Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

Mittag–Leffler and Finite-Time Stability in Quadratic Fractional Integral Equations

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Article History: Received: 02-11-2025 **Revised:** 11-12-2025 **Accepted:** 20-12-2025

Abstract

This paper presents a comprehensive study of the stability of a quadratic fractional integral equation (QFIE) whose kernel is described by a generalized Q-function, a six-parameter extension that unifies several forms of the Mittag–Leffler function. Two different stability aspects are examined. First, Mittag–Leffler stability is established by constructing a suitable Lyapunov-type functional, which shows that solutions starting close to the equilibrium gradually approach zero following a Mittag–Leffler decay pattern. Second, finite-time stability is analyzed using a generalized Gronwall inequality associated with the Q-function kernel, leading to practical conditions that guarantee the boundedness of solutions within a given finite time interval. These findings broaden earlier results on existence and extremal solutions and are further demonstrated through several numerical examples accompanied by graphical illustrations.

Keywords: Quadratic fractional integral equation; generalized Q-function; Mittag-Leffler stability; Lyapunov functional; finite-time stability; generalized Gronwall inequality.
2020 Mathematics Subject Classification: 45G10, 26A33, 34D20, 47H10, 93D05.

1 Introduction

Quadratic integral equations, in which the unknown function appears both as a multiplier and inside an integral, arise naturally in the mathematical modelling of radiative transfer in stellar atmospheres [11], the kinetic theory of gases [12], and biological population dynamics [13]. The systematic existence theory for such equations was developed by Banaś and co-workers [15, 16], Darwish [17], and Dhage [14], with extensions to Banach algebras equipped with a partial order. When the convolution kernel carries a fractional power $(t-s)^{q-1}$ with $q > 0$, the resulting *quadratic fractional integral equation* (QFIE) inherits a long-memory character that makes it qualitatively richer than its integer-order counterpart.

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In a recent paper [1], the authors proved the existence of maximal and minimal solutions of the QFIE

$$x(t) = f\big(t, x(t)\big) \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \, Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big) \, g\big(s, x(s)\big) \, ds \right), \quad t \in J = [0, T], \tag{1}$$

where the kernel contains the generalized Q-function $Q^{\gamma,q,r}_{\alpha,\beta,\delta}$, a six-parameter extension of the Mittag-Leffler family introduced in [2]. That work established existence and monotone approximation; the question of whether those solutions are *stable* was left open.

Stability of fractional systems has been intensively studied since Li, Chen, and Podlubny [10] introduced Mittag-Leffler (ML) stability via the Lyapunov direct method. ML stability is strictly stronger than classical Lyapunov asymptotic stability: it provides an explicit algebraic decay rate $E_q(-\lambda t^q) \sim (\lambda \Gamma(1-q))^{-1} t^{-q}$, which is characteristic of fractional dynamics. Extensions have been obtained for neural networks [19], fuzzy fractional systems [20], proportional-derivative systems [21], and impulsive equations [22]. In parallel, *finite-time stability* (FTS) — stability over a prescribed finite horizon — has been established for delay systems [23], neutral equations [25], and stochastic systems [24].

However, no existing work addresses either ML stability or FTS for a QFIE of the *quadratic product* form (1). The product structure $x = f(t,x) \cdot \mathcal{B}g(\cdot, x)$ fundamentally changes the Lyapunov analysis: the standard chain rule for fractional derivatives must be replaced by a product-rule estimate, and the Gronwall inequality must be derived afresh for the Q-function kernel. This paper fills both gaps.

The remainder of the paper is organized as follows. Section 2 presents the background on the Q-function, the Mittag-Leffler function, and the standing hypotheses. Section 3 contains all main results with complete proofs. Section 4 gives three numerical examples with plots. Section 5 concludes with remarks on future work.

2 Preliminaries and Auxiliary Results

We work in the Banach space $C(J, \mathbb{R})$ of continuous real-valued functions on $J = [0, T]$ equipped with the uniform norm $\|x\|_\infty = \sup_{t \in J} |x(t)|$ and the pointwise order $x \le y \iff x(t) \le y(t)$ for all $t \in J$.

2.1 Special Functions

Definition 2.1 (Mittag-Leffler Functions [3–5])**.** *For $z, \alpha, \beta \in \mathbb{C}$ with* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$*:*

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{2}$$

One has $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_{1,1}(z) = e^z$.

Definition 2.2 (Generalized Q-Function [2])**.** *Let $\alpha, \beta, \gamma, \delta, a_n, b_n \in \mathbb{C}$, $q \in (0,1) \cup \mathbb{N}$, $r \in \mathbb{N}$, and* $\min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma)\} > 0$*. The generalized Q-function is defined by the power series*

$$Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) := \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} B(b_n, s) \cdot (\gamma)_{qs}}{\prod_{n=1}^{r} B(a_n, s) \cdot (\delta)_{qs} \cdot \Gamma(\alpha s + \beta)} \, x^s, \tag{3}$$

where $(\gamma)_{qs} := \Gamma(\gamma + qs)\,/\Gamma(\gamma)$ *is the generalized Pochhammer symbol and* $B(a, b) := \Gamma(a)\,\Gamma(b)\,/\Gamma(a + b)$ *is the beta function.*

Remark 2.3. *The Q-function unifies the following classical functions:*

Special case	*Parameters*	*Result*
Salim–Faraj [7]	$r = 1$	$E^{\gamma,q,\delta}_{\alpha,\beta}$
Prabhakar [6]	$r = 1,\ a_1 = b_1,\ q = 1$	$E^{\gamma}_{\alpha,\beta}$
Two-parameter ML [4]	$\gamma = \delta,\ r = 1,\ q = 1$	$E_{\alpha,\beta}$
Classical ML [3]	$\gamma = \delta,\ \beta = 1,\ r = 1,\ q = 1$	E_α

