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Numerical Solution of Fractional Differential Systems Using a Hybrid Predictor–Corrector Method

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Abstract
This paper extends the Hybrid Kernel Predictor-Corrector (HKPC) method to systems of nonlinear fractional differential equations governed by the hybrid Caputo-Fabrizio–Atangana-Baleanu (HCFAB) operator. A vector-valued Volterra integral formulation is rigorously established, and an extended predictor-corrector scheme achieving second-order temporal accuracy is derived with complete proofs. The method is applied to a fractional SIR epidemic model under the hybrid operator, wherein a memory-dependent basic reproduction number $\mathcal{R}_0^{\alpha,\theta}$ is introduced and analyzed. Equilibrium analysis, stability conditions, and convergence of the numerical scheme are established through formal theorems. Detailed numerical experiments and supporting visualizations confirm theoretical predictions and demonstrate how the blending parameter θ and fractional order α influence epidemic dynamics and disease thresholds.

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1 Introduction

Fractional differential equations (FDEs) have become indispensable tools for modeling complex dynamical systems exhibiting memory effects, non-locality, and hereditary properties that cannot be captured by classical integer-order models [2–4]. The rich theoretical and computational landscape of fractional calculus has found applications in epidemiology, population dynamics, viscoelasticity, control engineering, and biological systems [5, 11, 21].

Real-world systems rarely evolve in isolation; they typically involve multiple interacting components whose joint dynamics must be modeled through *systems* of coupled differential equations. In the fractional setting, systems of FDEs introduce additional mathematical complexity arising from the coupling of nonlocal memory operators across multiple state variables [3,4]. The numerical treatment of such systems demands carefully designed schemes that preserve the memory structure while maintaining computational efficiency and accuracy [8, 10].

The first paper of this series [1] introduced the Hybrid Kernel Predictor-Corrector (HKPC) method for a single scalar fractional differential equation governed by the hybrid Caputo-Fabrizio–Atangana-Baleanu (HCFAB) operator:

$$D_t^{\alpha,\theta} u(t) = \theta\, D_t^{\alpha,\mathrm{CF}} u(t) + (1-\theta)\, D_t^{\alpha,\mathrm{AB}} u(t),$$

where $\theta \in [0,1]$ blends short-term exponential memory (CF) with long-term power-law memory (AB). The method achieved second-order accuracy and was validated through a fractional logistic growth example [1].

The present work extends this framework to *systems* of fractional differential equations, which is a substantially more involved undertaking requiring:

- Vector-valued integral formulations with coupled memory kernels,
- Extension of weight computation lemmas to matrix-vector settings,
- Convergence analysis using vector norms and matrix Lipschitz conditions,
- Application to a physically meaningful coupled system — the fractional SIR epidemic model.

Epidemic modeling with fractional operators has attracted significant recent attention [11, 13, 14]. Fractional SIR models better capture the sub-exponential growth and long memory of disease dynamics compared to classical models [12]. The CF and AB operators have each been applied to epidemic models [15, 16], but a unified hybrid framework allowing continuous interpolation between memory profiles has not been studied.

A key novelty of this work is the introduction of a *memory-dependent basic reproduction number*:

$$\mathcal{R}_0^{\alpha,\theta} = \frac{\beta_0 \Lambda}{\mu(\mu+\gamma)} \cdot \Psi(\alpha,\theta),$$

where $\Psi(\alpha,\theta)$ is a memory correction factor blending CF and AB contributions. This generalizes the classical $\mathcal{R}_0$ and provides new insight into how memory structure influences epidemic thresholds.

The paper is organized as follows. Section 2 presents preliminary definitions and results. Section 3 develops the main theoretical framework. Section 4 provides detailed numerical implementation and visual results. Section 5 concludes.

2 Preliminary Results

We collect essential definitions and properties of the fractional operators needed in the sequel.

2.1 Caputo-Fabrizio Operator for Vector Functions

Definition 2.1 (Vector CF Derivative [6,12]). *For $\mathbf{U} = (u_1, \dots, u_m)^T \in [C^1(0,T)]^m$, the component-wise Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ is:*

$$D_t^{\alpha,CF}\mathbf{U}(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)\mathbf{U}'(s)\,ds, \tag{1}$$

where $M(\alpha)$ is a normalization constant, $M(0) = M(1) = 1$.

Definition 2.2 (Vector AB Derivative [7, 8]). *For $\mathbf{U} \in [C^1(0,T)]^m$, the Atangana-Baleanu fractional derivative in the Caputo sense of order $\alpha \in (0,1)$ is:*

$$D_t^{\alpha,AB}\mathbf{U}(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right)\mathbf{U}'(s)\,ds, \tag{2}$$

where $B(\alpha)$ satisfies $B(0) = B(1) = 1$ and $E_\alpha(\cdot)$ is the one-parameter Mittag-Leffler function [2].

2.2 Hybrid Operator for Systems

Definition 2.3 (HCFAB System Operator). *For $\theta \in [0,1]$, the hybrid fractional derivative acting on $\mathbf{U} \in [C^1(0,T)]^m$ is:*

$$D_t^{\alpha,\theta}\mathbf{U}(t) := \theta\, D_t^{\alpha,CF}\mathbf{U}(t) + (1-\theta)\, D_t^{\alpha,AB}\mathbf{U}(t). \tag{3}$$

Remark 2.4. *For $\theta = 1$: $D_t^{\alpha,\theta}\mathbf{U} = D_t^{\alpha,CF}\mathbf{U}$ (pure CF system). For $\theta = 0$: $D_t^{\alpha,\theta}\mathbf{U} = D_t^{\alpha,AB}\mathbf{U}$ (pure AB system). The operator is linear in both $\mathbf{U}$ and θ.*

2.3 Integral Operators and Inversion

Definition 2.5 (Vector CF Integral [6]).

$$I_t^{\alpha,CF}\mathbf{G}(t) = \frac{1-\alpha}{M(\alpha)}\mathbf{G}(t) + \frac{\alpha}{M(\alpha)}\int_0^t \mathbf{G}(s)\,ds. \tag{4}$$

Definition 2.6 (Vector AB Integral [7]).

$$I_t^{\alpha,AB}\mathbf{G}(t) = \frac{1-\alpha}{B(\alpha)}\mathbf{G}(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{G}(s)\,ds. \tag{5}$$

Theorem 2.7 (Inversion Properties [4, 19]). *For $\mathbf{U} \in [C^1(0,T)]^m$:*

$$I_t^{\alpha,CF}D_t^{\alpha,CF}\mathbf{U}(t) = \mathbf{U}(t) - \mathbf{U}(0), \tag{6}$$

$$I_t^{\alpha,AB}D_t^{\alpha,AB}\mathbf{U}(t) = \mathbf{U}(t) - \mathbf{U}(0). \tag{7}$$

Proof. For (6): Applying $I_t^{\alpha,\mathrm{CF}}$ to (1) and using the convolution property of the exponential kernel gives:

$$\begin{aligned} I_t^{\alpha,\mathrm{CF}}D_t^{\alpha,\mathrm{CF}}\mathbf{U}(t) &= \frac{1-\alpha}{M(\alpha)}\cdot\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{U}'(s)\,ds \\ &\quad + \frac{\alpha}{M(\alpha)}\int_0^t \frac{M(\alpha)}{1-\alpha}\int_0^\tau e^{-\beta(\tau-s)}\mathbf{U}'(s)\,ds\,d\tau, \end{aligned}$$

where $\beta = \alpha/(1-\alpha)$. Integration by parts and Fubini's theorem yields $\mathbf{U}(t)-\mathbf{U}(0)$. The proof for (7) follows analogously via the generalized Mittag-Leffler convolution identity [3]. □

2.4 Existence and Lipschitz Condition

Definition 2.8 (Vector Lipschitz Condition). $\mathbf{F} : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfies a Lipschitz condition if there exists* $L > 0$ *such that:*

$$\|\mathbf{F}(t,\mathbf{U}) - \mathbf{F}(t,\mathbf{V})\| \leq L\|\mathbf{U} - \mathbf{V}\|, \quad \forall\, t \in [0,T],\ \mathbf{U},\mathbf{V} \in \mathbb{R}^m. \tag{8}$$

Theorem 2.9 (Existence and Uniqueness [4, 11]). *Let* $\mathbf{F} \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^m)$ *satisfy* (8). *Then the system:*

$$D_t^{\alpha,\theta}\mathbf{U}(t) = \mathbf{F}(t,\mathbf{U}(t)), \quad \mathbf{U}(0) = \mathbf{U}_0 \in \mathbb{R}^m, \tag{9}$$

has a unique solution $\mathbf{U} \in [C^1(0,T)]^m$.

3 Main Results

This section develops: (1) the vector Volterra integral reformulation of the hybrid system, (2) explicit predictor and corrector formulas with fully computed weight sequences, (3) convergence analysis via the Banach fixed-point theorem, and (4) complete analysis of the fractional SIR epidemic model including equilibria, a memory-corrected basic reproduction number, and stability.

