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Brownian-Swish: A Hybrid Stochastic Activation for Deep Neural Networks

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Abstract:

Activation functions play an important role in deep learning, yet through the use of conventional deterministic activation functions, such as ReLU, Swish, and Mish, the ability to achieve smooth optimization and robustness to uncertainties can be difficult. Exploration of stochastic alternative forms of activation functions have been undertaken; however, they suffered from a lack of source literature establishing rigorous theoretical underpinnings, in combination with empirical evidence supporting consistent improvement in performance. In this paper, we discuss the introduction of a hybrid activation function known as Brownian-Swish that combines the smooth, nonlinear characteristics of Swish, with stochastic perturbations derived adaptively via Brownian motion. We establish key properties of Brownian-Swish: differentiability, convergence to Swish, gradient stability, and implicit regularization. In a large number of experiments in various domains, such as vision (CIFAR-10, CIFAR-100), natural language processing (IMDB sentiment, AG News), and sequential forecasting, Brownian-Swish consistently produced greater accuracy, greater stability to noise, and lower generalization gap compared to both deterministic and stochastic baselines; findings were further supported through ablation studies demonstrating similar convergence behaviour. These results underscore the Brownian-Swish activation function as a foundational component of stochastic deep-learning processes, and provide the basis for developing more robust and generalizable models.

Keywords: Brownian motion, Stochastic activation functions, Nonlinear optimization, Reliability modelling, Gradient stability.

1. Introduction:

Activation functions are fundamental to the success of deep neural networks, directly influencing gradient flow, convergence behavior, and the ability to approximate complex nonlinear mappings. Despite the widespread adoption of deterministic smooth activations such as Swish and Mish, these functions remain limited in their ability to capture stochasticity inherent in real-world data, particularly in noisy or uncertain domains such as financial forecasting, climate modeling, and reinforcement learning. Conversely, stochastic activations such as Brownian ReLU introduce randomness that enhances robustness but suffer from

computational inefficiency and lack of smoothness. This research is important and necessary because it addresses the gap between smooth deterministic activations and stochastic activations, aiming to unify their strengths. By embedding Brownian motion into the Swish function, we propose a hybrid activation that is both differentiable and uncertainty-aware, offering practical benefits in domains where noise and variability are intrinsic.

Classical activations such as ReLU and Tanh have well-documented limitations, including vanishing gradients, saturation, and the "dying neuron" problem. Extensions such as Leaky ReLU, PReLU, and GELU improved gradient stability but remain deterministic. More recent smooth activations like Swish and Mish demonstrated superior empirical performance due to their differentiability and non-monotonicity, yet they fail to incorporate stochastic behavior. On the other hand, stochastic activations such as Brownian ReLU and Noisy ReLU introduced randomness to improve robustness, but they lack smoothness and impose computational overhead due to Monte Carlo sampling. Against this background, the scientific novelty of our work lies in the introduction of Brownian-Swish, a hybrid activation function that integrates the smooth nonlinearity of Swish with stochastic perturbations derived from Brownian motion. This design uniquely combines differentiability, adaptability, and stochastic robustness, filling a clear gap in the activation function landscape.

The contributions of this paper are threefold:

1. **New Activation Function:** We propose **Brownian-Swish**, a hybrid stochastic-smooth activation that unifies Swish's differentiability with Brownian motion's stochastic adaptability.
2. **Theoretical Analysis:** We provide formal proofs of differentiability, gradient stability, convergence under Monte Carlo sampling, and implicit regularization effects, establishing the mathematical foundation of Brownian-Swish.
3. **Empirical Validation:** Through experiments on vision (CIFAR-10, ImageNet), NLP (IMDB sentiment, Transformer tasks), and sequential datasets (financial time series, climate data), we demonstrate that Brownian-Swish consistently improves generalization, reduces overfitting, and enhances robustness compared to both deterministic and stochastic baselines.

Together, these contributions establish Brownian-Swish as a principled and effective activation function for uncertainty-aware deep learning.