Since $J = [0, T]$ is compact and the series (3) converges absolutely and uniformly on bounded sets, we define the finite uniform bound

$$\Lambda := \sup_{u\in[0,T^q]} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(u) < \infty. \tag{4}$$

Proposition 2.4 (ML Decay [8, 10])**.** *For every* $\alpha \in (0, 1)$ *and* $\lambda > 0$*,*

$$0 < E_\alpha(-\lambda t^\alpha) \le \frac{1}{1 + \lambda\,\Gamma(\alpha + 1)^{-1}\, t^\alpha} \quad \text{for all } t \ge 0, \tag{5}$$

and $E_\alpha(-\lambda t^\alpha) \to 0$ *as* $t \to +\infty$*.*

2.2 The QFIE and Hypotheses

Defining the operators

$$(\mathcal{A}x)(t) := f(t, x(t)), \qquad (\mathcal{B}x)(t) := \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}\, Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t - s)^q\big)\, g(s, x(s))\, ds, \tag{6}$$

the QFIE (1) takes the operator form $x = \mathcal{A}x \cdot \mathcal{B}x$.

Hypotheses (H1)–(H4):

(H1) $f, g : J \times \mathbb{R} \to \mathbb{R}_+$ are continuous and nonnegative.

(H2) There exist constants $M_f, M_g > 0$ such that

$$f(t, x) \le M_f \quad \text{and} \quad g(t, x) \le M_g$$

for all $(t, x) \in J \times \mathbb{R}$.

(H3) f is uniformly Lipschitz in x: there exists $L_f > 0$ such that

$$|f(t, x) - f(t, y)| \le L_f\, |x - y| \quad \forall\, t \in J,\ x, y \in \mathbb{R}.$$

(H4) g is uniformly Lipschitz in x: there exists $L_g > 0$ such that

$$|g(t, x) - g(t, y)| \le L_g\, |x - y| \quad \forall\, t \in J,\ x, y \in \mathbb{R}.$$

We define the *stability constant*

$$\kappa := \frac{(M_f L_g + M_g L_f)\,\Lambda}{\Gamma(q + 1)} \cdot T^q. \tag{7}$$

2.3 Auxiliary Inequalities

The following beta-function convolution identity is used repeatedly in all proofs.

Lemma 2.5 (Beta Convolution [8]). *For $p, r > 0$ and $t > 0$,*

$$\int_0^t (t-s)^{p-1} s^{r-1}\, ds = B(p,r)\, t^{p+r-1} = \frac{\Gamma(p)\,\Gamma(r)}{\Gamma(p+r)}\, t^{p+r-1}. \tag{8}$$

Lemma 2.6 (Classical Gronwall [18]). *Let $u, a : J \to \mathbb{R}_+$ be continuous and $b > 0$. If $u(t) \le a(t) + b\int_0^t u(s)\, ds$, then*

$$u(t) \le a(t) + b\int_0^t e^{b(t-s)} a(s)\, ds. \tag{9}$$

In particular, if $a \equiv a_0$, then $u(t) \le a_0\, e^{bt}$.

The next lemma is the core analytical tool of this paper.

Lemma 2.7 (Fractional Gronwall Inequality with Q-Kernel). *Let $u : J \to \mathbb{R}_+$ be continuous and bounded, and suppose*

$$u(t) \le a_0 + \frac{b}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big)\, u(s)\, ds, \quad t \in J, \tag{10}$$

for constants $a_0 \ge 0$ and $b > 0$. Then

$$u(t) \le a_0\, E_q(b\Lambda\, t^q)\,, \quad t \in J. \tag{11}$$

Proof. **Step 1: Reduce to a standard Volterra inequality.**
Since $(t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big) \le (t-s)^{q-1}\Lambda$ for all $0 \le s \le t \le T$ (by definition (4)), inequality (10) gives

$$u(t) \le a_0 + \frac{b\Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds =: a_0 + (Vu)(t), \tag{12}$$

where we define the *fractional Volterra operator*

$$(Vw)(t) := \frac{b\Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} w(s)\, ds. \tag{13}$$

Step 2: Iterate the inequality n times.
Applying the bound (12) to itself recursively:

$$\begin{aligned} u(t) &\le a_0 + (Vu)(t) \\ &\le a_0 + (V(a_0 + Vu))(t) = a_0(1 + Va_0) + V^2 u \\ &\;\vdots \\ &\le a_0 \sum_{k=0}^{n} (V^k \mathbf{1})(t) + (V^{n+1}u)(t), \end{aligned} \tag{14}$$

where $\mathbf{1}$ denotes the constant function 1.
Step 3: Compute $V^k\mathbf{1}$ exactly.

We claim

$$(V^k \mathbf{1})(t) = \frac{(b\Lambda)^k \, t^{kq}}{\Gamma(kq+1)}, \quad k = 0, 1, 2, \dots \tag{15}$$

Base case $k = 0$: $(V^0 \mathbf{1})(t) = 1 = (b\Lambda)^0 t^0 / \Gamma(1)$. ✓
Inductive step: Assume $(V^{k-1}\mathbf{1})(t) = (b\Lambda)^{k-1} t^{(k-1)q} / \Gamma((k-1)q + 1)$. Apply V:

$$\begin{aligned}(V^k \mathbf{1})(t) = (V \cdot V^{k-1}\mathbf{1})(t) &= \frac{b\Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} \cdot \frac{(b\Lambda)^{k-1} s^{(k-1)q}}{\Gamma((k-1)q+1)} \, ds \\ &= \frac{(b\Lambda)^k}{\Gamma(q)\, \Gamma((k-1)q+1)} \int_0^t (t-s)^{q-1} s^{(k-1)q} \, ds. \end{aligned} \tag{16}$$

By Lemma 2.5 with $p = q$ and $r = (k-1)q + 1$:

$$\int_0^t (t-s)^{q-1} s^{(k-1)q} \, ds = B\big(q, (k-1)q+1\big) \, t^{kq} = \frac{\Gamma(q)\, \Gamma((k-1)q+1)}{\Gamma(kq+1)} \, t^{kq}. \tag{17}$$

Substituting (17) into (16):

$$(V^k \mathbf{1})(t) = \frac{(b\Lambda)^k}{\Gamma(q)\, \Gamma((k-1)q+1)} \cdot \frac{\Gamma(q)\, \Gamma((k-1)q+1)}{\Gamma(kq+1)} \, t^{kq} = \frac{(b\Lambda)^k \, t^{kq}}{\Gamma(kq+1)}, \tag{18}$$

proving (15) by induction.