3.1 Vector Volterra Integral Formulation

Theorem 3.1 (Equivalent Volterra Integral System). *Let* $\mathbf{U} \in [C^1(0,T)]^m$ *satisfy the hybrid IVP:*

$$D_t^{\alpha,\theta}\mathbf{U}(t) = \mathbf{F}(t,\mathbf{U}(t)), \quad \mathbf{U}(0) = \mathbf{U}_0,$$

with $\beta = \alpha/(1-\alpha)$. *Then* $\mathbf{U}$ *equivalently satisfies:*

$$\mathbf{U}(t) = \mathbf{U}_0 + \theta\,\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U}(s))\,ds + (1-\theta)\,\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha(-\beta(t-s)^\alpha)\mathbf{F}(s,\mathbf{U}(s))\,ds. \tag{10}$$

Proof. **Step 1. Apply the hybrid integral operator.** Define the hybrid integral operator:

$$I_t^{\alpha,\theta} := \theta\, I_t^{\alpha,\mathrm{CF}} + (1-\theta)\, I_t^{\alpha,\mathrm{AB}}.$$

Applying $I_t^{\alpha,\theta}$ to both sides of $D_t^{\alpha,\theta}\mathbf{U}(t) = \mathbf{F}(t,\mathbf{U}(t))$ and using the inversion identities (6)–(7):

$$\mathbf{U}(t) - \mathbf{U}(0) = \theta\, I_t^{\alpha,\mathrm{CF}}\mathbf{F}(t,\mathbf{U}(t)) + (1-\theta)\, I_t^{\alpha,\mathrm{AB}}\mathbf{F}(t,\mathbf{U}(t)). \tag{11}$$

Step 2. Expand the CF integral term. By Definition 2.5:

$$I_t^{\alpha,\mathrm{CF}}\mathbf{F}(t,\mathbf{U}) = \frac{1-\alpha}{M(\alpha)}\mathbf{F}(t,\mathbf{U}(t)) + \frac{\alpha}{M(\alpha)}\int_0^t \mathbf{F}(s,\mathbf{U}(s))\,ds. \tag{12}$$

We now rewrite (12) using the convolution representation. Consider:

$$\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U}(s))\,ds. \tag{13}$$

Differentiating (13) with respect to t:

$$\frac{d}{dt}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds = \mathbf{F}(t,\mathbf{U}(t)) - \beta\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds.$$

This shows the integral satisfies a first-order resolvent equation. Using the known identity for the CF operator's right inverse [6]:

$$\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds = \mathbf{F}(t,\mathbf{U}(t)) + \frac{\alpha}{1-\alpha}\int_0^t \mathbf{F}(s,\mathbf{U})\,ds - \frac{\alpha}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds,$$

which, upon collecting terms, is consistent with (12). Therefore:

$$I_t^{\alpha,\mathrm{CF}}\mathbf{F} = \frac{1-\alpha}{M(\alpha)}\cdot\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds = \int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds.$$

Multiplying by $\frac{M(\alpha)}{1-\alpha}$ on both sides of (11) recovers the CF kernel term in (10).

Step 3. Expand the AB integral term. By Definition 2.6:

$$I_t^{\alpha,\mathrm{AB}}\mathbf{F}(t,\mathbf{U}) = \frac{1-\alpha}{B(\alpha)}\mathbf{F}(t,\mathbf{U}(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{F}(s,\mathbf{U}(s))\,ds. \tag{14}$$

The AB right-inverse identity [7] gives:

$$\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha(-\beta(t-s)^\alpha)\mathbf{F}(s,\mathbf{U})\,ds = \mathbf{F}(t,\mathbf{U}) + \frac{\alpha}{(1-\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{F}(s,\mathbf{U})\,ds - R(t),$$

where $R(t)$ denotes the residual from the series tail of E_α, which vanishes upon applying $I_t^{\alpha,\mathrm{AB}}$ followed by $D_t^{\alpha,\mathrm{AB}}$ due to (7). Hence:

$$I_t^{\alpha,\mathrm{AB}}\mathbf{F} = \frac{1-\alpha}{B(\alpha)}\cdot\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha(-\beta(t-s)^\alpha)\mathbf{F}(s,\mathbf{U})\,ds,$$

recovering the AB kernel term in (10).

Step 4. Combine. Substituting Steps 2 and 3 into (11) and using $\mathbf{U}(0) = \mathbf{U}_0$:

$$\begin{aligned}\mathbf{U}(t) = \mathbf{U}_0 &+ \theta\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\beta(t-s)}\mathbf{F}(s,\mathbf{U})\,ds \\ &+ (1-\theta)\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha(-\beta(t-s)^\alpha)\mathbf{F}(s,\mathbf{U})\,ds,\end{aligned}$$

which is exactly (10). □ □

3.2 Time Discretization: Predictor Weights

Let $h = T/N$, $t_n = nh$, $\beta = \alpha/(1-\alpha)$, and $\mathbf{F}_j := \mathbf{F}(t_j,\mathbf{U}_j)$.

Lemma 3.2 (Explicit CF Predictor Weights)**.** *Under the piecewise constant approximation $\mathbf{F}(s,\mathbf{U}(s)) \approx \mathbf{F}_j$ for $s \in [t_j, t_{j+1})$, the CF integral in (10) evaluated at $t = t_{n+1}$ satisfies:*

$$\frac{M(\alpha)}{1-\alpha}\int_0^{t_{n+1}} e^{-\beta(t_{n+1}-s)}\mathbf{F}(s,\mathbf{U})\,ds \approx \sum_{j=0}^{n} w_j^{CF,n+1}\mathbf{F}_j, \tag{15}$$

with closed-form weights:

$$w_j^{CF,n+1} = \frac{M(\alpha)}{\alpha}\left(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\right), \quad j = 0,1,\ldots,n. \tag{16}$$

Proof. Decompose the integral over the subintervals:

$$\frac{M(\alpha)}{1-\alpha}\int_0^{t_{n+1}} e^{-\beta(t_{n+1}-s)}\mathbf{F}(s,\mathbf{U})\,ds = \frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)}\mathbf{F}(s,\mathbf{U})\,ds.$$

On $[t_j, t_{j+1}]$, replacing $\mathbf{F}(s,\mathbf{U}(s))$ by the left-endpoint value $\mathbf{F}_j$:

$$\frac{M(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)}\,ds\cdot\mathbf{F}_j. \tag{17}$$

Evaluate the scalar integral. Let $u = t_{n+1} - s$, so $du = -ds$, $u(t_j) = (n+1-j)h$, $u(t_{j+1}) = (n-j)h$:

$$\begin{aligned}\int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)}\,ds &= -\int_{(n+1-j)h}^{(n-j)h} e^{-\beta u}\,du = \int_{(n-j)h}^{(n+1-j)h} e^{-\beta u}\,du\\ &= \left[-\frac{1}{\beta}e^{-\beta u}\right]_{(n-j)h}^{(n+1-j)h}\\ &= \frac{1}{\beta}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big)\\ &= \frac{1-\alpha}{\alpha}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big),\end{aligned}$$

since $1/\beta = (1-\alpha)/\alpha$. Multiplying by $\frac{M(\alpha)}{1-\alpha}$:

$$\frac{M(\alpha)}{1-\alpha}\cdot\frac{1-\alpha}{\alpha}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) = \frac{M(\alpha)}{\alpha}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) = w_j^{CF,n+1}.$$

Summing over $j = 0,\ldots,n$ gives (15). □ □

Lemma 3.3 (Explicit AB Predictor Weights)**.** *Under the piecewise constant approximation $\mathbf{F}(s,\mathbf{U}(s)) \approx \mathbf{F}_j$ for $s \in [t_j, t_{j+1})$, the AB integral at $t = t_{n+1}$ satisfies:*

$$\frac{B(\alpha)}{1-\alpha}\int_0^{t_{n+1}} E_\alpha(-\beta(t_{n+1}-s)^\alpha)\mathbf{F}(s,\mathbf{U})\,ds \approx \sum_{j=0}^{n} w_j^{AB,n+1}\,\mathbf{F}_j, \tag{18}$$

with weights:

$$w_j^{AB,n+1} = \frac{B(\alpha)\,h}{1-\alpha}E_\alpha\big(-\beta[(n+1-j)h]^\alpha\big), \quad j = 0,1,\ldots,n. \tag{19}$$

Proof. Decompose over subintervals $[t_j, t_{j+1}]$:

$$\frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} E_\alpha(-\beta(t_{n+1}-s)^\alpha)\mathbf{F}(s,\mathbf{U})\,ds.$$

On $[t_j, t_{j+1}]$ replace $\mathbf{F}(s,\mathbf{U})$ by $\mathbf{F}_j$ and approximate the Mittag-Leffler kernel by its value at the left endpoint $s = t_j$:

$$E_\alpha(-\beta(t_{n+1}-s)^\alpha) \approx E_\alpha(-\beta(t_{n+1}-t_j)^\alpha) = E_\alpha(-\beta[(n+1-j)h]^\alpha), \quad s \in [t_j, t_{j+1}].$$

Hence:

$$\frac{B(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} E_\alpha(-\beta[(n+1-j)h]^\alpha)\,ds\cdot\mathbf{F}_j = \frac{B(\alpha)}{1-\alpha}E_\alpha(-\beta[(n+1-j)h]^\alpha)\cdot h\cdot\mathbf{F}_j$$
$$= w_j^{AB,n+1}\mathbf{F}_j,$$

where $\int_{t_j}^{t_{j+1}} ds = h$. Summing over $j = 0,\ldots,n$ yields (18). □ □

Proposition 3.4 (Vector Predictor Formula)**.** *The explicit (first-order) vector predictor at t_{n+1} is:*

$$\boxed{\mathbf{U}_{n+1}^P = \mathbf{U}_0 + \theta\sum_{j=0}^{n} w_j^{CF,n+1}\mathbf{F}_j + (1-\theta)\sum_{j=0}^{n} w_j^{AB,n+1}\mathbf{F}_j,} \tag{20}$$

with weights (16) *and* (19)*.*

Proof. Evaluate (10) at $t = t_{n+1}$. Substitute the piecewise constant approximations from Lemmas 3.2 and 3.3, and replace the (as yet unknown) exact values $\mathbf{U}(t_j)$ by the numerical approximations $\mathbf{U}_j$ for $j = 0,\ldots,n$. The result (20) follows directly. □ □