2. Related Work:

Activation functions are central to deep learning, shaping gradient flow, convergence, and model expressivity. Early nonlinearities such as the sigmoid [1] and tanh [2] enabled the first generation of neural networks but suffered from vanishing gradients. The introduction of the Rectified Linear Unit (ReLU) [3] marked a breakthrough, offering simplicity and improved convergence, though it introduced the "dying neuron" problem.

Extensions such as Leaky ReLU [4] and Parametric ReLU (PReLU) [5] addressed neuron inactivity by introducing fixed or trainable slopes for negative inputs. Exponential Linear Unit (ELU) [6] and Scaled Exponential Linear Unit (SELU) [7] improved gradient flow and self-normalization. Softplus [8] provided a smooth approximation to ReLU. More recent smooth

activations such as Gaussian Error Linear Unit (GELU) [9], Swish [10], and Mish [11] emphasized differentiability and non-monotonicity, yielding superior empirical performance across vision and language tasks.

Parallel research explored stochastic activations. Noisy ReLU [12] injected Gaussian noise to improve robustness, while Randomized Leaky ReLU [13] introduced randomness in the negative slope. Brownian ReLU (Br-ReLU) [14] leveraged Brownian motion to generate stochastic paths for negative inputs, enhancing gradient stability in sequential tasks. These stochastic approaches improved uncertainty modeling but often suffered from computational overhead and lack of smoothness.

Recent works (2024–2025) have revisited activation design. Agarwal [15] compared classical activations on MNIST, reaffirming Swish's superior generalization. Subramanian et al. [16] introduced APALU, a trainable adaptive activation function, highlighting the need for flexibility in specialized tasks. A 2024 IEEE review [17] emphasized bridging theory and application, noting unresolved challenges in balancing smoothness, adaptability, and computational efficiency. These studies underscore the ongoing relevance of activation research and the demand for hybrid approaches.

Against this background, the **novelty of our work** lies in the introduction of **Brownian-Swish**, a hybrid activation that unifies Swish's smoothness with Brownian motion's stochastic adaptability. Unlike prior methods, Brownian-Swish is differentiable everywhere, introduces controlled stochasticity, and acts as implicit regularization, offering both theoretical elegance and practical robustness.

3. Proposed Method:

The proposed **Brownian-Swish activation function** integrates the smooth nonlinearity of the Swish function with stochastic perturbations derived from Brownian motion, creating a hybrid activation that is both differentiable and uncertainty-aware. Formally, Brownian-Swish is defined as

$$f(x;\alpha,\beta,M) = \frac{x}{1+e^{-\beta x}} - \alpha\frac{1}{M}\sum_{k=1}^{M} B^k(|x|),$$

where the first term represents the Swish backbone and the second term introduces stochasticity through Monte Carlo sampling of Brownian paths. The parameters α and β are learnable, controlling the strength of stochastic perturbation and the steepness of the Swish curve, respectively, while M determines the number of sample paths. This design ensures smooth differentiability across the input domain, prevents vanishing gradients in negative regions, and acts as implicit regularization by injecting input-dependent noise. Importantly, Brownian-Swish reduces to standard Swish when $\alpha = 0$ and approximates Brownian-ReLU as $\beta \to \infty$, thereby unifying deterministic and stochastic activation paradigms. By combining these properties, Brownian-Swish offers improved robustness, generalization, and adaptability in noisy or high-variance learning environments.

4. Theoretical Analysis:

Theorem 4.1 (Differentiability of Brownian-Swish)

The Brownian-Swish activation function is differentiable everywhere with probability 1.

Proof:

The first term, Swish is differentiable everywhere since it is a composition of smooth functions (x and the logistic sigmoid). The second term is a Monte Carlo average of Brownian increments $B^k(|x|)$. Each $B^k(|x|)$ is normally distributed with mean 0 variance $|x|$.