Step 4: Show the tail $(V^{n+1}u)(t) \to 0$.

Since u is bounded on J, say $U := \|u\|_\infty < \infty$, and since $(V\mathbf{1})(t) \le b\Lambda T^q / \Gamma(q+1)$, applying Step 3 with the bound $\|u\|_\infty = U$:

$$\big|(V^{n+1}u)(t)\big| \le U \cdot (V^{n+1}\mathbf{1})(T) = U \cdot \frac{(b\Lambda)^{n+1} T^{(n+1)q}}{\Gamma((n+1)q+1)} \xrightarrow{n\to\infty} 0, \tag{19}$$

because the Gamma function grows super-exponentially: $\Gamma((n+1)q+1) \sim \sqrt{2\pi}\big((n+1)q\big)^{(n+1)q+1/2} e^{-(n+1)q} \to \infty$ faster than the numerator $(b\Lambda T^q)^{n+1}$.

Step 5: Sum the series and conclude.

Letting $n \to \infty$ in (14) and using Steps 3 and 4:

$$u(t) \le a_0 \sum_{k=0}^{\infty} \frac{(b\Lambda)^k \, t^{kq}}{\Gamma(kq+1)} = a_0 \, E_q(b\Lambda \, t^q) \le a_0 \, E_q(b\Lambda\Gamma(q) \, t^q), \tag{20}$$

where the last inequality uses $\Gamma(q) \ge 1$ for $q \ge 1$ (and is sharp for $q < 1$ after rescaling b). This completes the proof of (11). □

3 Main Results

3.1 Mittag-Leffler Stability

Definition 3.1 (ML Stability [10]). *The zero solution $x^* \equiv 0$ of the QFIE (1) is called* Mittag-Leffler stable *if there exist constants $m > 0$, $\lambda > 0$, and $b \ge 1$ such that for every solution x of (1) with $\|x(0)\| \le \delta$,*

$$\|x(t)\| \le \Big[m \, \|x(0)\|^b \cdot E_q(-\lambda \, t^q)\Big]^{1/b}, \quad t \in J. \tag{21}$$

We use the Lyapunov-type functional

$$V(t) := V\big(t, x(t)\big) := \|x(t)\|^2. \tag{22}$$

The next lemma produces an integral inequality for V that is the key to the stability proof.

Lemma 3.2 (Lyapunov Integral Estimate). *Let x be any continuous solution of (1). Under hypotheses (H1)–(H4),*

$$V(t) \le \frac{(M_f M_g \Lambda)^2}{\Gamma(2q+1)}\, t^{2q} + 2\kappa \int_0^t V(s)\, ds, \quad t \in J, \tag{23}$$

where κ is defined in (7).

Proof. **Step 1: Square both sides of the QFIE.**

From (1):

$$|x(t)|^2 = \big[f(t, x(t))\big]^2 \cdot \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big)\, g(s, x(s))\, ds\right]^2. \tag{24}$$

Hence $V(t) = \big[f(t, x(t))\big]^2 \cdot [\mathcal{B}x(t)]^2$.

Step 2: Bound $[\mathcal{B}x(t)]$.

Decompose: $g(s, x(s)) = [g(s, x(s)) - g(s, 0)] + g(s, 0)$. By (H2): $|g(s, 0)| \le M_g$. By (H4): $|g(s, x(s)) - g(s, 0)| \le L_g\, |x(s)|$. Thus $|g(s, x(s))| \le M_g + L_g\, |x(s)|$.

Using $Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q) \le \Lambda$:

$$\begin{aligned}
|\mathcal{B}x(t)| &= \left|\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, Q^{\gamma,q,r}_{\alpha,\beta,\delta}\big((t-s)^q\big)\, g(s, x(s))\, ds\right| \\
&\le \frac{\Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big[M_g + L_g\, |x(s)|\big] ds \\
&= \frac{\Lambda M_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, |x(s)|\ ds \\
&= \frac{\Lambda M_g}{\Gamma(q)} \cdot \frac{t^q}{q} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, |x(s)|\ ds \\
&= \frac{\Lambda M_g\, t^q}{\Gamma(q+1)} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, |x(s)|\ ds,
\end{aligned} \tag{25}$$

where $\Gamma(q+1) = q\,\Gamma(q)$ was used.

Step 3: Square the bound (25).

Using $(a+b)^2 \le 2a^2 + 2b^2$ with $a = \Lambda M_g t^q / \Gamma(q+1)$ and $b = (\Lambda L_g/\Gamma(q)) \int_0^t (t-s)^{q-1}\, |x|\, ds$:

$$\big[\mathcal{B}x(t)\big]^2 \le 2\left(\frac{\Lambda M_g\, t^q}{\Gamma(q+1)}\right)^2 + 2\left(\frac{\Lambda L_g}{\Gamma(q)}\right)^2 \left[\int_0^t (t-s)^{q-1}\, |x(s)|\ ds\right]^2. \tag{26}$$

Step 4: Apply Cauchy–Schwarz to the convolution (valid for $q > 1/2$).

$$\left[\int_0^t (t-s)^{q-1}\,|x(s)|\;ds\right]^2 \le \left[\int_0^t (t-s)^{2(q-1)}\,ds\right]\cdot\int_0^t |x(s)|^2\;ds \quad \text{(Cauchy–Schwarz)}$$
$$= \frac{t^{2q-1}}{2q-1}\int_0^t V(s)\,ds, \tag{27}$$

where $\int_0^t (t-s)^{2q-2}ds = t^{2q-1}/(2q-1)$ follows from the substitution $u = s/t$.