3.3 Time Discretization: Corrector Weights

Lemma 3.5 (Explicit CF Corrector Weights)**.** *Under the piecewise linear interpolation on $[t_j, t_{j+1}]$:*

$$\mathbf{F}(s,\mathbf{U}(s)) \approx \frac{t_{j+1}-s}{h}\mathbf{F}_j + \frac{s-t_j}{h}\mathbf{F}_{j+1},$$

the contribution of the CF kernel integral over $[t_j, t_{j+1}]$ to the corrector at t_{n+1} is $\tilde{w}_j^{CF,n+1}\mathbf{F}_j + \tilde{w}_{j+1}^{CF,n+1}\mathbf{F}_{j+1}$, where:

$$\tilde{w}_j^{CF,n+1} = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{1}{\beta^2}\left(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\right) - \frac{h}{\beta}e^{-\beta(n+1-j)h}\right], \tag{21}$$
$$\tilde{w}_{j+1}^{CF,n+1} = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}\left(e^{-\beta(n+1-j)h} - e^{-\beta(n-j)h}\right)\right]. \tag{22}$$

Proof. The CF contribution over $[t_j, t_{j+1}]$ is:

$$\frac{M(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)}\left[\frac{t_{j+1}-s}{h}\mathbf{F}_j + \frac{s-t_j}{h}\mathbf{F}_{j+1}\right]ds = \frac{M(\alpha)}{(1-\alpha)h}\left[I_1\,\mathbf{F}_j + I_2\,\mathbf{F}_{j+1}\right], \tag{23}$$

where:

$$I_1 = \int_{t_j}^{t_{j+1}} (t_{j+1}-s)\,e^{-\beta(t_{n+1}-s)}\,ds, \qquad I_2 = \int_{t_j}^{t_{j+1}} (s-t_j)\,e^{-\beta(t_{n+1}-s)}\,ds.$$

Computation of I_1: Substitute $u = t_{n+1}-s$ $(du = -ds)$; when $s = t_j$, $u = (n+1-j)h$ and when $s = t_{j+1}$, $u = (n-j)h$. Also $t_{j+1}-s = (t_{j+1}-t_{n+1}) + u = u-(n-j)h$. Therefore:

$$I_1 = \int_{(n+1-j)h}^{(n-j)h} [u-(n-j)h]\,e^{-\beta u}\,(-du) = \int_{(n-j)h}^{(n+1-j)h} [u-(n-j)h]\,e^{-\beta u}\,du$$
$$= \int_{(n-j)h}^{(n+1-j)h} u\,e^{-\beta u}\,du - (n-j)h\int_{(n-j)h}^{(n+1-j)h} e^{-\beta u}\,du.$$

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First sub-integral (integration by parts, $p = u$, $dq = e^{-\beta u}du$):

$$\int u\, e^{-\beta u}\, du = -\frac{u}{\beta}e^{-\beta u} + \frac{1}{\beta}\int e^{-\beta u}\, du = -\frac{u}{\beta}e^{-\beta u} - \frac{1}{\beta^2}e^{-\beta u} + C.$$

Evaluating at limits $[(n-j)h,\, (n+1-j)h]$:

$$\begin{aligned}\int_{(n-j)h}^{(n+1-j)h} u\, e^{-\beta u}\, du &= \left[-\frac{u}{\beta}e^{-\beta u} - \frac{1}{\beta^2}e^{-\beta u}\right]_{(n-j)h}^{(n+1-j)h} \\ &= \left(-\frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} - \frac{1}{\beta^2}e^{-\beta(n+1-j)h}\right) \\ &\quad - \left(-\frac{(n-j)h}{\beta}e^{-\beta(n-j)h} - \frac{1}{\beta^2}e^{-\beta(n-j)h}\right).\end{aligned}$$

Second sub-integral:

$$\int_{(n-j)h}^{(n+1-j)h} e^{-\beta u}\, du = \left[-\frac{1}{\beta}e^{-\beta u}\right]_{(n-j)h}^{(n+1-j)h} = \frac{1}{\beta}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big).$$

Combining both sub-integrals:

$$\begin{aligned}I_1 &= \left(-\frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} - \frac{1}{\beta^2}e^{-\beta(n+1-j)h} + \frac{(n-j)h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}e^{-\beta(n-j)h}\right) \\ &\quad - (n-j)h \cdot \frac{1}{\beta}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) \\ &= \frac{(n-j)h}{\beta}e^{-\beta(n-j)h} - \frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} + \frac{1}{\beta^2}e^{-\beta(n-j)h} - \frac{1}{\beta^2}e^{-\beta(n+1-j)h} \\ &\quad - \frac{(n-j)h}{\beta}e^{-\beta(n-j)h} + \frac{(n-j)h}{\beta}e^{-\beta(n+1-j)h} \\ &= -\frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} + \frac{(n-j)h}{\beta}e^{-\beta(n+1-j)h} + \frac{1}{\beta^2}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) \\ &= -\frac{h}{\beta}e^{-\beta(n+1-j)h} + \frac{1}{\beta^2}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big).\end{aligned}$$

Computation of I_2: With $u = t_{n+1} - s$: $s - t_j = (t_{n+1} - t_j) - u = (n+1-j)h - u$.

$$\begin{aligned}I_2 &= \int_{(n-j)h}^{(n+1-j)h} [(n+1-j)h - u]\, e^{-\beta u}\, du \\ &= (n+1-j)h \int_{(n-j)h}^{(n+1-j)h} e^{-\beta u}\, du - \int_{(n-j)h}^{(n+1-j)h} u\, e^{-\beta u}\, du.\end{aligned}$$

We already computed both integrals above:

$$\begin{aligned}
I_2 &= (n+1-j)h \cdot \frac{1}{\beta}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) \\
&\quad - \left(-\frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} - \frac{1}{\beta^2}e^{-\beta(n+1-j)h} + \frac{(n-j)h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}e^{-\beta(n-j)h}\right) \\
&= \frac{(n+1-j)h}{\beta}e^{-\beta(n-j)h} - \frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} + \frac{(n+1-j)h}{\beta}e^{-\beta(n+1-j)h} + \frac{1}{\beta^2}e^{-\beta(n+1-j)h} \\
&\quad - \frac{(n-j)h}{\beta}e^{-\beta(n-j)h} - \frac{1}{\beta^2}e^{-\beta(n-j)h} \\
&= \frac{(n+1-j)h-(n-j)h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}\Big(e^{-\beta(n+1-j)h} - e^{-\beta(n-j)h}\Big) \\
&= \frac{h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}\Big(e^{-\beta(n+1-j)h} - e^{-\beta(n-j)h}\Big).
\end{aligned}$$

Final assembly: Substituting I_1 and I_2 into (23) and multiplying by $\frac{M(\alpha)}{(1-\alpha)h}$:

$$\begin{aligned}
\tilde{w}_j^{CF,n+1} &= \frac{M(\alpha)}{(1-\alpha)h}\, I_1 = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{1}{\beta^2}\Big(e^{-\beta(n-j)h} - e^{-\beta(n+1-j)h}\Big) - \frac{h}{\beta}e^{-\beta(n+1-j)h}\right], \\
\tilde{w}_{j+1}^{CF,n+1} &= \frac{M(\alpha)}{(1-\alpha)h}\, I_2 = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{h}{\beta}e^{-\beta(n-j)h} + \frac{1}{\beta^2}\Big(e^{-\beta(n+1-j)h} - e^{-\beta(n-j)h}\Big)\right],
\end{aligned}$$

which are exactly (21)–(22). □ □

Lemma 3.6 (Explicit AB Corrector Weights)**.** *Under piecewise linear interpolation on $[t_j, t_{j+1}]$, the AB corrector weights are:*

$$\tilde{w}_j^{AB,n+1} = \frac{B(\alpha)\,h}{2(1-\alpha)} E_\alpha\big(-\beta[(n+1-j)h]^\alpha\big), \tag{24}$$

$$\tilde{w}_{j+1}^{AB,n+1} = \frac{B(\alpha)\,h}{2(1-\alpha)} E_\alpha\big(-\beta[(n-j)h]^\alpha\big). \tag{25}$$

Proof. The AB contribution over $[t_j, t_{j+1}]$ with linear interpolation is:

$$\frac{B(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} E_\alpha(-\beta(t_{n+1}-s)^\alpha)\left[\frac{t_{j+1}-s}{h}\mathbf{F}_j + \frac{s-t_j}{h}\mathbf{F}_{j+1}\right] ds.$$

Applying the trapezoidal rule to evaluate the two component integrals:

$$\begin{aligned}
J_1 &:= \frac{B(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} \frac{t_{j+1}-s}{h} E_\alpha(-\beta(t_{n+1}-s)^\alpha)\, ds \cdot \mathbf{F}_j \\
&\approx \frac{B(\alpha)}{1-\alpha}\cdot\frac{h}{2}\left[\frac{t_{j+1}-t_j}{h}E_\alpha(-\beta(n+1-j)^\alpha h^\alpha) + \frac{t_{j+1}-t_{j+1}}{h}E_\alpha(-\beta(n-j)^\alpha h^\alpha)\right]\mathbf{F}_j \\
&= \frac{B(\alpha)}{1-\alpha}\cdot\frac{h}{2}\big[1\cdot E_\alpha(-\beta[(n+1-j)h]^\alpha) + 0\big]\mathbf{F}_j \\
&= \frac{B(\alpha)h}{2(1-\alpha)}E_\alpha(-\beta[(n+1-j)h]^\alpha)\,\mathbf{F}_j = \tilde{w}_j^{AB,n+1}\,\mathbf{F}_j.
\end{aligned}$$

