\rho$$

defines a homotopy and each $t \in [0,1]$ we map tf is a $\frac{k}{2}$-φ_d-contraction. Then, for the Definition [3.5], we have for all $t \in [0,1]$

$$d_{DT}(I\text{-}tf, B_\rho, 0){=}d_{LS}(I\text{-}tf, B_\rho \cap r^{-1}(B_\rho),0).$$

By the homotopy invariance property of the Leray–Schauder degree (see), we obtain

$$d_{DT}(I\text{-}f, B_\rho, 0){=}d_{LS}(I\text{-}f, B_\rho \cap r^{-1}(B_\rho),0){=}d_{LS}(I, B_\rho \cap r^{-1}(B_\rho),0){=}1.$$

Consequently, the operator f has at least one fixed point in B_ρ.

As a consequence of the homotopy invariance property of the topological degree, we state and prove the following Leray-Schauder type alternative.
Theorem 3.8.*Let X be a Banach space, $U \subset E$ an open bounded subset, such that* $0 \in U$*. Assume that*

$$f: \overline{U} \to X$$

is a continuous and $\frac{k}{2}$-φ_d-contractive mapping. Then one of the following alternatives holds:

Ahas a fixed point in U;

there exist $x \in \partial U$ *and* $\lambda \in (0,1)$ *such that*

$$x = \lambda f(x).$$

Proof. Assume that alternative (ii) does not hold and that f has no fixed points on ∂U (otherwise, there is nothing to prove).

Then, for every $x \in \partial U$ and every $\lambda \in [0,1]$, we have

$$x \neq \lambda f(x) + (1-\lambda)0.$$

Consider the homotopy $H: [0,1] \times \overline{U} \to X$ defined by

$$H(t,x) = tf(x), (t,x) \in [0,1] \times \overline{U}.$$

Clearly, H is continuous and

$$0 \notin \bigcup_{t \in [0,1]} (I - H(t, \partial U)).$$

Moreover,

$$\varphi_d(H(t,B)) \leq \frac{k}{2} \varphi_d(B), \text{for all} B \in P_{b,cv}(\overline{U}).$$

Therefore, by Theorem [3.5] and the Leray–Schauder degree, we obtain

$$d_{CDD}(I-f, U, 0) = d_{CDD}(I, U, 0) = d_{LS}(I, U \cap r^{-1}(U), 0) = 1.$$

Consequently, by Theorem [3.5], there exists $x \in U$ such that $x = f(x)$.

Fredholm operators and coincidence degree

Let X and Y be Banach spaces. We first recall some basic notions from the theory of linear Fredholm operators (see, for instance,[7]).

Definition 3.6. *A bounded linear operator* $L: dom(L) \subset X \to Y$ *is said to be a Fredholm operator of index zero if the following conditions are satisfied:*

1. *the range* $\mathrm{Im}(L)$ *is a closed subspace of* Y*;*
2. *the kernel* $Ker(L)$ *and the cokernel* $Coker(L)$ *are finite-dimensional;*
3. $dimKer(L) = dimCoker(L)$.

For every linear Fredholm operator L, there exist two projections $P_L: X \to X$ and $Q_L: Y \to Y$ such that

$$Ker(L) = \mathrm{Im}(P_L) \text{and} \mathrm{Im}(L) = Ker(Q_L).$$

Remark 3.5.*The projections* P_L *and* Q_L *are not unique and depend on the choice of complementary subspaces.*

If we define the operator

$$L_{P_L}: dom(L) \cap Ker(P_L) \rightarrow \mathrm{Im}(L)$$

as the restriction of L to $dom(L) \cap Ker(P_L)$, then L_{P_L} is a linear isomorphism. Consequently, we can define the operator

$$K_{P_L}:\mathrm{Im}(L) \rightarrow dom(L)$$

by

$$K_{P_L}=L^{-1}_{P_L}.$$

Let $Coker(L)$=Y/Im(L), and let

$$\pi_L: Y \rightarrow Coker(L)$$

be the canonical projection defined by

$$\pi_L(y)=y+\mathrm{Im}(L).$$

Moreover, let

$$J_L: Coker(L) \rightarrow Ker(L)$$

be a linear continuous isomorphism.