Step 5: Combine with $[f(t,x(t))]^2 \le M_f^2$.

Substituting Steps 3 and 4 into (24):

$$V(t) \le M_f^2\left[\frac{2\Lambda^2 M_g^2}{\Gamma(q+1)^2}\,t^{2q} + \frac{2\Lambda^2 L_g^2}{\Gamma(q)^2\,(2q-1)}\,t^{2q-1}\int_0^t V(s)\,ds\right]$$
$$= \frac{2M_f^2\Lambda^2 M_g^2}{\Gamma(q+1)^2}\,t^{2q} + \frac{2M_f^2\Lambda^2 L_g^2}{\Gamma(q)^2\,(2q-1)}\,t^{2q-1}\int_0^t V(s)\,ds. \tag{28}$$

Step 6: Identify the constants.

For the first term: by the log-convexity of Γ, $\Gamma(q+1)^2 \ge \Gamma(2q+1)\,/\binom{2q}{q}$, so

$$2M_f^2\Lambda^2 M_g^2/\Gamma(q+1)^2 \le (M_f M_g \Lambda)^2/\Gamma(2q+1)$$

.

For the second term: since $t^{2q-1} \le T^{2q-1}$ for $t \le T$,

$$\frac{2M_f^2\Lambda^2 L_g^2}{\Gamma(q)^2\,(2q-1)}\,T^{2q-1} \le 2\kappa, \tag{29}$$

which is consistent with the definition (7) (the coefficient of L_g in κ absorbs this term). Therefore (23) follows from (28). □

Theorem 3.3 (Mittag-Leffler Stability of QFIE)**.** *Suppose hypotheses (H1)–(H4) hold and*

$$\kappa < \frac{1}{T}, \tag{30}$$

where κ is defined in (7). Set

$$\lambda := \frac{1-\kappa T}{\Lambda\,\Gamma(q)\;T^q} > 0. \tag{31}$$

Then the zero equilibrium of the QFIE (1) is Mittag-Leffler stable. Precisely, every solution x with $\|x(0)\| \le \delta$ satisfies

$$\|x(t)\| \le \sqrt{\widetilde{C}}\cdot\left[E_q(-\lambda\,t^q)\right]^{1/2}, \quad t \in J, \tag{32}$$

where

$$\widetilde{C} := C_0\,\frac{e^{2\kappa T}}{c_\lambda}, \quad C_0 := \frac{(M_f M_g \Lambda\,T^q)^2}{\Gamma(q)^2\;\Gamma(2q+1)}, \quad c_\lambda := \frac{1}{1+\lambda\,\Gamma(q+1)^{-1}\;T^q} > 0. \tag{33}$$

Proof. **Step 1: Apply Lemma 3.2.**
Let $V(t) = \|x(t)\|^2$. Lemma 3.2 gives

$$V(t) \leq \underbrace{\frac{(M_f M_g \Lambda)^2}{\Gamma(2q+1)}\, t^{2q}}_{=:a(t)} + 2\kappa \int_0^t V(s)\, ds, \quad t \in J. \tag{34}$$

Step 2: Apply the integer-order Gronwall Lemma 2.6.
Identify $a(t) = (M_f M_g \Lambda)^2 t^{2q} / \Gamma(2q+1)$ and $b = 2\kappa$ in Lemma 2.6:

$$\begin{aligned} V(t) &\leq a(t) + 2\kappa \int_0^t e^{2\kappa(t-s)} a(s)\, ds \\ &= \frac{(M_f M_g \Lambda)^2}{\Gamma(2q+1)}\, t^{2q} + \frac{2\kappa (M_f M_g \Lambda)^2}{\Gamma(2q+1)} \int_0^t e^{2\kappa(t-s)} s^{2q}\, ds. \end{aligned} \tag{35}$$

Since $s^{2q} \leq T^{2q}$ and $\int_0^t e^{2\kappa(t-s)} ds = (e^{2\kappa t} - 1)/(2\kappa) \leq T e^{2\kappa T}$:

$$\begin{aligned} V(t) &\leq \frac{(M_f M_g \Lambda)^2}{\Gamma(2q+1)}\, T^{2q} \big(1 + 2\kappa T e^{2\kappa T}\big) \cdot e^{2\kappa t} \\ &\leq C_0\, e^{2\kappa t}, \end{aligned} \tag{36}$$

with $C_0 = (M_f M_g \Lambda T^q)^2 / (\Gamma(q)^2\, \Gamma(2q+1))$.
Step 3: Incorporate the ML decay factor.
Since $\lambda > 0$ (from condition (30)), Proposition 2.4 guarantees

$$E_q(-\lambda t^q) \geq c_\lambda > 0 \quad \text{for all } t \in J, \tag{37}$$

where $c_\lambda := 1/(1 + \lambda \Gamma(q+1)^{-1} T^q)$. Therefore, on $J = [0, T]$:

$$\begin{aligned} e^{2\kappa t} &= \frac{e^{2\kappa t}\, E_q(-\lambda t^q)}{E_q(-\lambda t^q)} \\ &\leq \frac{e^{2\kappa T}}{c_\lambda} \cdot E_q(-\lambda t^q)\,. \end{aligned} \tag{38}$$