Differentiability of $B^k(|x|)$ w.r.to $|x|$ holds almost surely because Brownian paths are continuous and differentiable in distribution. Thus, the sum of differentiable functions is differentiable almost surely.

Theorem 4.2: (Gradient Stability)

Brownian-Swish prevents vanishing gradients for negative inputs by maintaining non-zero stochastic derivatives.

Proof:

For $x > 0$, the gradient reduces to the Swish derivative:

$$\frac{\partial f}{\partial x} = \sigma(\beta x) + \beta x.\sigma(\beta x)\big(1 - \sigma(\beta x)\big)$$

Which is strictly positive for all $x > 0$.

For $x \leq 0$, the gradient includes a stochastic term:

$$\frac{\partial f}{\partial x} = \sigma(\beta x) + \beta x.\sigma(\beta x)\big(1 - \sigma(\beta x)\big) - \alpha \frac{1}{M}\sum_{k=1}^{M} \frac{\partial B^k(|x|)}{\partial x}$$

Since $\frac{\partial B^k(|x|)}{\partial x}$ is normally distributed with variance proportional to $|x|$, the gradient has non zero variance. Therefore, the probability of the gradient being exactly zero is negligible, ensuring neurons remain active.

Theorem 4.3: (Convergence of Monte Carlo Approximation)

As $M \to \infty$, the stochastic term in Brownian Swish converges to zero almost surely.

Proof:

By definition,

$$M \to \infty$$

$$\frac{1}{M}\sum_{k=1}^{M} B^k(|x|) \to E[B|x|] = 0$$

Since $B(|x|)\ N(0, |x|)$.

By the Strong law of large numbers, the sample mean converges almost surely to the expectation. Thus,

$$f(x; \alpha, \beta, M) = \frac{x}{1 + e^{-\beta x}}$$

Which is the deterministic Swish function.

Theorem 4.4:(Implicit Regularization)

Brownian Swish acts as an implicit regularizer by injecting input dependent noise.

Proof:

The stochastic term $-\alpha \frac{1}{M} \sum_{k=1}^{M} B^k(|x|)$ introduces variance proportional to $\alpha^2 |x|/M$. The variance scales with input magnitude, meaning larger inputs receive stronger perturbations. Such perturbations mimic noise injection strategies (e.g., dropout), but adaptively depend on input scale. Therefore, Brownian Swish implicitly regularizes the network by preventing overconfidence in large magnitude activations.

5. Empirical Setup:

The empirical results demonstrate that Brownian-Swish consistently outperforms both deterministic and stochastic activation baselines across vision, NLP, and time-series tasks. On CIFAR-10 and IMDB, Brownian-Swish achieves the highest test accuracy, surpassing Swish and Mish by 1–3 percentage points and outperforming ReLU by more than 5%. Robustness evaluations under adversarial perturbations and random noise confirm its stability, with relative accuracy maintained above 99%, while other activations degrade significantly. Most importantly, Brownian-Swish exhibits the smallest generalization gap—around 3.5% compared to 5–7% for baselines—highlighting its implicit regularization effect and reduced overfitting. Ablation studies on the number of Monte Carlo samples M further validate the theoretical convergence property, showing accuracy improvements as M increases. Together, these findings confirm that Brownian-Swish delivers superior accuracy, robustness, and generalization, making it a principled and practical activation function for modern deep learning applications.

6. Results and Discussion:

Brownian-Swish consistently delivers higher accuracy across vision, NLP, and time-series tasks compared to both classical and modern activations. It shows superior robustness under noise, maintaining stable performance even when inputs are perturbed, while other activations degrade. Most importantly, it achieves the smallest generalization gap, confirming its implicit regularization effect and reduced overfitting. These properties make Brownian-Swish not just a theoretical innovation but a practical tool: it ensures smoother optimization, stronger resilience in noisy environments, and better generalization to unseen data. In short, Brownian-Swish should be used because it combines the strengths of Swish with adaptive stochasticity, offering a principled and high-performing activation function for modern deep learning.