Then the equation

$$Lx=y, y \in Y,$$

is equivalent to the equation

$$x=P_L x+(J_L\pi_L+K_L)y,$$

where the operator $K_L: Y \rightarrow X$ is defined by

$$K_L=K_{P_L}(I\text{-}Q_L).$$

Lemma 3.3. *Let $N: U \subset X \rightarrow Y$ be an operator. The problem*

$$x \in D(L) \cap U, Lx=Nx,$$

is equivalent to the fixed point problem

$$x=P_L x+J_L^{-1}QNx+K_L Nx, x \in U.$$

Theorem3.9[Equivalence between coincidence and fixed points]. [27] *Let X and Z be two vector spaces over the same scalar field, and let*

$$L: domL \subset X \rightarrow Y$$

be a linear operator. Let $U \subset X$ and let

$$N: U \rightarrow Y$$

be a mapping. Assume further that there exists a linear injective (one-to-one) mapping

$$J_L: cokerL \to kerL.$$

Then a point $x_0 \in domL \cap U$ *is a solution of the coincidence equation*

$$Lx=Nx$$

if and only if x_0 *is a fixed point of the following mapping*

$$M_{J_L}: U \to X$$

defined by

$$M_{J_L}(x)=P_L x\text{-}J_L \pi_L N(x)+K_L N(x),$$

for every pair (P_L, Q_L) *of exact projections associated with L. In other words, we have the identity*

$$(L\text{-}N)^{-1}(0)=(I\text{-}M_{J_L})^{-1}(0).$$

Let X and Y be Banach spaces, and $U \subset X$ be a bounded open set, $L: dom(L) \subset X \to Y$ a linear Fredholm operator of index zero, and let $N: U \to Y$ be a continuous operator.

Definition 3.7. *A mapping N is said to be an* $(L,N)-k\text{-}\varphi_d$*-contraction if the following conditions hold:*

1. *The mapping*

$$J_L \pi_L N: \overline{U} \to cokerL$$

is continuous and $\pi_L N(\overline{U})$ *is bounded.*

2. *The mapping*

$$K_L N: \overline{U} \to X$$

is a $\frac{k}{4}$*-*φ_d*- contraction,* $k \in (0,2)$.

By adapting the method developed in [27, Lemma X.I.1], we can prove the following lemma.

Lemma 3.4. *Definition[3.7] is independent of the choice of the projectors* P_L *and* Q_L.

Definition 3.8. *Two continuous suitable mappings*

$$\Psi, \Psi': cokerL \to kerL$$

are said to be suitably homotopic if there exists a continuous mapping

$$H: cokerL \times [0,1] \to kerL$$

such that, for every $t\lambda \in [0,1]$*, the mapping*

$$H(\cdot, \lambda): cokerL \to kerL$$

is suitable, and

$$\Psi = H(\cdot, 0), \Psi' = H(\cdot, 1).$$

Corollary 3.2. *Let X be a Banach space, let $N: X \to X$ be a continuous operator that maps bounded sets into bounded sets, and let $A \in L(X)$.*

Assume that there exists a constant k>0 such that

$$\varphi_d(N(B)) \leq k\varphi_d(B), \text{for all} B \in P_{b,cv}(X).$$

Then, for every $B \in P_{b,cv}(X)$, we have

$$\varphi_d((A \circ N)(B)) \leq \| A \| k\varphi_d(B),$$

and

$$\varphi_d((N \circ A)(B)) \leq \| A \| k\varphi_d(B).$$

Proof. By Lemma [3.1], we have

$$\varphi_d((A \circ N)(B)) \leq \| A \| k\varphi_d(N(B)), \text{for all} B \in P_b(X).$$

Hence, for every, $B \in P_b(X)$,we obtain

$$\varphi_d((N \circ A)(B)) \leq \| A \| k\varphi_d(B).$$

Since$N is k\text{-}\varphi_d\text{-}contractive$,

$$\varphi_d((N \circ A)(B)) \leq k\varphi_d(A(B)), \text{for all} B \in P_b(X).$$

Then

$$\varphi_d((N \circ A)(B)) \leq \| A \| k\varphi_d(B), \text{for all} B \in P_b(X).$$

Proposition 3.10.*Assume that hypotheses (A_1)--(A_2) are satisfied, and let $M_{J_L}: \overline{U} \to X$ be the mapping defined as Theorem [3.9], associated with a continuous linear isomorphism*

$$J_L: cokerL \to kerL.$$

Then, M_{J_L} is a continuous and k–φ_d-contractive mapping.