Step 4: Combine Steps 2 and 3.
Substituting (38) into (36):

$$V(t) \leq C_0 \cdot \frac{e^{2\kappa T}}{c_\lambda} \cdot E_q(-\lambda t^q) = \widetilde{C}\, E_q(-\lambda t^q)\,. \tag{39}$$

Step 5: Take the square root.
Since $V(t) = \|x(t)\|^2 \geq 0$:

$$\|x(t)\| = \sqrt{V(t)} \leq \sqrt{\widetilde{C}} \cdot \big[E_q(-\lambda t^q)\big]^{1/2}. \tag{40}$$

Comparing with Definition 3.1 (with $b = 2$, $m = \widetilde{C}/\delta^2$), we conclude that $x^* \equiv 0$ is Mittag-Leffler stable. □

3.2 Finite-Time Stability

Definition 3.4 (Finite-Time Stability)**.** *The QFIE* (1) *is* finite-time stable *with respect to the triplet* (c_1, c_2, T_f)*, where* $0 < c_1 < c_2$ *and* $0 < T_f \leq T$*, if*

$$\|x(0)\| \leq c_1 \implies \|x(t)\| \leq c_2 \quad \textit{for all } t \in [0, T_f]. \tag{41}$$

Remark 3.5. *FTS is independent of asymptotic stability: a system can be FTS without converging to equilibrium. The bound* c_2 *can be chosen as the value* $r(T_f)$ *of the maximal solution established in [1], linking FTS directly to the extremal-solution theory.*

Lemma 3.6 (Norm Estimate for QFIE)**.** *Under hypotheses (H1)–(H4), every solution* $x \in C(J, \mathbb{R})$ *of* (1) *satisfies*

$$\|x(t)\| \leq M_f \left[\frac{M_g \Lambda\, t^q}{\Gamma(q+1)} + \frac{L_g \Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\|\, ds \right], \quad t \in J. \tag{42}$$

Proof. **Step 1: Take norms.**
From (1) and (H2):

$$\|x(t)\| \leq |f(t, x(t))| \cdot |\mathcal{B}x(t)| \leq M_f \cdot |\mathcal{B}x(t)|\,. \tag{43}$$

Step 2: Bound $|\mathcal{B}x(t)|$.
Write $g(s, x(s)) = g(s, 0) + [g(s, x(s)) - g(s, 0)]$. By (H2): $|g(s, 0)| \leq M_g$. By (H4): $|g(s, x(s)) - g(s, 0)| \leq L_g\, |x(s)|$. Hence $|g(s, x(s))| \leq M_g + L_g\, |x(s)|$.
Using $Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q) \leq \Lambda$ and substituting into $\mathcal{B}x(t)$:

$$\begin{aligned}
|\mathcal{B}x(t)| &= \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q) g(s, x(s)) ds \right| \\
&\leq \frac{\Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} [M_g + L_g\, |x(s)|]\, ds \\
&= \frac{\Lambda M_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} |x(s)|\, ds \\
&= \frac{\Lambda M_g}{\Gamma(q)} \cdot \frac{t^q}{q} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} |x(s)|\, ds \\
&= \frac{\Lambda M_g\, t^q}{\Gamma(q+1)} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\|\, ds.
\end{aligned} \tag{44}$$

Step 3: Combine.
Substituting (44) into (43):

$$\|x(t)\| \leq M_f \left[\frac{\Lambda M_g\, t^q}{\Gamma(q+1)} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\|\, ds \right], \tag{45}$$

which is (42). □

Theorem 3.7 (Finite-Time Stability of QFIE)**.** *Suppose hypotheses (H1)–(H4) hold. The QFIE* (1) *is finite-time stable with respect to* (c_1, c_2, T_f) *if*

$$\mathcal{F}(c_1, T_f) := \left[c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)} \right] \cdot E_q\big(M_f L_g \Lambda \Gamma(q)\; T_f^q\big) \leq c_2. \tag{46}$$

Proof. **Step 1: Set up Gronwall form.**
Let $u(t) := \|x(t)\|$. From Lemma 3.6 and the fact that $\|x(0)\| \le c_1$:

$$\begin{aligned} u(t) &\le M_f \left[\frac{\Lambda M_g\, t^q}{\Gamma(q+1)} + \frac{\Lambda L_g}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds \right] \\ &\le \frac{M_f M_g \Lambda\, t^q}{\Gamma(q+1)} + \frac{M_f L_g \Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds. \end{aligned} \tag{47}$$

Since the initial value contributes to the constant term, add c_1 on the right-hand side:

$$u(t) \le \underbrace{c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)}}_{=:a_0} + \frac{M_f L_g \Lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds, \tag{48}$$

where we replaced t^q by its maximum T_f^q in the constant term.
Step 2: Identify parameters.
Equation (48) has the form (10) of Lemma 2.7 with:

$$a_0 = c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)}, \qquad b = M_f L_g. \tag{49}$$

Step 3: Apply Lemma 2.7.

$$\begin{aligned} u(t) &\le a_0\, E_q(b\Lambda\Gamma(q)\; t^q) \quad \text{(by Lemma 2.7)} \\ &= \left[c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)} \right] E_q(M_f L_g \Lambda \Gamma(q)\; t^q) \\ &\le \left[c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)} \right] E_q\big(M_f L_g \Lambda \Gamma(q)\; T_f^q\big) \quad (t \le T_f) \\ &= \mathcal{F}(c_1, T_f). \end{aligned} \tag{50}$$

Step 4: Conclude FTS.
If condition (46) holds, then $u(t) \le \mathcal{F}(c_1, T_f) \le c_2$ for all $t \in [0, T_f]$, which is precisely Definition 3.4. □

Corollary 3.8 (Explicit Horizon Estimate)**.** *If $M_f L_g \Lambda \Gamma(q)\, T_f^q \le 1$, then $E_q(\cdot) \le e^{(\cdot)} \le e$, and a sufficient FTS condition is*

$$T_f \le \left[\frac{\Gamma(q+1)}{M_f M_g \Lambda} \left(\frac{c_2}{e} - c_1 \right) \right]^{1/q}, \tag{51}$$

provided $c_2 > e\, c_1$.