Table 1Test Accuracy Comparison (%)

Activation Function	CIFAR-10	IMDB Sentiment	Time-Series Forecasting
ReLU	84.2	81.5	78.9
Leaky ReLU	85	82.1	79.3
PReLU	85.6	82.4	79.7

ELU	86.1	83	80.2
GELU	87.3	84.2	81
Swish	88.5	85.6	82.3
Mish	88.1	85.2	82
Brownian ReLU	87	84.5	81.2
Brownian-Swish	**90.3**	**88.7**	**85.1**

Table 2 Robustness Under Noise (Relative Accuracy %)

Activation Function	**CIFAR-10 (Noise)**	**IMDB (Noise)**
ReLU	91.2	89
Swish	94.5	92.3
Mish	94.1	92
Brownian ReLU	95	92.8
Brownian-Swish	**100.8**	**99.5**

Tabl 3 Generalization Gap (%)

Activation Function	**CIFAR-10**	**IMDB Sentiment**	**Time-Series**
ReLU	6.8	6.2	6.5
Swish	5	4.8	4.7
Mish	5.2	4.9	4.8
Brownian ReLU	4.6	4.4	4.5
Brownian-Swish	**3.5**	**3.4**	**3.6**

Brownian-Swish always manages to stay at least marginally ahead of both traditional and state-of-the-art activation functions in vision, language, and time series tasks. If we observe Table 1, Brownian-Swish achieves 90.3% on CIFAR-10, 88.7% on IMDB sentiment classification, and 85.1% on time series forecasting. Brownian-Swish is also more accurate than Swish and Mish by around 1-3%, and more accurate than ReLU by more than 5%. This further shows the effectiveness of hybrid activation functions. From the above experiments, we have seen that Brownian-Swish, which is a fusion of Brownian motion and Swish operator, is effective in improving both optimization stability and accuracy. Brownian-Swish is not just an activation function; it is more than that.

Further tests of robustness in the presence of noise continue to prove the robustness of the Brownian-Swish activation function. In reference to Table 2 above, the relative accuracy of the Brownian-Swish activation function in the presence of noise is above 99%, whereas the relative accuracy of the other activation functions drops significantly. For instance, the relative accuracy of the Swish activation function, Mish activation function, and the ReLU activation function drops to 92-94% and below 91%, respectively. However, the relative accuracy of the

Brownian-Swish activation function remains unchanged. Therefore, the Brownian-Swish activation function can be used as an implicit safety net.
Lastly, the generalization results demonstrate the regularization property of Brownian-Swish. As summarized in Table 3, Brownian-Swish achieves the lowest generalization gap at 3.5%, whereas deterministic alternatives achieve 5-7% and stochastic alternatives achieve 4.4-4.6%. The reduction in overfitting is due to the stochastic component, which introduces an adaptive form of noise to prevent models from becoming overconfident about their training data. This smoothness of optimization, robustness, and generalization to new samples is an advantage of Brownian-Swish, which can be concluded as a principled activation function that combines smoothness with stochastic adaptability.

7. Conclusion:

In this paper, we introduced **Brownian-Swish**, a novel activation function that integrates the smooth non-linearity of Swish with adaptive stochastic perturbations inspired by Brownian motion. Through rigorous theoretical analysis, we established its differentiability, convergence to Swish, implicit regularization, and gradient stability. Empirical evaluations across vision, NLP, and time-series tasks confirmed these properties, showing that Brownian-Swish consistently achieves higher accuracy, superior robustness under noise, and smaller generalization gaps compared to both deterministic and stochastic baselines. Together, these findings demonstrate that Brownian-Swish is not only a theoretical innovation but also a practical activation function that improves training stability and generalization in modern deep learning.
Looking ahead, several promising directions emerge. First, adaptive sampling strategies could dynamically adjust the number of Monte Carlo samples M during training, balancing efficiency and robustness. Second, extending the framework to alternative stochastic processes beyond Brownian motion may yield new families of hybrid activations with distinct regularization properties. Third, integrating Brownian-Swish into large-scale architectures such as Transformers, diffusion models, and multimodal networks could further validate its scalability and impact. Finally, exploring theoretical links to Bayesian deep learning may provide deeper insights into uncertainty-aware optimization. These avenues suggest that Brownian-Swish is not only a high-performing activation today but also a foundation for future research in hybrid deterministic–stochastic activation design.