Proof. Clear that P_L, Q_L, K_P, J_L, and π are linear continuous operators. Next, observe that P_L is a continuous linear operator with finite-dimensional range. Consequently, P_L is compact and hence is a 0–φ_d-contractive mapping. Furthermore, since $J_L\pi_L N(U)$ is a bounded subset of a finite-dimensional subspace, it is relatively compact for any subset $B \in P_b(\overline{U})$. We now show that the operator

$$(J_L\pi_L + K_P(I\text{-}Q_L))N$$

is a $\frac{k}{4}$–φ_d-contraction. Let $B \in P_{b,cv}(\overline{U})$. Then,

$$\varphi_d((J_L \circ \pi + K_P(I\text{-}Q_L))N(B)) \leq 2\chi(J_L \circ \pi N(B)) + 2\chi(K_P(I\text{-}Q_L)N(B))$$
$$\leq 2\varphi_d(J_L \circ \pi N(B)) + 2\varphi_d(K_P(I\text{-}Q_L)N(B)),$$

by the subadditivity of φ_d. Since $J_L \circ \pi N(B)$ is relatively compact, we have

$$\varphi_d(J_L \circ \pi N(B)) = 0,$$

and therefore

$$\varphi\big((J_L \circ \pi + K_{P_L}(I\text{-}Q_L))N(B)\big) \leq \frac{k}{2}\varphi_d(B), \text{for all} B \in P_{b,cv}(\overline{U}).$$

Therefore, M_{J_L} is a k–φ-contractive mapping from $\overline{U}$ into X.

Now, give the definition of the coincidence degree based on degree of non-densifiability and Leary-Schauder degree;

Definition 3.9. *Let $N: X \to Y$ be a continuous operator, (L, N)–k-φ_d-contraction, $U \subset X$ is bounded open convex subset and the following condition holds*

1. $Lx \neq N(x)$ *for all* $x \in \partial U \cap dom(L)$.

Then we define the coincidence degree, of the couple (L, N) in U to be the integer

$$d_{CDD}((L, N), U) = d_{DT}(I\text{-}M_J, U, 0),$$

where

$$M_L = P_L\text{-}(J_L \pi_L + K_L)Nx.$$

Now, we present the fundamental properties of the coincidence degree.

Theorem 3.11 [Existence and Excision]

1. ***Existence.*** *If*

   $$d_{CDD}((L, N), U) \neq 0,$$

   then

   $$0 \in (L\text{-}N)(dom(L) \cap U),$$

   that is, the equation $Lx = N(x)$ admits at least one solution in $dom(L) \cap U$.

2. ***Excision.*** *Let $U_1 \subset U$ be an open subset such that*

   $$(L\text{-}N)^{-1}(0) \subset U_1.$$

   Then

   $$d_{CDD}((L, N), U) = d_{CDD}((L, N), U_1).$$

3. ***(Homotopy invariance property).*** *Let $U \subset X$ be an open bounded convex set and let*

   $$\hat{N}_\lambda : U \times [0,1] \to Z, (x, \lambda) \mapsto \hat{N}_\lambda(x),$$

be a homotopy such that $\hat{N}_\lambda$ *is* $(L,\hat{N})$-k-φ_d*–contraction on* $U \times [0,1]$. *Assume moreover that, for every* $\lambda \in [0,1]$, *the following condition holds:*

$$0 \notin (L\text{-}\hat{N}_\lambda)(domL \cap \partial U).$$

Then the coincidence degree

$$d_{CDD}\big((L,\hat{N}_\lambda),U\big)$$

is independent of $\lambda \in [0,1]$. *In particular, one has*

$$d_{CDD}\big((L,\hat{N}_1),U\big)=d_{CDD}\big((L,\hat{N}_0),U\big).$$

Prrof.

(1) If

$$d_{DT}\big(I\text{-}M_{J_L},U,0\big)=d_{CDD}((L,N),U) \neq 0,$$

thus

$$d_{DT}\big(I\text{-}M_{J_L},U,0\big) \neq 0,$$

then by Theorem [3.5], there exists $x \in U$ such that

$$(I\text{-}M_{J_L})x=0.$$

In fact, we have $x \in domL \cap U$. Moreover, by Theorem [3.9], it follows that

$$Lx=Nx,$$

that is,

$$0 \in (L\text{-}N)(domL \cap U).$$

(2) Assume that $U_0 \cap r^{-1}(U_0) \subset U \cap r^{-1}(U)$ is an open set such that

$$(L\text{-}N)^{-1}(0) \subset U_0 \cap r^{-1}(U_0),$$

where r is a retraction given in Definition [3.5]. Then, by Theorem [3.9], we also have

$$(I\text{-}M_{J_L})^{-1}(0) \subset U_0.$$

Therefore, by the excision property of the Leray–Schauder degree,

$$d_{LS}\big(I\text{-}M_{J_L},U \cap r^{-1}(U),0\big)=d_{LS}\big(I\text{-}M_{J_L},U_0 \cap r^{-1}(U_0),0\big).$$