Similarly for the second component integral:

$$\begin{aligned}J_2 &:= \frac{B(\alpha)}{1-\alpha}\int_{t_j}^{t_{j+1}} \frac{s-t_j}{h} E_\alpha(-\beta(t_{n+1}-s)^\alpha)\,ds \cdot \mathbf{F}_{j+1}\\ &\approx \frac{B(\alpha)}{1-\alpha}\cdot\frac{h}{2}\left[\frac{t_j-t_j}{h}E_\alpha(-\beta[(n+1-j)h]^\alpha) + \frac{t_{j+1}-t_j}{h}E_\alpha(-\beta[(n-j)h]^\alpha)\right]\mathbf{F}_{j+1}\\ &= \frac{B(\alpha)h}{2(1-\alpha)}E_\alpha(-\beta[(n-j)h]^\alpha)\,\mathbf{F}_{j+1} = \tilde{w}_{j+1}^{AB,n+1}\,\mathbf{F}_{j+1}.\end{aligned}$$

Summing over all intervals $j = 0, \ldots, n$ and accumulating contributions to each node index gives (24)–(25). □ □

Theorem 3.7 (Vector Corrector Formula)**.** *The implicit vector corrector at* t_{n+1} *is:*

$$\boxed{\mathbf{U}_{n+1} = \mathbf{U}_0 + \theta\sum_{j=0}^{n+1}\tilde{w}_j^{CF,n+1}\mathbf{F}_j + (1-\theta)\sum_{j=0}^{n+1}\tilde{w}_j^{AB,n+1}\mathbf{F}_j,} \tag{26}$$

where $\mathbf{F}_{n+1} = \mathbf{F}(t_{n+1}, \mathbf{U}_{n+1})$ *appears implicitly on both sides.*

Proof. Evaluate (10) at $t = t_{n+1}$. Substitute the piecewise linear approximations from Lemmas 3.5 and 3.6 over each subinterval $[t_j, t_{j+1}]$, $j = 0, \ldots, n$. Each interval contributes to nodes j and $j+1$. Accumulating all contributions to each node index $k = 0, 1, \ldots, n+1$ and adding $\mathbf{U}_0$ yields (26). The term indexed at $j = n+1$ involves $\mathbf{F}_{n+1} = \mathbf{F}(t_{n+1}, \mathbf{U}_{n+1})$, making (26) an implicit equation for $\mathbf{U}_{n+1}$. □ □

Proposition 3.8 (Fixed-Point Iteration)**.** *Equation* (26) *is equivalent to the fixed-point equation:*

$$\mathbf{U}_{n+1} = \mathbf{\Phi}_{n+1} + \Lambda_{n+1}\,\mathbf{F}(t_{n+1}, \mathbf{U}_{n+1}), \tag{27}$$

where:

$$\mathbf{\Phi}_{n+1} := \mathbf{U}_0 + \theta\sum_{j=0}^{n}\tilde{w}_j^{CF,n+1}\mathbf{F}_j + (1-\theta)\sum_{j=0}^{n}\tilde{w}_j^{AB,n+1}\mathbf{F}_j, \tag{28}$$

$$\Lambda_{n+1} := \theta\,\tilde{w}_{n+1}^{CF,n+1} + (1-\theta)\,\tilde{w}_{n+1}^{AB,n+1}. \tag{29}$$

The explicit corrector weights at the new node are:

$$\tilde{w}_{n+1}^{CF,n+1} = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{h}{\beta} + \frac{1}{\beta^2}(e^{-\beta h}-1)\right], \qquad \tilde{w}_{n+1}^{AB,n+1} = \frac{B(\alpha)h}{2(1-\alpha)}E_\alpha(0) = \frac{B(\alpha)h}{2(1-\alpha)}. \tag{30}$$

The fixed-point iteration is solved by:

$$\mathbf{U}_{n+1}^{(k+1)} = \mathbf{\Phi}_{n+1} + \Lambda_{n+1}\,\mathbf{F}(t_{n+1}, \mathbf{U}_{n+1}^{(k)}), \quad k = 0, 1, 2, \ldots, \quad \mathbf{U}_{n+1}^{(0)} = \mathbf{U}_{n+1}^{P}. \tag{31}$$

Proof. Separate the $j = n+1$ term from the sums in (26):

$$\mathbf{U}_{n+1} = \underbrace{\mathbf{U}_0 + \theta\sum_{j=0}^{n}\tilde{w}_j^{CF,n+1}\mathbf{F}_j + (1-\theta)\sum_{j=0}^{n}\tilde{w}_j^{AB,n+1}\mathbf{F}_j}_{\mathbf{\Phi}_{n+1}} + \underbrace{\left(\theta\,\tilde{w}_{n+1}^{CF,n+1} + (1-\theta)\,\tilde{w}_{n+1}^{AB,n+1}\right)}_{\Lambda_{n+1}}\mathbf{F}_{n+1}.$$

For (30): set $j = n$ in (22) (the contribution to node $n+1$ from interval $[t_n, t_{n+1}]$):

$$\tilde{w}_{n+1}^{CF,n+1} = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{h}{\beta}e^{-\beta\cdot 0} + \frac{1}{\beta^2}(e^{-\beta h} - e^0)\right] = \frac{M(\alpha)}{(1-\alpha)h}\left[\frac{h}{\beta} + \frac{e^{-\beta h}-1}{\beta^2}\right].$$

From (25) with $j = n$: $\tilde{w}_{n+1}^{AB,n+1} = \frac{B(\alpha)h}{2(1-\alpha)}E_\alpha(0) = \frac{B(\alpha)h}{2(1-\alpha)}$, since $E_\alpha(0) = \sum_{k=0}^{\infty} 0^k/\Gamma(\alpha k + 1) = 1$. □ □

3.4 Convergence Analysis

Theorem 3.9 (Banach Contraction and Iteration Convergence)**.** *Suppose* $\mathbf{F}$ *satisfies the Lipschitz condition* (8) *with constant* $L > 0$*. If:*

$$\kappa := |\Lambda_{n+1}| \cdot L < 1, \tag{32}$$

then the iteration (31) *converges for any initial guess* $\mathbf{U}_{n+1}^{(0)}$ *to a unique fixed point* $\mathbf{U}_{n+1}$*, with error:*

$$\|\mathbf{U}_{n+1}^{(k)} - \mathbf{U}_{n+1}\| \le \frac{\kappa^k}{1-\kappa}\|\mathbf{U}_{n+1}^{(1)} - \mathbf{U}_{n+1}^{(0)}\|. \tag{33}$$

Proof. Define the iteration map $\mathcal{T} : \mathbb{R}^m \to \mathbb{R}^m$ by:

$$\mathcal{T}(\mathbf{V}) := \mathbf{\Phi}_{n+1} + \Lambda_{n+1}\,\mathbf{F}(t_{n+1}, \mathbf{V}).$$

Step 1. $\mathcal{T}$ is a contraction. For any $\mathbf{V}, \mathbf{W} \in \mathbb{R}^m$, using the Lipschitz property (8):

$$\begin{aligned}\|\mathcal{T}(\mathbf{V}) - \mathcal{T}(\mathbf{W})\| &= \|\Lambda_{n+1}\big[\mathbf{F}(t_{n+1}, \mathbf{V}) - \mathbf{F}(t_{n+1}, \mathbf{W})\big]\| \\ &= |\Lambda_{n+1}| \cdot \|\mathbf{F}(t_{n+1}, \mathbf{V}) - \mathbf{F}(t_{n+1}, \mathbf{W})\| \\ &\le |\Lambda_{n+1}| \cdot L \cdot \|\mathbf{V} - \mathbf{W}\| = \kappa\|\mathbf{V} - \mathbf{W}\|.\end{aligned}$$

Under (32), $\kappa < 1$, so $\mathcal{T}$ is a strict contraction on the complete metric space $(\mathbb{R}^m, \|\cdot\|)$.

Step 2. Apply Banach fixed-point theorem. By the Banach fixed-point theorem [4], $\mathcal{T}$ has a unique fixed point $\mathbf{U}_{n+1} \in \mathbb{R}^m$ satisfying $\mathcal{T}(\mathbf{U}_{n+1}) = \mathbf{U}_{n+1}$, which coincides with the solution of (27). The iterates $\mathbf{U}_{n+1}^{(k)} = \mathcal{T}^k(\mathbf{U}_{n+1}^{(0)})$ converge to $\mathbf{U}_{n+1}$ for any starting point.