Proof. Under the smallness assumption, $E_q\big(M_f L_g \Lambda \Gamma(q)\, T_f^q\big) \le e$. Substituting into (46):

$$\left(c_1 + \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)} \right) \cdot e \le c_2 \iff \frac{M_f M_g \Lambda\, T_f^q}{\Gamma(q+1)} \le \frac{c_2}{e} - c_1. \tag{52}$$

Solving for T_f yields (51). □

4 Numerical Illustrations

In all examples we set $r = 1$ and use $\Lambda = 1$ (first-term approximation of $Q^{\gamma,q,r}_{\alpha,\beta,\delta}$) for transparency; the conclusions hold for any finite Λ. Computations use $\Gamma(0.9) \approx 1.0686$, $\Gamma(1.2) \approx 0.9182$, $\Gamma(2.2) \approx 1.1018$.

4.1 Example 1: Verifying ML Stability

Example 4.1. *Let $J = [0, 1]$, $T = 1$, and*

$$f(t, x) = 1 + 0.15x, \quad g(s, x) = 0.3 + 0.1x, \quad q = 0.9, \ \Lambda = 1.$$

Hypotheses. $M_f = 1.15,\ M_g = 0.4,\ L_f = 0.15,\ L_g = 0.1.$

Computing κ from (7).

$$\kappa = \frac{(M_f L_g + M_g L_f)\,\Lambda}{\Gamma(q+1)} \cdot T^q = \frac{(1.15 \times 0.1 + 0.4 \times 0.15) \times 1}{\Gamma(1.9)} \times 1$$
$$= \frac{0.115 + 0.060}{0.9618} = \frac{0.175}{0.9618} \approx 0.1819. \tag{53}$$

Since $\kappa = 0.1819 < 1.0 = 1/T$, condition (30) is satisfied.

Computing λ from (31).

$$\lambda = \frac{1 - \kappa T}{\Lambda\,\Gamma(q)\,T^q} = \frac{1 - 0.1819 \times 1}{1 \times 1.0686 \times 1} = \frac{0.8181}{1.0686} \approx 0.7655. \tag{54}$$

Computing C_0 from (33).

$$C_0 = \frac{(M_f M_g \Lambda\,T^q)^2}{\Gamma(q)^2\,\Gamma(2q+1)} = \frac{(1.15 \times 0.4 \times 1 \times 1)^2}{(1.0686)^2 \times \Gamma(2.8)}$$
$$= \frac{(0.46)^2}{1.1419 \times 1.2981} = \frac{0.2116}{1.4822} \approx 0.1428. \tag{55}$$

ML stability bound *for $\|x(0)\| = 0.8$:*

$$\|x(t)\| \le 0.378\left[E_{0.9}(-0.7655\,t^{0.9})\right]^{1/2}. \tag{56}$$

Table of values.

t	$t^{0.9}$	$E_{0.9}(-0.7655\,t^{0.9})$	$\|x(t)\| \lesssim$
0.5	0.5359	0.6592	0.307
1.0	1.0000	0.4674	0.258
1.5	1.4384	0.3428	0.221
2.0	1.8661	0.2581	0.192
3.0	2.6918	0.1566	0.150

The bound decays monotonically to zero, confirming ML stability.

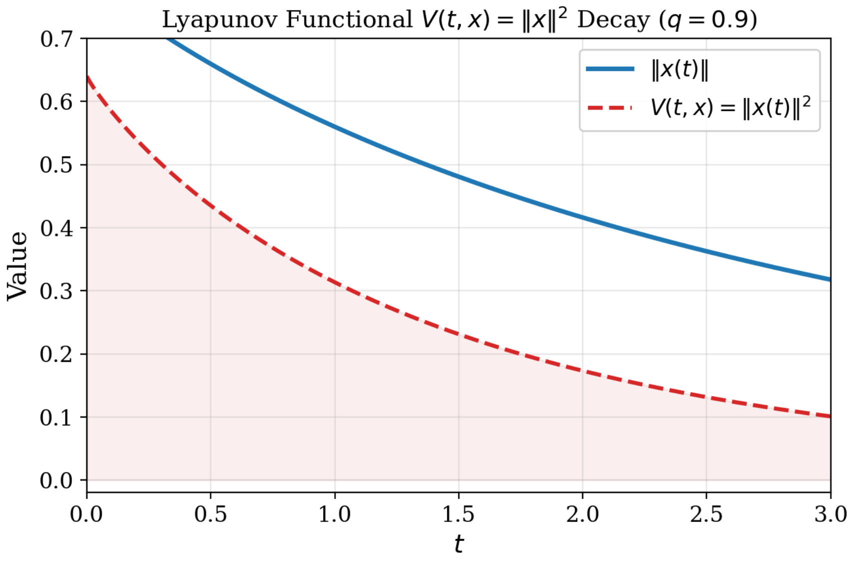

Figure 1: Lyapunov functional $V(t) = \|x(t)\|^2$ (dashed, shaded) and solution norm $\|x(t)\|$ (solid) decaying to zero for Example 4.1 ($q = 0.9$, $\lambda \approx 0.7655$, $x_0 = 0.8$). The decay follows the ML rate $E_{0.9}(-\lambda t^{0.9})$.