Hence,

$$d_{DT}\big(I\text{-}M_{J_L},U \cap r^{-1}(U),0\big)=d_{DT}\big(I\text{-}M_{J_L},U_0 \cap r^{-1}(U_0),0\big).$$

Consequently, by the definition of the coincidence degree, we obtain

$$d_{CDD}((L,N),U)=d_{CDD}((L,N),U_0).$$

(3) Let $J_L: cokerL \to kerL$ be an orientation-preserving continuous isomorphism. Define the following mapping

$$\hat{M}_{J_L}: [0,1] \times U \to X$$

by

$$\hat{M}_{J_L}(\lambda.x)=P_Lx+(J_L\pi_L+K_L(I-Q_L))\hat{N}(\lambda,x), (\lambda.x) \in [0,1] \times U.$$

Then,by Proposition [3.10], the mapping $\hat{M}_{J_L}$ satisfies the hypotheses of Theorem [3.5]. Hence, by the definition of the coincidence degree, we obtain

$$d_{CDD}\big((L,\hat{N}(0,\cdot)),U\big)=d_{CDD}\big((L,\hat{N}(1,\cdot)),U\big).$$

Now, for any $\lambda \in [0,1]$, set $\beta=t\lambda$ and apply the preceding argument to the family of operators $\hat{N}_\beta(t)=\hat{N}(\beta,t), t \in [0,1]$. It follows that

$$d_{CDD}\big((L,\hat{N}(\lambda,\cdot)),U\big)=d_{CDD}\big((L,\hat{N}(1,\cdot)),U\big)=d_{CDD}\big((L,\hat{N}(0,\cdot)),U\big), \forall\lambda \in [0,1].$$

Therefore, the coincidence degree

$$d_{CDD}\big((L,\hat{N}(\lambda,\cdot)),U\big)$$

is independent of $\lambda \in [0,1].square$

By using the homotopy invariance property of the coincidence degree, we now state a version of Mawhin's theorem, which provides a powerful tool for studying the existence of solutions to the coincidence equation. First, we present the following auxiliary lemma that will be used in the proof of the continuation theorem.

Lemma3.5.[35]*Let $N: X \to Y$ be a mapping. The problem*

$$Lx=Nx, x \in dom(L) \cap U,$$

is equivalent to the fixed point problem

$$x=P_Lx+J^{-1}QNx+K_{P_LQ_L}Nxx \in U,$$

where $J: \Im(Q_L) \to KerL$ ia isomorphism.

We give the second definition of (L,N)-k-φ_d-contraction by some why of Definition [3.7].

Definition 3.10.*Let U be an open convex and bounded subset of X such that $dom(L) \cap U \neq \varnothing$. The mapping $N: X \to Y$ is said to be (L,N)-k-φ_d-contraction. if $Q_LN(U)$ is bounded and*

$$K_{P_LQ_L}N: U \to X$$

is (L,N)-k-φ_d-contraction, where k satisfies the same condition as in Definition [3.7].

Remark 3.6.*Using Definition [3.10], the notion of coincidence degree introduced in Definition [3.9] can also be formulated by replacing the operator M_J defined there with the operator*

$$M_{J^{-1}}=P_L+J^{-1}QN+K_{P_LQ_L}N.$$

Theorem 3.12. *Let* $U\subset X$ *be a bounded open convex set. Assume that* $L: domL\subset X\rightarrow Y$ *is a Fredholm operator of index zero and that* N *is* (L,N)-$\frac{k}{4}$-φ_d*–contraction. Suppose that the following conditions hold:*

1. $Lx\neq\lambda Nx$*for every* $x\in\partial U\cap(domLL)$ *and every* $\lambda\in(0,1)$*;*
2. $Nx\notin \mathrm{Im}L$*for every* $x\in kerL\cap\partial U$*;*

$$deg_B\big(JQ_LN|_{kerL},U\cap kerL,0\big)\neq 0,$$

where $Q_L: Y\rightarrow Y$ *is a continuous projection satisfying* ImL=$kerQ_L$ *and* deg *is the Brouwer degree.*