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References:

1. Rumelhart, D. E., Hinton, G. E., & Williams, R. J. (1986). Learning representations by back-propagating errors. *nature*, *323*(6088), 533-536.
2. LeCun, Y., Bottou, L., Orr, G. B., & Müller, K. R. (2002). Efficient backprop. In *Neural networks: Tricks of the trade* (pp. 9-50). Berlin, Heidelberg: Springer Berlin Heidelberg.
3. Nair, V., & Hinton, G. E. (2010). Rectified linear units improve restricted boltzmann machines. In *Proceedings of the 27th international conference on machine learning (ICML-10)* (pp. 807-814).
4. Maas, A. L., Hannun, A. Y., & Ng, A. Y. (2013, June). Rectifier nonlinearities improve neural network acoustic models. In *Proc. icml* (Vol. 30, No. 1, p. 3).
5. He, K., Zhang, X., Ren, S., & Sun, J. (2015). Delving deep into rectifiers: Surpassing human-level performance on imagenet classification. In *Proceedings of the IEEE international conference on computer vision* (pp. 1026-1034).
6. Clevert, D. A., Unterthiner, T., & Hochreiter, S. (2015). Fast and accurate deep network learning by exponential linear units (elus). *arXiv preprint arXiv:1511.07289*, *4*(5), 11.
7. Klambauer, G., Unterthiner, T., Mayr, A., & Hochreiter, S. (2017). Self-Normalizing Neural Networks. Advances in Neural Information Processing Systems (NeurIPS).
8. Glorot, X., Bordes, A., & Bengio, Y. (2011, June). Deep sparse rectifier neural networks. In *Proceedings of the fourteenth international conference on artificial intelligence and statistics* (pp. 315-323). JMLR Workshop and Conference Proceedings.
9. Hendrycks, D., & Gimpel, K. (2016). Gaussian error linear units (gelus). *arXiv preprint arXiv:1606.08415*.
10. Ramachandran, P., Zoph, B., & Le, Q. V. (2017). Searching for activation functions. *arXiv preprint arXiv:1710.05941*.
11. Misra, D. (2019). Mish: A self regularized non-monotonic activation function. *arXiv preprint arXiv:1908.08681*.
12. Gulcehre, C., Moczulski, M., Denil, M., & Bengio, Y. (2016, June). Noisy activation functions. In *International conference on machine learning* (pp. 3059-3068). PMLR.
13. Xu, Bing & Wang, Naiyan & Chen, Tianqi & Li, Mu. (2015). Empirical Evaluation of Rectified Activations in Convolutional Network.
14. Awiakye-Marfo, G., Agbosu, E., Barns, V. M., & Gyamerah, S. A. (2026). Brownian ReLU (Br-ReLU): A New Activation Function for a Long-Short Term Memory (LSTM) Network. *arXiv preprint arXiv:2601.16446*.
15. Agarwal, M. (2024). Comparison of activation functions in neural networks. *International Journal of Advanced Education and Research*, *9*(3), 10–16.
16. Subramanian, B., Jeyaraj, R., & Ugli, R. A. A. (2024). APALU: a trainable, adaptive activation function for deep learning networks. *arXiv preprint arXiv:2402.08244*.
17. Thakur, A., & Dhawale, C. (2024, June). Activation Functions: Bridging the Gap Between Theory and Application in Deep Learning. In *2024 International Conference on Innovations and Challenges in Emerging Technologies (ICICET)* (pp. 1-6). IEEE.