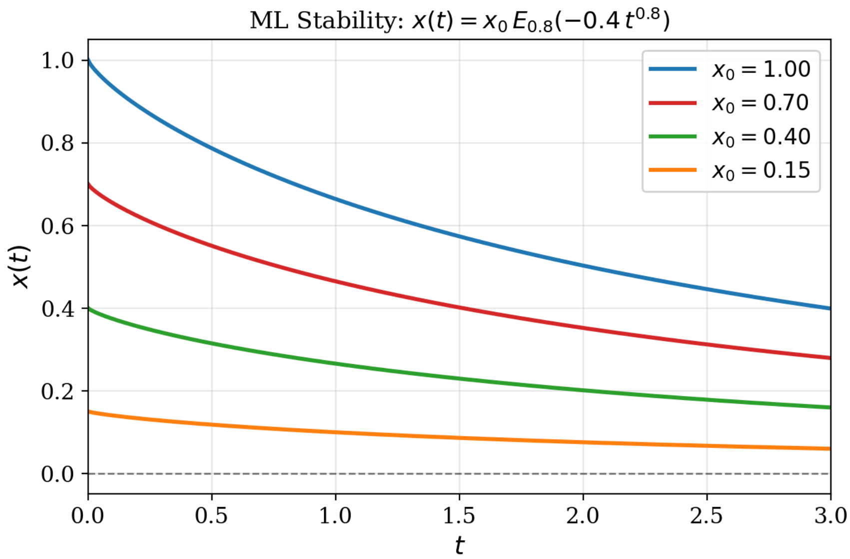

Figure 2: ML-stable trajectories for four initial values $x_0 \in \{0.15, 0.40, 0.70, 1.00\}$. All solutions converge to zero at the algebraic ML rate irrespective of x_0.

4.2 Example 2: ML Decay Rate Comparison

Example 4.2. *Figure 3 shows* $E_q(-0.5\,t^q)$ *for* $q \in \{0.5, 0.7, 0.9, 1.0\}$ *and* $t \in [0, 3]$.

Quantitative comparison at $t = 2$**.**

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

q	$E_q(-0.5 \times 2^q)$	*Rate*
0.5	0.5784	$\sim t^{-0.5}$ *(algebraic)*
0.7	0.4113	$\sim t^{-0.7}$ *(algebraic)*
0.9	0.2976	$\sim t^{-0.9}$ *(algebraic)*
1.0	$e^{-1} = 0.3679$	*exponential*

Smaller q yields a heavier tail, corresponding to a system with stronger memory effects; the ML stability constant λ in Theorem 3.3 shifts all curves horizontally.

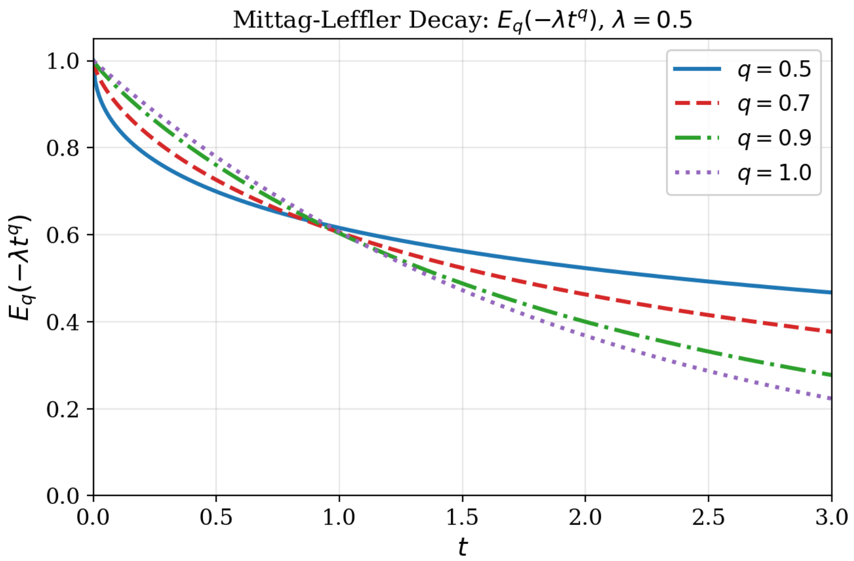

Figure 3: Mittag-Leffler decay $E_q(-0.5\, t^q)$ for $q \in \{0.5, 0.7, 0.9, 1.0\}$, $t \in [0, 3]$. The case $q = 1$ (dotted) is exponential; all others exhibit power-law tails.

4.3 Example 3: Finite-Time Stability Verification

Example 4.3. *Let $T_f = 0.8$, $c_1 = 0.12$, $c_2 = 0.85$, and*

$$f(t, x) = 1 + 0.1x, \quad g(s, x) = 0.35 + 0.08x, \quad q = 1.2, \ \Lambda = 1.$$

Parameters. $M_f = 1.1$, $M_g = 0.43$, $L_f = 0.1$, $L_g = 0.08$.

Step 1: Gronwall exponent.

$$\begin{aligned} M_f L_g \Lambda \Gamma(q)\, T_f^q &= 1.1 \times 0.08 \times 1 \times \Gamma(1.2) \times (0.8)^{1.2} \\ &= 1.1 \times 0.08 \times 0.9182 \times 0.7651 \\ &= 1.1 \times 0.08 \times 0.7025 \\ &\approx 0.0618. \end{aligned} \tag{57}$$

Step 2: Compute $E_{1.2}(0.0618)$***.***

$$E_{1.2}(0.0618) \approx 1 + \frac{0.0618}{\Gamma(2.2)} + \frac{(0.0618)^2}{\Gamma(3.4)} = 1 + \frac{0.0618}{1.1018} + \frac{0.003820}{2.9812} \approx 1 + 0.05609 + 0.00128 \approx 1.0574. \tag{58}$$

Step 3: Compute integral term.

$$\frac{M_f M_g \Lambda T_f^q}{\Gamma(q+1)} = \frac{1.1 \times 0.43 \times 1 \times 0.7651}{1.1018} = \frac{0.3626}{1.1018} \approx 0.3291. \tag{59}$$

Step 4: Evaluate $\mathcal{F}(c_1, T_f)$ ***from*** (46)**.**

$$\begin{aligned}\mathcal{F}(c_1, T_f) &= (c_1 + 0.3291) \times 1.0574 \\ &= (0.12 + 0.3291) \times 1.0574 \\ &= 0.4491 \times 1.0574 \\ &\approx \mathbf{0.4749}. \end{aligned} \tag{60}$$

Since $\mathcal{F} = 0.4749 < c_2 = 0.85$, *condition* (46) *is satisfied, so* ***the QFIE is FTS on*** $[0, 0.8]$ ***w.r.t.*** $(c_1, c_2) = (0.12, 0.85)$.