Then the equation

$$Lx=Nx.$$

has at least one solution in $domL\cap\overline{U}$*.*

Proof. For $\lambda\in[0,1]$, consider the family of problems

$$Lx=\lambda Nx+(1\text{-}\lambda)Nx, x\in dom(L)\cap U.$$

Consider the homotopy $H:[0,1]\times\overline{U}\rightarrow Y$ given by

$$H(t,x)\quad=\quad P_Lx+J^{-1}Q_LN(x)+(K_L[t(I\text{-}Q_L)N])(x).$$

By Lemma [3.5], the problem [3.4] is equivalent to the following fixed point problem:

$$x=P_Lx+tJ^{-1}Q_LNx+(1\text{-}t)J^{-1}Q_LNx+tK_{P_L,Q_L}Q_LNx+(1\text{-}t)K_{P_L,Q_L}Q_LNx$$
$$=:H(t,x), x\in\overline{U}.$$

If there exists $x\in\partial U$ such that Lx=Nx, then we are done. Now suppose that

$$Lx\neq Nx \text{ for all} x\in dom(L)\cap\partial U,$$

and, on the other hand,

$$Lx\neq tNx+(1\text{-}t)QNx \text{ for all}(t,x)\in(0,1)\times(dom(L)\cap\partial U).$$

If

$$Lx=tNx+(1\text{-}t)Q_LNx \text{ for some } (t,x)\in(0,1)\times(dom(L)\cap\partial U),$$

then, applying Q_L to both sides of the above equality, we obtain

$$Q_LNx=0, Lx=tNx.$$

The first of these equalities and condition (ii) imply that

$$x \notin kerL \cap \partial U,$$

that is,

$$x \in \partial U \cap (dom(L)L),$$

and therefore the second equality contradicts condition (i). Using once again (ii), it follows that

$$Lx \neq Q_L Nx \text{for all} x \in dom(L) \cap \partial U.$$

In view of [1.4], [1.5], and [1.6], we deduce that

$$x \neq H(t,x) \text{for all} (t,x) \in [0,1] \times \partial U.$$

Thus, for every $B \in P_{cv}(U)$

$$\varphi(H(t,B)) \quad \leq \quad \varphi_d(P_L(B)) + \varphi_d(J^{-1} Q_L N(B)) + \varphi_d(tK_{P_L Q_L} N(B)).$$

We can easily show that $\overline{P_L(B)}, \overline{J^{-1} Q_L N(B)}$ are compacts, hence

$$\begin{aligned} \varphi\big(H(t,B)\big) &\leq \varphi_d\left(tK_{P_L Q_L} N(B)\right) \\ &\leq \frac{k}{2} \varphi_d(B) \end{aligned}$$

Then, N is (L,N)-k-φ_d–contraction. By the homotopy properties of the topological degree introduced in Theorem [3.11], we obtain

$$d_{DT}(I\text{-}H(1,\cdot), U, 0) = d_{DT}(I\text{-}H(0,\cdot), U, 0).$$

Therefore,

$$d_{LS}(I\text{-}H(1,\cdot), U, 0) = d_{LS}(I\text{-}H(0,\cdot), U, 0).$$

Then,

$$d_{LS}(I\text{-}H(1,\cdot), U, 0) = d_{LS}(I\text{-}P_L\text{-}J^{-1} Q_L N, U, 0).$$

But the range of $P_L + J^{-1} Q_L N$ is contained in $ker(L)$. Hence, by using the reduction property of the Leray–Schauder degree and the fact that $P_L|_{ker(L)} = I|_{ker(L)}$ and since $ker(L) = \mathrm{Im}(P_L) = ker(I\text{-}P_L)$, we obtain

$$d_{LS}(I\text{-}P_L\text{-}J^{-1} Q_L N, U, 0) = deg_B(I\text{-}P_L\text{-}J^{-1} Q_L N, U \cap ker(L), 0)$$

Then,

$$d_{DT}(I\text{-}P_L\text{-}J^{-1} Q_L N, U, 0) = d_{LS}(I\text{-}P_L\text{-}J^{-1} Q_L N, U, 0) = deg(J^{-1} Q_L N, U \cap ker(L), 0).$$

Thus,

$$d_{DT}(I\text{-}P_L\text{-}J^{-1}Q_LN, U, 0)=deg(J^{-1}Q_LN, U \cap ker(L),0).$$

In view of [1.7] and [1.10], it follows that

$$d_{CDD}((L, N), U) \neq 0.$$

Therefore, by the existence property of coincidence degree, there exists $x \in U$ such that

$$x=H(1, x),$$

that is, $x \in dom(L) \cap U$ and $Lx=Nx$.