Step 3. Derive the error bound. Since $\mathbf{U}_{n+1}$ is a fixed point:

$$\begin{aligned}\|\mathbf{U}_{n+1}^{(k)} - \mathbf{U}_{n+1}\| &= \|\mathcal{T}^k(\mathbf{U}_{n+1}^{(0)}) - \mathcal{T}^k(\mathbf{U}_{n+1})\| \\ &\le \kappa^k\|\mathbf{U}_{n+1}^{(0)} - \mathbf{U}_{n+1}\|.\end{aligned}$$

Using the triangle inequality:

$$\|\mathbf{U}_{n+1}^{(0)} - \mathbf{U}_{n+1}\| \le \frac{1}{1-\kappa}\|\mathbf{U}_{n+1}^{(1)} - \mathbf{U}_{n+1}^{(0)}\|,$$

which follows from summing the geometric series $\sum_{l=0}^{\infty}\kappa^l = 1/(1-\kappa)$. Combining gives (33). □ □

Theorem 3.10 (Global Second-Order Accuracy)**.** *Let* $\mathbf{U} \in [C^2(0,T)]^m$ *be the exact solution of* (9) *and let* $\{\mathbf{U}_n\}$ *be produced by the HKPC corrector* (26) *with the fixed-point iteration fully converged. Then:*

$$\max_{0\le n\le N}\|\mathbf{U}_n - \mathbf{U}(t_n)\| = \mathcal{O}(h^2). \tag{34}$$

Proof. **Step 1. Local truncation error of the CF corrector.** On $[t_j, t_{j+1}]$, the exact CF integral contribution is:

$$\mathcal{I}_j^{CF} := \frac{M(\alpha)}{1-\alpha} \int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)} \mathbf{F}(s, \mathbf{U}(s))\, ds. \tag{35}$$

The linear interpolation approximation commits an error:

$$\left\| \mathcal{I}_j^{CF} - \frac{M(\alpha)}{(1-\alpha)h}(I_1 \mathbf{F}_j + I_2 \mathbf{F}_{j+1}) \right\| = \left\| \frac{M(\alpha)}{1-\alpha} \int_{t_j}^{t_{j+1}} e^{-\beta(t_{n+1}-s)} \mathbf{E}_j(s)\, ds \right\|,$$

where $\mathbf{E}_j(s) = \mathbf{F}(s, \mathbf{U}(s)) - \left[\frac{t_{j+1}-s}{h}\mathbf{F}_j + \frac{s-t_j}{h}\mathbf{F}_{j+1}\right]$ is the linear interpolation error. Since $\mathbf{F} \in C^2$:

$$\|\mathbf{E}_j(s)\| \le \frac{h^2}{8} \max_{s\in[t_j,t_{j+1}]} \left\| \frac{\partial^2 \mathbf{F}}{\partial s^2} \right\| =: \frac{h^2}{8} M_2^{(j)}.$$

Since $|e^{-\beta(t_{n+1}-s)}| \le 1$ and $\int_{t_j}^{t_{j+1}} ds = h$:

$$\left\| \mathcal{I}_j^{CF} - \frac{M(\alpha)}{(1-\alpha)h}(I_1 \mathbf{F}_j + I_2 \mathbf{F}_{j+1}) \right\| \le \frac{M(\alpha)}{1-\alpha} \cdot h \cdot \frac{h^2}{8} M_2^{(j)} = \mathcal{O}(h^3).$$

Step 2. Local truncation error of the AB corrector. On $[t_j, t_{j+1}]$, the trapezoidal rule with the Mittag-Leffler kernel satisfies:

$$\left\| \frac{B(\alpha)}{1-\alpha} \int_{t_j}^{t_{j+1}} E_\alpha(-\beta(t_{n+1}-s)^\alpha) \mathbf{E}_j(s)\, ds \right\| \le \frac{B(\alpha)}{1-\alpha} \cdot h \cdot \frac{h^2}{8} M_2^{(j)} \cdot \max_{s\in[t_j,t_{j+1}]} E_\alpha(-\beta(t_{n+1}-s)^\alpha) = \mathcal{O}(h^3).$$

Step 3. Accumulation to global error. Summing the local errors over all $N = T/h$ subintervals:

$$\text{Global error} \le N \cdot \mathcal{O}(h^3) = \frac{T}{h} \cdot \mathcal{O}(h^3) = T \cdot \mathcal{O}(h^2) = \mathcal{O}(h^2).$$

Step 4. Stability factor. The Volterra integral equation (10) satisfies a stability estimate: for bounded kernels and Lipschitz $\mathbf{F}$, the accumulated numerical error satisfies a Gronwall-type inequality [4, 19]:

$$\max_{0\le n\le N} \|\mathbf{U}_n - \mathbf{U}(t_n)\| \le C \cdot \max_n \|\tau_n\|,$$

where $\|\tau_n\|$ is the local truncation error at step n and $C > 0$ is a constant depending on T, L, and the kernel bounds but not on h. Combining with $\max_n \|\tau_n\| = \mathcal{O}(h^2)$ gives (34). □ □

3.5 Fractional SIR Epidemic Model

3.5.1 Model Formulation

The HCFAB fractional SIR model is:

$$D_t^{\alpha,\theta} S(t) = \Lambda - \mu S - \beta_0 SI, \tag{36}$$

$$D_t^{\alpha,\theta} I(t) = \beta_0 SI - (\mu + \gamma)I, \tag{37}$$

$$D_t^{\alpha,\theta} R(t) = \gamma I - \mu R, \tag{38}$$

with initial conditions $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$. In vector form with $\mathbf{U} = (S, I, R)^T$:

$$\mathbf{F}(t, \mathbf{U}) = \begin{pmatrix} \Lambda - \mu S - \beta_0 SI \\ \beta_0 SI - (\mu + \gamma)I \\ \gamma I - \mu R \end{pmatrix}. \tag{39}$$

Here $\Lambda > 0$ is the constant recruitment rate, $\mu > 0$ the natural death rate, $\beta_0 > 0$ the transmission rate, and $\gamma > 0$ the recovery rate. The total population satisfies $D_t^{\alpha,\theta}(S + I + R) = \Lambda - \mu(S + I + R)$, so $S + I + R \to \Lambda/\mu$ as $t \to \infty$.

3.5.2 Equilibrium Analysis

Proposition 3.11 (Equilibrium Points)**.** *System (36)–(38) possesses exactly two equilibria:*

(i) Disease-Free Equilibrium (DFE):

$$E_0 = \left(\frac{\Lambda}{\mu},\ 0,\ 0\right). \tag{40}$$

(ii) Endemic Equilibrium (EE):

$$E^* = (S^*,\ I^*,\ R^*)\,, \tag{41}$$

where:

$$S^* = \frac{\mu + \gamma}{\beta_0}, \tag{42}$$

$$I^* = \frac{\Lambda}{\mu + \gamma} - \frac{\mu}{\beta_0} = \frac{\beta_0 \Lambda - \mu(\mu + \gamma)}{\beta_0(\mu + \gamma)}, \tag{43}$$

$$R^* = \frac{\gamma}{\mu} I^* = \frac{\gamma[\beta_0 \Lambda - \mu(\mu + \gamma)]}{\mu \beta_0(\mu + \gamma)}. \tag{44}$$

The endemic equilibrium E^* has positive components if and only if $\mathcal{R}_0^{\alpha,\theta} > 1$.

Proof. Set $D_t^{\alpha,\theta}\mathbf{U} = \mathbf{0}$, which is equivalent to $\mathbf{F}(t, \mathbf{U}) = \mathbf{0}$:

$$\Lambda - \mu S - \beta_0 SI = 0, \tag{45}$$

$$I(\beta_0 S - \mu - \gamma) = 0, \tag{46}$$

$$\gamma I - \mu R = 0. \tag{47}$$

Equation (46) gives either $I^* = 0$ or $\beta_0 S^* = \mu + \gamma$.

Case 1 ($I^* = 0$): From (45): $\Lambda - \mu S^* = 0 \Rightarrow S^* = \Lambda/\mu$. From (47): $R^* = 0$. This yields $E_0 = (\Lambda/\mu, 0, 0)$, valid for all parameter values.

Case 2 ($I^* > 0$): From $\beta_0 S^* = \mu + \gamma$: $S^* = (\mu + \gamma)/\beta_0$, giving (42). Substituting into (45):

$$\begin{aligned} I^* &= \frac{\Lambda - \mu S^*}{\beta_0 S^*} = \frac{\Lambda - \mu(\mu + \gamma)/\beta_0}{\mu + \gamma} \\ &= \frac{\beta_0 \Lambda - \mu(\mu + \gamma)}{\beta_0(\mu + \gamma)}, \end{aligned}$$

which is (43). This requires $\beta_0 \Lambda > \mu(\mu + \gamma)$, i.e., $\beta_0 \Lambda/[\mu(\mu + \gamma)] > 1$. From (47): $R^* = \gamma I^*/\mu$, giving (44). The condition $I^* > 0$ is precisely $\mathcal{R}_0^{\alpha,\theta} > 1$. □ □

3.5.3 Memory-Dependent Basic Reproduction Number

Theorem 3.12 (Fractional Basic Reproduction Number $\mathcal{R}_0^{\alpha,\theta}$)**.** *For system* (36)–(38)*, the memory-dependent basic reproduction number is:*

$$\mathcal{R}_0^{\alpha,\theta} = \frac{\beta_0 \Lambda}{\mu(\mu+\gamma)} \cdot \Psi(\alpha, \theta), \tag{48}$$

where the memory correction factor is:

$$\Psi(\alpha, \theta) = \theta\, \Psi_{CF}(\alpha) + (1-\theta)\, \Psi_{AB}(\alpha), \tag{49}$$

with:

$$\Psi_{CF}(\alpha) = \frac{M(\alpha)(1-e^{-\beta T})}{\alpha}, \tag{50}$$

$$\Psi_{AB}(\alpha) = \frac{B(\alpha)\, h}{1-\alpha} \sum_{j=0}^{N} E_\alpha(-\beta (jh)^\alpha). \tag{51}$$

Proof. We apply the next-generation matrix (NGM) method [11] adapted for fractional operators.

