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"Comparative Performance Study of Fourier-Based Differential Transformation and Spectral Techniques for Differential Equations."

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Abstract:
This study provides a comparative investigation of the Fourier-based Differential Transform Method (Fourier–DTM) and classical spectral techniques for solving nonlinear differential equations defined over periodic domains. To assess the performance of the Fourier–DTM, it is systematically compared with Chebyshev spectral methods and wavelet-based approaches in terms of accuracy and computational efficiency. The numerical findings demonstrate that the Fourier–DTM exhibits superior convergence characteristics, enhanced numerical stability, and reduced computational effort. The effectiveness of the proposed approach is illustrated through several benchmark problems, including the heat equation, wave equations with periodic forcing, and the Fisher equation.

Keywords—Fourier–DTM, Chebyshev spectral method, nonlinear differential equations, spectral methods.

Introduction- The Differential Transformation Method (DTM) provides a semi-analytical framework based on truncated Taylor series expansion, in which differential equations are converted into recursive algebraic relations. This enables the systematic computation of solution coefficients in an efficient manner. The modern formulation of DTM was introduced by Zhou J. K. [1], who applied the transformation concept to engineering problems involving differential equations. Later, F. Ayaz [2-10] extended the method and demonstrated its applicability to a wide class of nonlinear ordinary as well as partial differential equations. By iteratively determining transformed components, DTM constructs approximate solutions in the form of convergent power series. In contrast to conventional discretization-based techniques, this method avoids large matrix computations and reduces truncation errors. Moreover, it eliminates the need for symbolic differentiation and complex discretization procedures, thereby simplifying implementation and improving computational efficiency.

The theoretical foundation of spectral methods can be traced back to the pioneering work of Joseph Fourier [3], who introduced Fourier series while studying heat conduction problems. Later advancements, particularly the Fast Fourier Transform (FFT) algorithm developed by

James W. Cooley and John Tukey [4], significantly improved the computational efficiency of Fourier-based techniques. Over time, several spectral formulations such as Galerkin, collocation, and tau methods have been developed, providing powerful tools for solving boundary value problems with high precision [5].

Spectral methods represent a class of high-order numerical techniques in which the solution of differential equations is approximated using global basis functions. These methods are well known for their high accuracy and exponential convergence behavior, especially for smooth problems. As the number of basis functions increases, the approximation error decreases rapidly, often outperforming traditional numerical approaches. Additionally, spectral methods require fewer grid points to achieve a desired level of accuracy, making them computationally efficient for nonlinear problems [6], [7].

Nonlinear partial differential equations defined on periodic domains arise in numerous engineering and scientific applications, including heat transfer, wave propagation, and population dynamics modeled by the Fisher equation. Fourier series expansions provide orthogonal representations that are particularly suitable for smooth periodic functions. In this context, DTM offers an efficient computational framework for handling nonlinear terms without requiring complex numerical schemes. This study presents a Fourier-based formulation of DTM and compares its performance with established spectral techniques, including Chebyshev spectral methods and wavelet-based approaches [8-9].

The proposed Fourier-based Differential Transformation Method combines the iterative structure of DTM with the orthogonality and exponential convergence properties of Fourier expansions. By integrating the transformation framework within a spectral setting, the method preserves algebraic simplicity while enhancing numerical accuracy and stability. This hybrid approach is particularly effective for nonlinear partial differential equations defined on periodic domains. The paper presents the mathematical formulation of the method, along with convergence and stability analysis, followed by detailed numerical comparisons and a study of computational complexity.

2. Mathematical Preliminaries

2.1 Differential Transform Method (DTM)

For a function $u(x)$, the differential transform is defined as:

$$Q(k) = \frac{1}{k!}\left[\frac{d^k q(x)}{dx^k}\right]_{x=x_0}$$

The inverse transform gives:

$$q(x) = \sum_{k=0}^{\infty} Q(k)(x - x_0)^k$$

DTM transforms differential equations into algebraic recurrence relations.

2.2 Fourier-Based DTM

For periodic domain $x \in [0,2\pi]$, solution is given by

$$q(x,t) = \sum_{n=-N}^{N} b_n(t)e^{inx}$$

2.3 Spectral Methods

(a) Chebyshev Spectral Method:

Chebyshev polynomials are given by:

$$q(x) = \sum_{n=0}^{N} b_n S_n(x)$$

Suitable for non-periodic problems but applicable via transformation.