By the homotopy properties of the coincidence degree and the properties of the Leray–Schauder degree, we obtain the following continuation theorem.

Theorem4.5 . *Let $U \subset Y$ be an open bounded set. Assume that L is a Fredholm operator of index zero and that N is (L, N)-k-contraction on U in sense of Definition 3.3 .*

Suppose that the following conditions hold:

1. $Lx \neq \mu Nx$*for every* $x \in \partial U \cap domL$ *and every* $\mu \in (0,1)$*;*
2. $Q_LNx \neq 0$*for every* $x \in \partial U \cap kerL$*;*
3. $deg_B(J^{-1}Q_LN, U \cap kerL, 0) \neq 0$.

Then the coincidence equation

$$Lx=Nx$$

has at least one solution in $\overline{U} \cap domL$.

Consider the operator Φ_ϑ defined by

$$H(t, x):=Px+JQNx+tKP(I\text{-}Q)Nx, \vartheta \in [0,1],$$

and observe that $H(1,\cdot)=H$, while $H(0,\cdot)$ has finite-dimensional range in $kerL$.

The assumptions of the lemma imply that $u \neq H(t, x)$ for all $x \in \partial U$ and all $t \in [0,1]$. By the homotopy invariance and the reduction property of the Leray–Schauder degree, we obtain

$$d_{DT}((L, N), U)=deg_{LS}(I\text{-}H(1.\cdot), U, 0) \quad = \quad deg_{LS}(I\text{-}H(0,\cdot), U, 0)=$$

Hence, the Theorem is proved.

Periodic Boundary Value Problem in a Hilbert Space

In this section, we apply the theoretical framework developed in the previous sections to establish the existence of solutions for second order differential equations in Hilbert spaces subject to periodic boundary value conditions. These applications demonstrate the practical utility of our topological and coincidence degree theories.

We conclude this paper by presenting some applications of the previous results to the solvability of certain boundary value problems for nonlinear second-order differential equations in Hilbert spaces.

$$\begin{cases} -x''(t)=f(t,x(t)),\ t \in [0,1], \\ x(0)-x(1)=0,\ x'(0)-x'(1)=0, \end{cases} \tag{4,10}$$

where $x'=\frac{dx}{dt}$ and $f:[0,1] \times H \to H$, with H being a real Hilbert space endowed with the inner product $\langle \cdot,\cdot \rangle$ and the associated norm $\|\cdot\|$.

One may also consider the Neumann boundary conditions

$$x'(0)=0=x'(1),$$

which, like the periodic conditions, lead to a non-invertible linear part in the abstract formulation of problem (4.10). Let $\hat{H}=L^2([0,1],H)=X=Y$ be the Hilbert space endowed with the inner product

$$(u,v)=\int_0^1 \langle u(s),v(s)\rangle ds,$$

and the corresponding norm $\|\cdot\|$. We define the operator N on $\hat{H}$ by

$$(Nx)(s)=f(s,x(s))\text{for a.e.}s \in [0,1],$$

for every $x \in \hat{H}$. We define

$$domL=\{x \in \hat{H}: x \in AC([0,1],H), x',x'' \in \hat{H},\text{and} x(0)-x(1)=x'(0)-x'(1)=0\},$$

where

$$AC([0,1],H)\text{is a space of absolutely continuous on}[0,1].$$

Then $domL$ is a dense subspace of $\hat{H}$. Define the linear operator $L: domL \subset \hat{H} \to \hat{H}$ by

$$Lx=-x''.$$

Then L is a closed operator and

$$kerL=\{x \in domL: x\text{is constant on}[0,1]\}.$$

Moreover,

$$\text{Im}L=\left\{x \in \hat{H}: \int_0^1 x\,(s)ds=0\right\}=(kerL)^{\perp}.$$

Furthermore, $kerL=\text{Im}P_L$, where $P:\hat{H} \to \hat{H}$ is the orthogonal projection onto $kerL$, defined by

$$P_L x=\int_0^1 x\,(s)ds=Q_L x$$

and the inverts K_L:Im$L \to domL$ given by

$$K_{P_L}x=\int_0^1 G\,(s,t)x(s)ds,$$

where the Green function associated with the linear part of the problem (4.10) is

$$G(s,t)=\begin{cases}\dfrac{s(1\text{-}2t+s)}{2}, & 0\leq s<t,\\ \dfrac{(1\text{-}s)(2t\text{-}s)}{2}, & 0\leq t\leq s\leq 1.\end{cases}$$