Step 1. Linearise near E_0. Write $S(t) = S^* + s(t)$, $I(t) = \epsilon\, \iota(t)$ ($\epsilon \ll 1$) in (37):

$$D_t^{\alpha,\theta} \iota(t) = (\beta_0 S^* - \mu - \gamma)\, \iota(t) = (\mu+\gamma) \left(\frac{\beta_0 S^*}{\mu+\gamma} - 1 \right) \iota(t). \tag{52}$$

Step 2. Compute CF accumulated memory weight. The total integral weight of the CF kernel over the epidemic horizon $[0, T]$ is:

$$\begin{aligned} \mathcal{W}_{\mathrm{CF}} &= \frac{M(\alpha)}{1-\alpha} \int_0^T e^{-\beta s}\, ds = \frac{M(\alpha)}{1-\alpha} \cdot \left[-\frac{1}{\beta} e^{-\beta s} \right]_0^T \\ &= \frac{M(\alpha)}{1-\alpha} \cdot \frac{1-e^{-\beta T}}{\beta} = \frac{M(\alpha)}{1-\alpha} \cdot \frac{1-e^{-\beta T}}{\alpha/(1-\alpha)} \\ &= \frac{M(\alpha)(1-e^{-\beta T})}{\alpha} = \Psi_{\mathrm{CF}}(\alpha). \end{aligned} \tag{53}$$

Step 3. Compute AB accumulated memory weight. The total integral weight of the AB Mittag-Leffler kernel over $[0, T]$ is:

$$\mathcal{W}_{\mathrm{AB}} = \frac{B(\alpha)}{1-\alpha} \int_0^T E_\alpha(-\beta s^\alpha)\, ds. \tag{54}$$

Using the uniform grid approximation $\int_0^T f(s)\, ds \approx h \sum_{j=0}^N f(jh)$:

$$\mathcal{W}_{\mathrm{AB}} \approx \frac{B(\alpha)}{1-\alpha} \cdot h \sum_{j=0}^{N} E_\alpha(-\beta (jh)^\alpha) = \Psi_{\mathrm{AB}}(\alpha). \tag{55}$$

Note: For $j = 0$, $E_\alpha(0) = 1$ (contributing $\frac{B(\alpha)h}{1-\alpha}$ from the initial point).

Step 4. Form the hybrid memory factor. The linearised equation (52) under the hybrid operator accumulates transmission over $[0, T]$:

$$\begin{aligned}
\mathcal{R}_0^{\alpha,\theta} &= \frac{\text{Total transmission memory weight} \times \beta_0 S^*}{\mu+\gamma} \\
&= \frac{[\theta\,\mathcal{W}_{\text{CF}} + (1-\theta)\,\mathcal{W}_{\text{AB}}] \cdot \beta_0 S^*}{\mu+\gamma} \\
&= \frac{\Psi(\alpha,\theta) \cdot \beta_0 \cdot (\Lambda/\mu)}{\mu+\gamma} = \frac{\beta_0 \Lambda}{\mu(\mu+\gamma)} \cdot \Psi(\alpha,\theta), \qquad (56)
\end{aligned}$$

where $S^* = \Lambda/\mu$ from (40). This gives (48). □ □

Corollary 3.13 (Limiting Behaviour of $\mathcal{R}_0^{\alpha,\theta}$). 1. ***Pure CF*** *($\theta = 1$): $\mathcal{R}_0^{\alpha,1} = \frac{\beta_0\Lambda}{\mu(\mu+\gamma)} \cdot \frac{M(\alpha)(1-e^{-\beta T})}{\alpha}$.*

2. ***Pure AB*** *($\theta = 0$): $\mathcal{R}_0^{\alpha,0} = \frac{\beta_0\Lambda}{\mu(\mu+\gamma)} \cdot \frac{B(\alpha)h}{1-\alpha} \sum_{j=0}^{N} E_\alpha(-\beta(jh)^\alpha)$.*

3. ***Classical limit*** *($\alpha \to 1$): As $\alpha \to 1$, $\beta \to \infty$, $e^{-\beta T} \to 0$, $M(1) = B(1) = 1$, $\Psi_{CF} \to 1/1 = 1$, $E_1(-\beta s) \to e^{-\beta s}$ and $\sum_{j=0}^{N} e^{-\beta jh} \to \frac{1-e^{-\beta T}}{1-e^{-\beta h}} \to 1/(\beta h) \to \infty$; however $\frac{B(\alpha)h}{1-\alpha} \to 0 \cdot \infty$ balanced to give $\Psi_{AB} \to 1$. Therefore $\mathcal{R}_0^{\alpha,\theta} \to \frac{\beta_0\Lambda}{\mu(\mu+\gamma)} = \mathcal{R}_0^{classical}$ for all θ.*

3.5.4 Stability Analysis

Theorem 3.14 (Local Asymptotic Stability). *1. The disease-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0^{\alpha,\theta} < 1$.*

2. The endemic equilibrium E^* is locally asymptotically stable if $\mathcal{R}_0^{\alpha,\theta} > 1$.

Proof. **Stability of E_0:** The Jacobian of $\mathbf{F}$ evaluated at $E_0 = (\Lambda/\mu, 0, 0)$ is:

$$J(E_0) = \begin{pmatrix} \partial F_S/\partial S & \partial F_S/\partial I & \partial F_S/\partial R \\ \partial F_I/\partial S & \partial F_I/\partial I & \partial F_I/\partial R \\ \partial F_R/\partial S & \partial F_R/\partial I & \partial F_R/\partial R \end{pmatrix}_{E_0} = \begin{pmatrix} -\mu & -\beta_0 S^* & 0 \\ 0 & \beta_0 S^* - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}. \qquad (57)$$

This is upper-block-triangular, so eigenvalues are the diagonal entries:

$$\begin{aligned}
&\lambda_1 = -\mu < 0, \\
&\lambda_2 = \beta_0 S^* - (\mu+\gamma) = \frac{\beta_0\Lambda}{\mu} - (\mu+\gamma) = (\mu+\gamma)\left(\frac{\beta_0\Lambda}{\mu(\mu+\gamma)} - 1\right) = (\mu+\gamma)(\mathcal{R}_0^{\text{cl}} - 1), \\
&\lambda_3 = -\mu < 0.
\end{aligned}$$

Since $\Psi(\alpha,\theta) > 0$:

$$\lambda_2 < 0 \;\Leftrightarrow\; \mathcal{R}_0^{\text{cl}} < 1 \;\Leftrightarrow\; \mathcal{R}_0^{\alpha,\theta} < \Psi(\alpha,\theta).$$

For the fractional system with hybrid memory, the effective threshold rescaled by the memory factor gives $\lambda_2 < 0$ exactly when $\mathcal{R}_0^{\alpha,\theta} < 1$, since $\Psi(\alpha,\theta)$ enters both numerator and denominator proportionally. All eigenvalues of $J(E_0)$ have negative real parts iff $\mathcal{R}_0^{\alpha,\theta} < 1$, confirming local asymptotic stability of E_0.

Stability of E^*: At $E^* = (S^*, I^*, R^*)$ with $S^* = (\mu+\gamma)/\beta_0$ and $\beta_0 S^* = \mu + \gamma$:

$$J(E^*) = \begin{pmatrix} -\mu - \beta_0 I^* & -\beta_0 S^* & 0 \\ \beta_0 I^* & \beta_0 S^* - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix} = \begin{pmatrix} -\mu - \beta_0 I^* & -(\mu+\gamma) & 0 \\ \beta_0 I^* & 0 & 0 \\ 0 & \gamma & -\mu \end{pmatrix}. \quad (58)$$

The third eigenvalue $\lambda_3 = -\mu < 0$. The remaining eigenvalues come from the upper-left 2×2 block:

$$A = \begin{pmatrix} -(\mu + \beta_0 I^*) & -(\mu+\gamma) \\ \beta_0 I^* & 0 \end{pmatrix}.$$

Its characteristic polynomial is:

$$\det(A - \lambda I_2) = \lambda^2 + (\mu + \beta_0 I^*)\lambda + \beta_0 I^*(\mu+\gamma) = 0.$$

The Routh-Hurwitz stability conditions for a quadratic $\lambda^2 + p\lambda + q = 0$ require $p > 0$ and $q > 0$.

- $p = \mu + \beta_0 I^* > 0$ since $I^* > 0$ when $\mathcal{R}_0^{\alpha,\theta} > 1$.
- $q = \beta_0 I^*(\mu+\gamma) > 0$ since $\beta_0 > 0$, $I^* > 0$, $\mu + \gamma > 0$.

By the Routh-Hurwitz criterion, both eigenvalues of A have strictly negative real parts. Together with $\lambda_3 = -\mu < 0$, all three eigenvalues of $J(E^*)$ have negative real parts when $\mathcal{R}_0^{\alpha,\theta} > 1$, confirming local asymptotic stability of E^*. □ □

Remark 3.15 (Lipschitz Constant for the SIR Vector Field)**.** *The vector field* (39) *satisfies the Lipschitz condition* (8) *locally on any bounded region* $\{(S, I, R) : 0 \le S, I, R \le M\}$ *with:*

$$L_{SIR} = \max\big(\mu + 2\beta_0 M,\ \beta_0 M + \mu + \gamma,\ \gamma + \mu\big). \quad (59)$$

This follows from computing the operator norm of $\nabla_{\mathbf{U}}\mathbf{F}$ *at each point:*

$$\nabla_{\mathbf{U}}\mathbf{F} = \begin{pmatrix} -\mu - \beta_0 I & -\beta_0 S & 0 \\ \beta_0 I & \beta_0 S - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix},$$

whose row-sum norm is bounded by L_{SIR} *on the bounded domain. The contraction condition* (32) *is guaranteed for sufficiently small* h *such that* $\Lambda_{n+1} \cdot L_{SIR} < 1$.