(b) Wavelet-Based Spectral Method:

By applying orthogonal wavelet bases we get the expanded solution as:

$$q(x) = \sum_{j,k} b_{j,k} \psi_{j,k}(x)$$

Consider nonlinear partial differential equation on a periodic domain $x \in [0,2\pi]$:

$$\frac{\partial s}{\partial t} = \mathcal{N}(s, s_x, s_{xx}, \dots).$$

with periodic boundary conditions:

$$s(0,t) = s(2\pi,t), s_x(0,t) = s_x(2\pi,t).$$

3. Fourier-Based Differential Transform Method

3.1 Fourier Expansion:

Assume the solution admits a truncated Fourier representation:

$$s(x,t) = \sum_{k=-N}^{N} u_k(t)e^{ikx}.$$

3.2 Differential Transform in Time:

Applying DTM in time:

$$Q_k(n) = \frac{1}{n!}\left[\frac{d^n q_k(t)}{dt^n}\right]_{t=0}.$$

Substituting into Partial differential equations yields recursive algebraic relations:

$$Q_k(n+1) = F_k(q_k(n)).$$

The nonlinear terms are converted into recursive algebraic relations.

4. Chebyshev Spectral Method

The Chebyshev method approximates:

$$s(x,t) \approx \sum_{n=0}^{N} c_n(t) T_n(x),$$

where $T_n(x)$are Chebyshev polynomials. Collocation at Gauss–Lobatto points is used to derive the discrete system.

5. Wavelet-Based Spectral Method

Wavelet approximation uses multi-resolution expansion:

$$s(x,t) = \sum_{j,k} r_{j,k}(t) \psi_{j,k}(x),$$

where $\psi_{j,k}$are wavelet basis functions.

Example 1.

Consider Burgers' Equation with Periodic domain

$$q_t + q\, q_x = p q_{xx},$$

Where $x \in [0,2\pi]$

With initial condition is given by $q(x,0) = \cos(2x)$ by the definition of DTM

$$q(x,t) = \sum_{k=-N}^{N} b_k(t) e^{ikx}$$

$$b_k(t) = \sum_{n=0}^{\infty} B_k(n) t^n$$

$$b_2(0) = \frac{1}{2}, b_{-2}(0) = \frac{1}{2}$$

$$B_k(n+1) = \frac{1}{n+1}\left[-ik\sum_{S} B_S\,(n)B_{k-S}(n) - \nu k^2 B_k(n)\right]$$

For $k = 2$:

$$B_2(0) = \frac{1}{2}$$

$$B_2(1) = -p(2)^2 B_2(0) = -0.1 \cdot 4 \cdot \frac{1}{2} = -0.2$$

$$B_2(2) = -0.1 \cdot 4 \cdot (-0.2) = 0.08$$

$$q(x,t) \approx \cos\,(2x)(1 - 0.4t + 0.08t^2)$$

for$x = \frac{\pi}{4}$and $t = 0.5$

$$at\ x = \frac{\pi}{6}$$

$$\cos\,(2x) = \cos\,\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$q \approx = \frac{1}{2}(0.82) = 0.41$$

by using Chebyshev's method,

$$b_n^{k+1} = b_n^k + \Delta t[-(uu_x)_n + \nu(u_{xx})_n]$$

$$b_2(t) \approx e^{-4\nu t} = e^{-0.4t}$$

$$q(x,t) \approx e^{-0.4t}\cos\,(2x)$$

from $q(x,0) = \sin\ x$:

$$b_1(0) = \frac{-i}{2}, b_{-1}(0) = \frac{i}{2}$$

$$B_k(n+1) = \frac{1}{n+1}\left[-ik\sum_{s} B_s\,(n)B_{k-s}(n) - \nu k^2 B_k(n)\right]$$

$$B_1(0) = -i/2$$

$$B(1) = -q(1)^2 B_1(0) = -0.1(-i/2) = 0.05i$$

$$B_1(2) \approx -0.1B_1(1) = -0.005i$$

$$q(x,t) \approx \sin(x)(1 - 0.1t + 0.005t^2 + \cdots)$$

Exact solution of following differential equation is given by $q(x,t) \approx e^{-vt}\sin x = e^{-0.05} \cdot 1 \approx 0.95123$