Then

$$K_{P_L}(I\text{-}Q)Nx=\int_0^1 G\,(s,t)\left[f(s,x(s))\text{-}\int_0^1 f\,(t,x(t))dt\right]ds.$$

Since $X=\hat{H}=Y=\hat{H}$, the operator J_L can be chosen as the identity operator I . We assume the following hypotheses:

($\hat{H}_1$)The operator $N:\hat{H}\to\hat{H}$ is continuous and $\frac{k}{2}$-φ_d-contractive.

($\hat{H}_2$)There exists a constant r>0 such that

$$\| Nx \|\leq r,\text{for all}x\in\hat{H}.$$

($\hat{H}_3$)The operator N is monotone, that is,

$$\int_0^1 \langle f(s,x(s))\text{-}f(s,y(s)),x(s)\text{-}y(s)\rangle ds\geq 0,$$

for all $x,y\in\hat{H}$.

($\hat{H}_4$)There exists a constant R>0 such that, for almost every $t\in[0,1]$ and for all $x\in H$ with $\| x \|\geq R$, the inequality

$$\langle f(t,x),x\rangle\geq\left(\frac{r}{2\pi}\right)^2 \tag{4.11}$$

holds.

Lemma 4.1. *Assume that conditions (H_2)–(H_4) are satisfied. Then there exists a constant M>0 such that every solution $x\in\hat{H}$ of the family of problems*

$$Lx=\lambda Nx+(1\text{-}\lambda)P_LNx,x\in domL, \tag{4.12}$$

is bounded by M, that is,

$$\| x \|_{\hat{H}}<M.$$

Proof. As in , all solutions of [36, Theorem 6.1] are *a priori* bounded. More precisely, there exists a constant M>0 such that

$$\| x \|_{\hat{H}} < M$$

for every solution x of (4.12).

We now apply Theorem 3.13 to establish the solvability of Problem (4.10).

Theorem 4.1. *Assume that conditions* $(\hbar_1)$–$(\hbar_4)$ *are satisfied. Moreover, assume that the following condition*

$(\hbar_5)$ *The Leary-Schauder degree*

$$d_{LS}(-P_L-Q_L N, U, 0) \neq 0,$$

where

$$U=\{x \in \hat{H}: \| x \|_{\hat{H}} < M\}.$$

Then problem (4.10) admits at least one solution.

Proof. We considerar the following homotopy

$$H(\lambda, x)=P_L x+Q_L Nx+\lambda K_{P_L}(I-Q_L)Nx, (\lambda, x) \in [0,1] \times \hat{H}.$$

By standard arguments, the equation (4.12) is equivalent to the fixed point equation

$$x=P_L x+\lambda Q_L Nx+(1-\lambda)Q_L Nx+\lambda K_{P_L,Q_L} Q_L Nx+(1-\lambda)K_{P_L,Q_L} Q_L Nx$$
$$=: H(\lambda, x).$$

Set

$$U=\{x \in \hat{H}: \| x \|_{\hat{H}} < M\},$$

is open convex bounded subset of $\hat{H}$ and $\partial U=\{x \in \hat{H}: \| x \|_{\hat{H}} = M\}$.

We first show that $H(\lambda, x) \neq x$ for all $x \in \partial U$ and $\lambda \in (0,1]$. Indeed, let $x \in \partial U$ and $\lambda \in (0,1)$ such that $Lx=\lambda Nx$. Since $\mathrm{Im}(P_L)=kerL$,$\mathrm{Im}L=kerP_L$, then

$$\lambda PNx=0, \lambda Nx=P_L Nx+\lambda(I-P_L)Nx,$$

hence

$$Lx=\lambda Nx+(1-\lambda)P_L Nx,$$

this implies

$$M= \| x \|_{\hat{H}} < M,$$

which contradiction.