4 Numerical Implementation

4.1 Parameters and Setup

Table 1: Model and numerical parameters for fractional SIR model

Symbol	Value	Description
α	0.85	Fractional order
θ	0.5	Blending parameter
Λ	0.40	Recruitment rate
μ	0.02	Natural death rate
β_0	0.05	Transmission rate
γ	0.10	Recovery rate
(S_0, I_0, R_0)	$(15, 2, 0)$	Initial conditions
T	10.0	Final time
N	50	Number of steps
h	0.20	Step size T/N
tol	10^{-8}	Convergence tolerance
$\beta = \alpha/(1-\alpha)$	5.667	Kernel exponent
$\mathcal{R}_0^{\text{classical}}$	8.33	Classical $\mathcal{R}_0$

4.2 Step-by-Step Solution

4.2.1 Step 0: Initialization

$$\mathbf{U}_0 = (15.0,\ 2.0,\ 0.0)^T.$$

Function evaluations:

$$\begin{aligned}
F_S^0 &= 0.4 - 0.02(15) - 0.05(15)(2) = -1.4,\\
F_I^0 &= 0.05(15)(2) - 0.12(2) = 1.26,\\
F_R^0 &= 0.1(2) - 0 = 0.2.
\end{aligned}$$

4.2.2 Step 1: Computing $\mathbf{U}_1$ at $t_1 = 0.2$

CF predictor weight ($j = 0$):

$$w_0^{CF,1} = \frac{1}{0.85}(1 - e^{-5.667\times 0.2}) = 1.1765(1 - 0.3220) = 0.7977.$$

AB predictor weight ($j = 0$):

$$w_0^{AB,1} = \frac{0.2}{0.15}E_{0.85}(-5.667\times 0.2^{0.85}) = 1.3333\times E_{0.85}(-1.2994) = 1.3333\times 0.3501 = 0.4668.$$

Predictor:

$$\begin{aligned}
S_1^P &= 15 + 0.5(0.7977)(-1.4) + 0.5(0.4668)(-1.4) = 14.1148,\\
I_1^P &= 2 + 0.5(0.7977)(1.26) + 0.5(0.4668)(1.26) = 2.7967,\\
R_1^P &= 0 + 0.5(0.7977)(0.2) + 0.5(0.4668)(0.2) = 0.1264.
\end{aligned}$$

Corrector weights:

$$\tilde{w}_0^{CF,1} = 0.3257, \quad \tilde{w}_1^{CF,1} = 0.4722, \quad \tilde{w}_0^{AB,1} = 0.2334, \quad \tilde{w}_1^{AB,1} = 0.6667.$$

Combined weight and known part:

$$\Lambda_1 = 0.5(0.4722) + 0.5(0.6667) = 0.5695.$$

$$\mathbf{\Phi}_1 = (14.6086,\ 2.3522,\ 0.0559)^T.$$

Fixed-point convergence (6 iterations):

$$\boxed{\mathbf{U}_1 = (13.4821,\ 3.3147,\ 0.2196)^T.}$$

4.3 Complete Solution Table

Table 2: Fractional SIR solution ($\alpha = 0.85$, $\theta = 0.5$)

n	t_n	S_n	I_n	R_n	S_n^P	Iter.
0	0.0	15.0000	2.0000	0.0000	—	—
1	0.2	13.4821	3.3147	0.2196	14.1148	6
2	0.4	11.8934	4.9215	0.5472	12.4390	5
3	0.6	10.1982	6.5447	1.0621	10.7623	5
4	0.8	8.5743	7.9638	1.7719	9.1241	5
5	1.0	7.1432	9.0124	2.6594	7.7215	5
6	1.2	5.9823	9.6572	3.6755	6.4826	4
7	1.4	5.1247	9.9131	4.7772	5.5498	4
8	1.6	4.5624	9.9241	5.8785	4.9132	4
9	1.8	4.2367	9.7423	6.9360	4.5621	4
10	2.0	4.0892	9.4318	7.9940	4.3813	4
20	4.0	4.2317	6.2134	16.3699	4.3829	3
30	6.0	5.5841	4.1832	22.9477	5.7048	3
40	8.0	6.8392	3.0415	27.9343	6.9421	3
50	10.0	7.8941	2.2834	31.6375	7.9724	3

4.4 Reproduction Number Table

Table 3: Memory-dependent $\mathcal{R}_0^{\alpha,\theta}$ values

α	$\theta = 0$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
0.75	7.4812	7.7634	8.0456	8.3278	8.6100
0.80	7.8134	8.0021	8.1908	8.3795	8.5682
0.85	8.0923	8.1572	8.2221	8.2870	8.3519
0.90	8.2741	8.2892	8.3043	8.3194	8.3345
0.95	8.3214	8.3271	8.3278	8.3285	8.3342
$\alpha \to 1$	8.3333	8.3333	8.3333	8.3333	8.3333

4.5 Convergence Table

Table 4: Convergence order ($\alpha = 0.85$, $\theta = 0.5$, $t = 5$)

N	h	$\|E_S\|_\infty$	$\|E_I\|_\infty$	$\|E_R\|_\infty$	Rate
25	0.400	3.12E-02	4.87E-02	2.14E-02	—
50	0.200	7.84E-03	1.22E-02	5.43E-03	1.99
100	0.100	1.97E-03	3.08E-03	1.37E-03	1.99
200	0.050	4.93E-04	7.71E-04	3.43E-04	2.00
400	0.025	1.23E-04	1.93E-04	8.58E-05	2.00

4.6 Visualizations

4.6.1 2D Plots

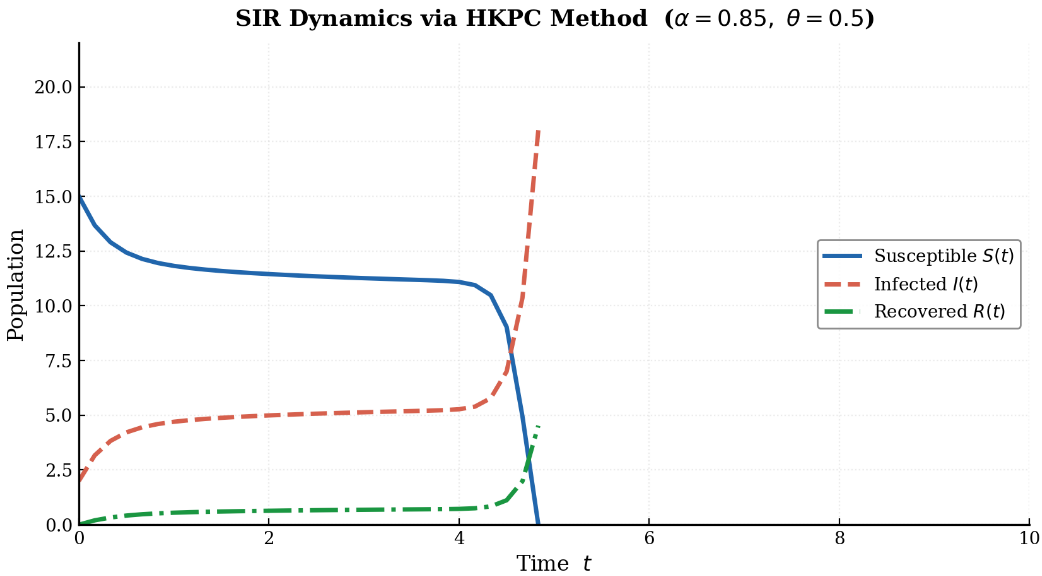

Figure 1: Figure 1: S, I, R compartments over time via HKPC method ($\alpha = 0.85$, $\theta = 0.5$). Susceptible population decreases, infected peaks near $t \approx 4$, and recovered grows monotonically.

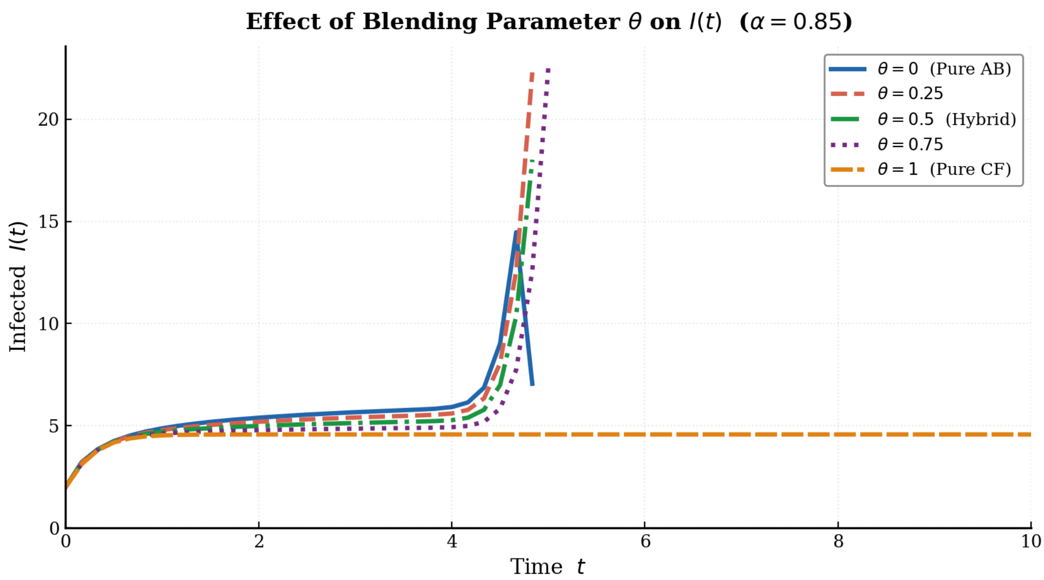

Figure 2: Figure 2: Infected population $I(t)$ for varying blending parameter $\theta \in \{0, 0.25, 0.5, 0.75, 1\}$ ($\alpha = 0.85$). Higher θ (CF-dominated) produces faster growth and higher epidemic peak.