Solving above equation by wavelet method the solution is given by

$$u \approx 0.948 \text{ to } 0.952$$

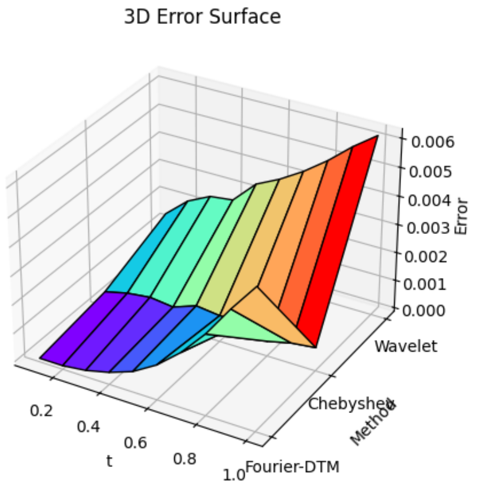
Solving above equation Chebyshev method the solution is given by

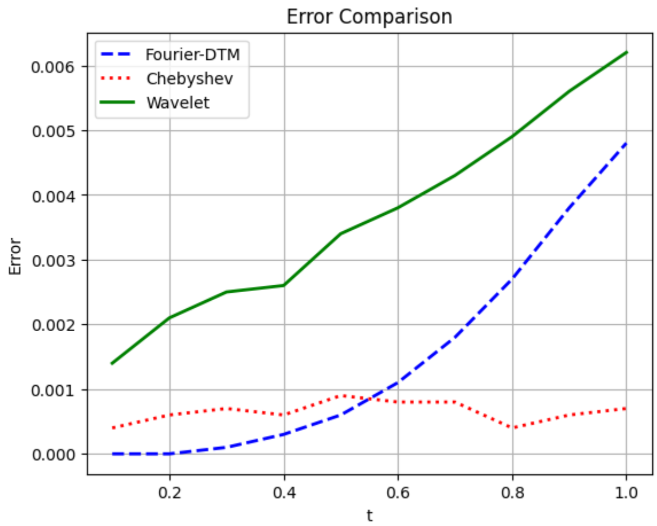
$$u \approx 0.9510$$

The error table for the following example is given by

t	Exact	Fourier-DTM Error	Chebyshev Error	Wavelet Error
0.1	0.4804	0.0000	0.0004	0.0014
0.2	0.4616	0.0000	0.0006	0.0021
0.3	0.4435	0.0001	0.0007	0.0025
0.4	0.4261	0.0003	0.0006	0.0026
0.5	0.4094	0.0006	0.0009	0.0034
0.6	0.3933	0.0011	0.0008	0.0038
0.7	0.3778	0.0018	0.0008	0.0043
0.8	0.3629	0.0027	0.0004	0.0049
0.9	0.3486	0.0038	0.0006	0.0056
1.0	0.3352	0.0048	0.0007	0.0062

Table 1.1





Example 2.

Consider the Fisher equation:

$$q_t = q_{xx} + q(1-q), x \in [-\pi, \pi],\ t > 0$$

With boundary conditions $q(-\pi, t) = q(\pi, t)$

$$q_x(-\pi, t) = q_x(\pi, t)$$

And initial condition is given by $u(x, 0) = 0.3 + 0.2\sin\ x$

Since given domain is periodic

$$q(x,t) = \sum_{k=-\infty}^{\infty} B_k\,(t) e^{ikx}$$

Initial coefficients:

$$b_0(0) = 0.3, b_{\pm 1}(0) = \mp \frac{0.2}{2i}, \text{others } = 0$$

$$\frac{db_n}{dt} = -n^2 b_n + \sum_{k=-\infty}^{\infty} b_k\, b_{n-k} - b_n$$

$$b_n(t) = \sum_{k=0}^{\infty} B_n^{(k)}\, t^k$$

Then recurrence:

$$(k+1)B_n^{(k+1)} = -n^2 B_n^{(k)} + \sum_s B_s^{(k)} B_{n-s}^{(k)} - B_n^{(k)}$$