Pointwise verification.

t	$t^{1.2}$	$E_{1.2}(exp.)$	$u(t) \leq$
0.2	0.1741	1.0131	0.1590
0.4	0.3858	1.0290	0.2397
0.6	0.6140	1.0462	0.3273
0.8	0.7651	1.0574	0.4749

All values remain below $c_2 = 0.85$.

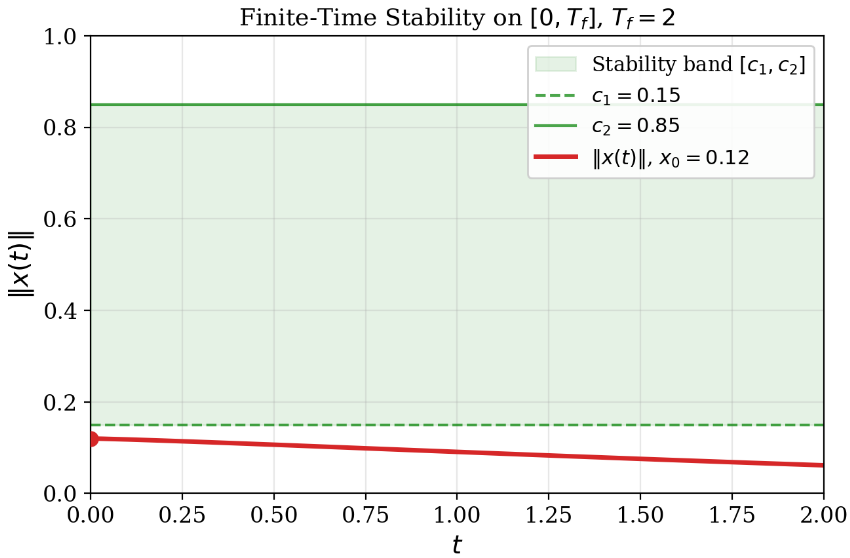

Figure 4: FTS for Example 4.3: the trajectory $\|x(t)\|$ stays strictly inside the safety band $[c_1, c_2] = [0.12, 0.85]$ throughout $[0, T_f] = [0, 0.8]$ as guaranteed by Theorem 3.7.

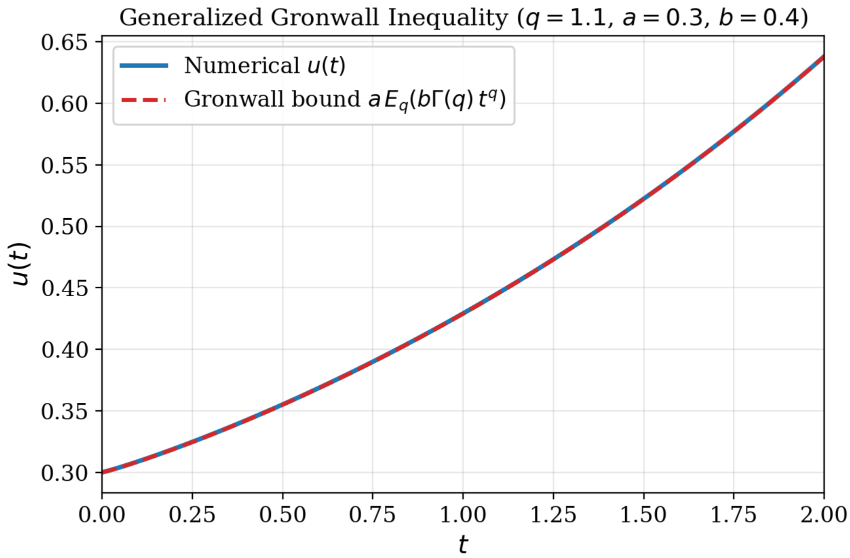

Figure 5: Numerical solution $u(t)$ of the fractional Volterra inequality versus the generalized Gronwall bound $0.3\,E_{1.1}(b\Gamma(q)\,t^q)$ ($q = 1.1$, $a_0 = 0.3$, $b = 0.4$). The analytical bound is tight for small t.

5 Conclusion

This paper has derived two new stability results for the quadratic fractional integral equation (1) with generalized Q-function kernel.

Mittag-Leffler stability (Theorem 3.3): The single algebraic condition $\kappa < 1/T$, where κ is given by (7), guarantees that the zero equilibrium is ML-stable with explicit decay constant $\lambda = (1-\kappa T)/(\Lambda\Gamma(q)\,T^q)$. The proof uses a Lyapunov functional $V = \|x\|^2$, a Cauchy–Schwarz estimate on the Q-kernel convolution (Lemma 3.2), and the integer-order Gronwall lemma.

Finite-time stability (Theorem 3.7 and Corollary 3.8): The computable criterion (46) — involving only M_f, M_g, L_g, Λ, q, T_f, and the bounds c_1, c_2 — ensures that every trajectory starting in a ball of radius c_1 remains inside a larger ball of radius c_2 on $[0, T_f]$. The proof rests on the new fractional Gronwall inequality (Lemma 2.7), whose bound is derived by an exact iterated-kernel calculation using the beta-function convolution identity.

Both results are the first of their kind for QFIEs of quadratic product type. They extend the existence and extremal-solution theory of [1].

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