If $x \in \partial U \cap kerL$ and $Q_L Nx=0$, then

$$L(x)=0,\ N(x) \in kerQ_L$$

and

$$N(x) \in ker(I\text{-}Q_L)=\mathrm{Im}Q_L.$$

Thus,

$$L(x)=N(x)=0, L(x)=\lambda Nx+(1\text{-}\lambda)P_L N(x)$$

which also contradiction.

We now show that the operator

$$H(\cdot,x)=P_L x+Q_L Nx+\lambda K_{P_L}(I\text{-}P_L)Nx$$

is a k–φ_d-contraction. Let $B \in P_{b,cv}(\overline{U})$. By Lemma 4.3, we obtain

$$\varphi_d(H(\lambda,B)) \leq \varphi_d(P_L(B))+\varphi_d(P_L \circ N(B))+\varphi_d(K_P(I\text{-}Q_L)N(B)),$$

by the subadditivity of φ_d. Since $P_L(B)$ and $P_L(N(B))$ are relatively compact, we have

$$\varphi_d(P_L(B))=\varphi_d(P_L(N(B)))=0.$$

Moreover, since $\| I\text{-}P_L \|_{\hat{H}} =1$, a straightforward computation shows that

$$\| K_L x \|_{\hat{H}} \leq \| x \|_{\hat{H}}.$$

Hence, by hypothesis ($\mathcal{H}_1$), we deduce that

$$\varphi_d(H(\lambda,N(B))) \leq \frac{k}{2}\varphi_d(B), \text{for all} B \in P_{b,cv}(\overline{U}).$$

Therefore, it follows that $H(\cdot,\cdot)$ is a $\frac{k}{2}$–φ-contractive mapping from $\overline{U}$ into $\hat{H}$.

Hence, by the homotopy invariance property of the degree,

$$d_{LS}(I\text{-}H(\cdot\,,1),U,0)=d_{LS}(I\text{-}H(\cdot\,,0),U,0).$$

But

$$H(\cdot\,,0)=P_L+Q_L N.$$

Since the range of this mapping is contained in the finite-dimensional space $kerL$, we obtain

$$d_{LS}(I\text{-}H(\cdot\,,0),U,0)=deg_B\big(I\text{-}(P+Q_L N)|_{kerL},U \cap kerL,0\big).$$

Hence

$$deg_B\big(I\text{-}(P_L+Q_L N)|_{kerL},U \cap kerL,0\big)=deg_B\big(\text{-}Q_L N|_{kerL},U \cap kerL,0\big).$$

By hypothesis ($\mathcal{H}_5$), we deduce that

$$deg_B\big(\text{-}Q_L N|_{kerL},U \cap kerL,0\big) \neq 0.$$

It follows that Theorem 4.5 applies.

Conclusions

In the present paper, we have introduced a topological degree and a version of Mawhin's coincidence degree based on the so-called contraction DND mappings. This approach provides a novel framework for establishing fixed point theorems that extends beyond the classical methods based on measures of noncompactness.

Several fixed point results, derived from existing topological degree theories, have been established. Many results based on the concept of DND mappings are of the Schauder and Darbo type in Banach spaces (see). Our work complements and extends these contributions by providing a unified framework that combines the degree of nondensifiability with classical topological degree theory.

In this work, we have proved a nonlinear alternative of Leray–Schauder type, as well as Schaefer's fixed point theorem and a continuation theorem. These results demonstrate that the degree of nondensifiability provides a flexible and powerful tool for analyzing nonlinear operators in Banach spaces.

The application to periodic boundary value problems for second order differential equations in Hilbert spaces illustrates the practical utility of our theoretical framework. The coincidence degree method, combined with the degree of nondensifiability, offers new approaches to establishing existence results for differential equations that may not be amenable to classical techniques.

The theoretical framework developed in this paper opens several avenues for future research. The extension of these results to fractional differential equations, partial differential equations, and other classes of nonlinear problems represents a promising direction for further investigation. Additionally, the development of computational methods based on the degree of nondensifiability could provide practical tools for numerical analysis of nonlinear problems.

Our results extend and unify several classical and recent contributions available in the literature, providing a coherent framework that bridges the gap between measure of noncompactness techniques and topological degree theory. This synthesis offers new perspectives on fixed point theory and its applications to nonlinear analysis.

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Author Contributions

The three authors participated in the research and in the drafting of the paper, and reviewed the final version.

Data Availability

Data sharing is not applicable to this article, as the research predominantly involves mathematical analysis and does not involve the generation, collection, or analysis of specific datasets. The results presented in this paper are derived from mathematical proofs.

Declaration

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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