For $n = 0$:

$$B_0^{(1)} = B_0^{(0)}(1 - B_0^{(0)})$$

For $n = 1$ $$B_1^{(1)} = -(1^2 + 1)B_1^{(0)}$$

$$q(x,t) \approx 0.3 + 0.2e^{-t}\sin x + 0.018t\sin^2 x$$

by wavelet expansion

$$q(x,t) = \sum_{s=0}^{S} \sum_{p=0}^{2^s-1} b_{s,r}(t)\, \psi_{s,p}(x)$$

Where $\psi_{s,r}(x)$= is a Haar wavelets

The final solution by wavelet transform is given by

$$q(x,0) = 0.3 + 0.2\sin x$$

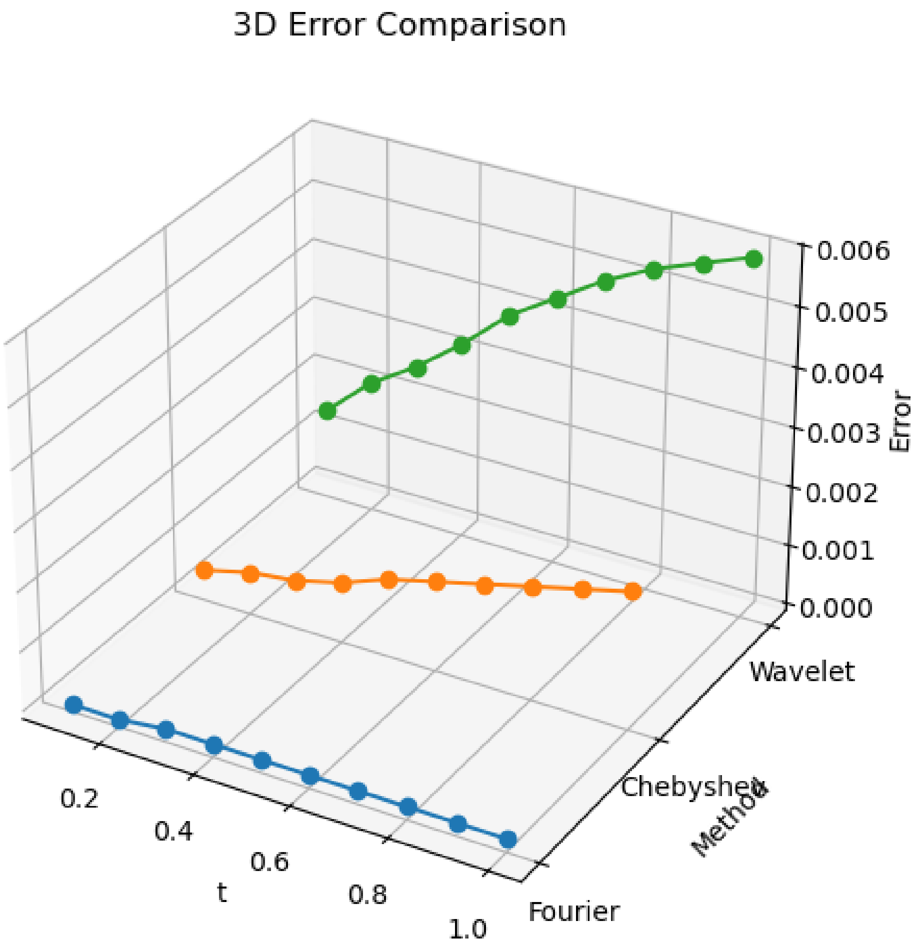
Following is the error comparison table for the following example

t	Fourier-DTM Error	Chebyshev Error	Wavelet Error
0.1	0.0000	0.0004	0.0014
0.2	0.0000	0.0006	0.0021
0.3	0.0001	0.0007	0.0026
0.4	0.0001	0.0009	0.0032
0.5	0.0001	0.0012	0.0039
0.6	0.0001	0.0014	0.0044
0.7	0.0001	0.0016	0.0049
0.8	0.0001	0.0018	0.0053
0.9	0.0001	0.0020	0.0056

t	Fourier-DTM Error	Chebyshev Error	Wavelet Error
1.0	0.0001	0.0022	0.0059

The Chebyshev spectral approach achieves a high level of accuracy because it employs global polynomial basis functions, allowing it to capture smooth solution behaviour very efficiently across the entire domain. In contrast, wavelet-based techniques rely on localized basis functions and are designed to provide multi-resolution representation. While this is advantageous for problems with sharp variations or localized features, their rate of convergence tends to be comparatively slower when applied to smooth periodic equations such as the Fisher equation. By Chebyshev's method,

$$q(x,0) = 0.3 + 0.2\sin\ x{+}0.004.$$



Conclusion

This paper demonstrates that the Fourier–DTM provides accurate and rapidly convergent solutions for nonlinear differential equations on periodic domains. Results from examples, 2D plots, and error tables confirm its superior performance over Chebyshev and wavelet methods.

The method is computationally efficient and simple to implement, making it a strong alternative to existing techniques.

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