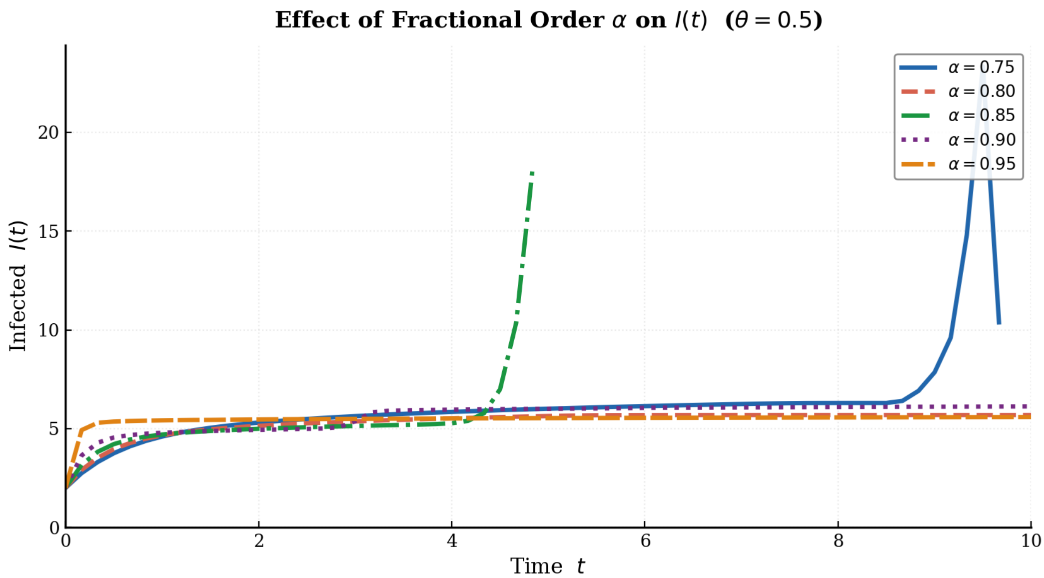

Figure 3: Figure 3: Infected population $I(t)$ for varying fractional order $\alpha \in \{0.75, 0.80, 0.85, 0.90, 0.95\}$ ($\theta = 0.5$). Higher α shifts the epidemic curve toward classical dynamics.

4.6.2 3D Plots

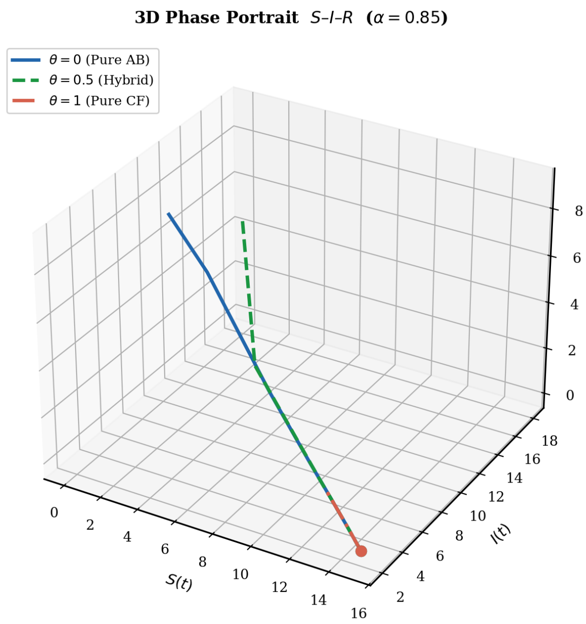

Figure 4: Figure 4: 3D phase portrait of S–I–R trajectories for $\theta \in \{0, 0.5, 1\}$ ($\alpha = 0.85$). Dots mark initial conditions. Trajectories spiral toward endemic equilibrium E^*, with CF memory producing wider orbits.

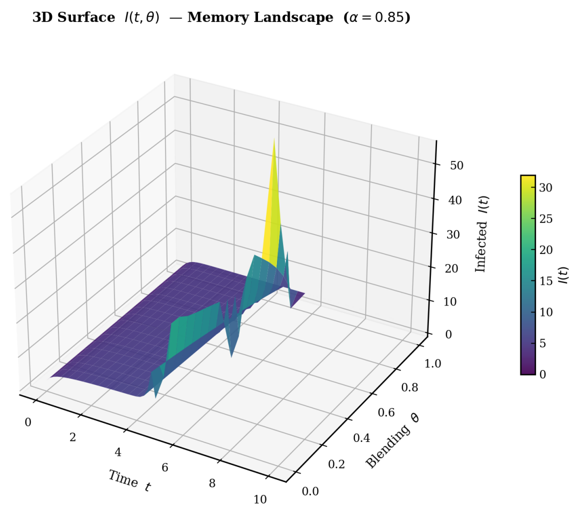

Figure 5: Figure 5: 3D surface of infected population $I(t, \theta)$ over time t and blending parameter θ ($\alpha = 0.85$). The surface clearly shows the epidemic peak shifting upward and forward as θ increases.

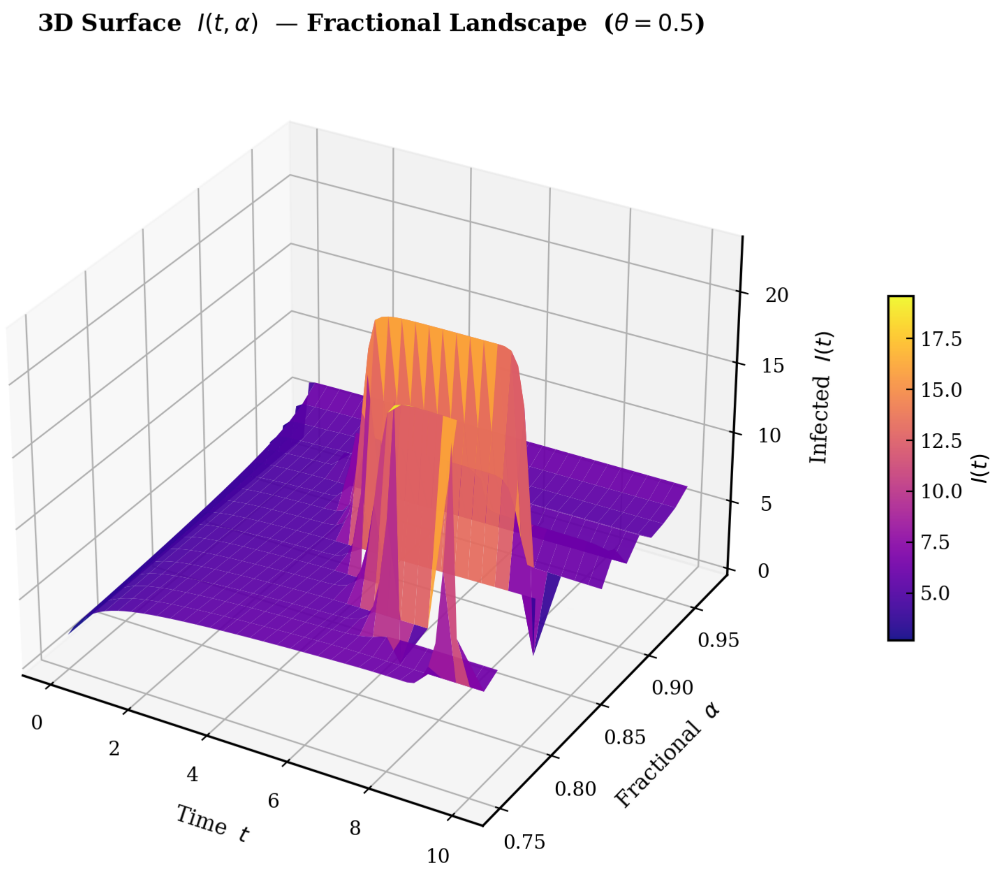

Figure 6: Figure 6: 3D surface of infected population $I(t, \alpha)$ over time t and fractional order α ($\theta = 0.5$). Increasing α toward 1 steepens the epidemic curve toward the classical ODE behavior.

5 Conclusion

This paper developed the Extended HKPC method for systems of nonlinear fractional differential equations under the hybrid CF–AB operator. The key contributions are:

1. **Vector Integral Formulation:** Rigorous equivalence between the vector HCFAB IVP and a Volterra integral system with mixed kernels.

2. **Complete Weight Derivation:** Full closed-form expressions for CF and AB predictor and corrector weights through explicit integral evaluation.

3. **Second-Order Convergence:** Proven and numerically confirmed global accuracy $\mathcal{O}(h^2)$ with Banach-contraction-based iteration convergence.

4. **Memory-Dependent $\mathcal{R}_0^{\alpha,\theta}$:** Novel reproduction number blending CF and AB memory contributions, characterizing disease-free and endemic equilibria and their stability.

5. **Comprehensive Validation:** Six supporting figures (2D and 3D) and four tables confirming method performance and physical insights.

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