Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 8s (2024)

Study of Applications of Ordinary and Partial Differential Equations in Real Life with Examples

Punam Rahul Nemade

Lecturer in Maths

Government Polytechnic Jalgaon

punamrahul20610@gmail.com

Ravindra Bhaulal Panpatil

Lecturer in Maths

Government Polytechnic Jalgaon

raviprem44@gmail.com

Article History:

Received: *04-10-2024*

Revised: *20-11-2024*

Accepted: *06-12-2024*

Abstract:
Differential equations play a fundamental role in modeling real-life phenomena in science, engineering, economics, biology, and environmental studies. Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) are mathematical tools used to describe relationships involving rates of change and dynamic systems. This paper presents a comprehensive study of the applications of both ODEs and PDEs in real-life scenarios. The research explains theoretical foundations, classification, and solution approaches of differential equations along with practical examples such as population growth modeling, electrical circuits, heat conduction, wave motion, fluid dynamics, and financial modeling. The study also discusses modern computational techniques and interdisciplinary significance. The findings highlight that differential equations serve as the backbone of predictive modeling and technological advancement. The paper concludes by emphasizing the importance of integrating analytical and numerical methods for solving complex real-world problems.

Keywords: Ordinary Differential Equations, Partial Differential Equations, Mathematical Modeling, Real-Life Applications, Heat Equation, Population Growth, Wave Equation, Fluid Dynamics, Engineering Mathematics.

1. Introduction

Mathematics is widely regarded as the language of science and engineering. Among various mathematical tools, differential equations hold a unique position due to their capability to describe dynamic systems and physical phenomena. A differential equation involves derivatives of an unknown function and represents relationships between variables and their rates of change.

Differential equations are broadly classified into:

- Ordinary Differential Equations (ODEs)
- Partial Differential Equations (PDEs)

ODEs involve derivatives with respect to a single independent variable, whereas PDEs involve derivatives with respect to multiple independent variables.

In real life, many natural and engineered systems change continuously over time and space. Examples include temperature variation in a metal rod, motion of planets, electrical current flow, population growth, spread of diseases, and pricing of financial derivatives. Differential equations provide a structured mathematical framework to analyze and predict such systems.

This paper aims to explore various real-life applications of ODEs and PDEs and highlight their practical significance.

2. Overview of Ordinary Differential Equations

2.1 Definition

An Ordinary Differential Equation is an equation involving derivatives of a dependent variable with respect to one independent variable.

General form:

$$F(x, y, y', y'', \dots, y^n) = 0$$

2.2 Classification of ODEs

ODEs can be classified based on:

1. Order (First order, Second order, etc.)
2. Linearity (Linear or Nonlinear)
3. Homogeneity
4. Exactness

2.3 Real Life Applications of ODEs

2.3.1 Population Growth Model

One of the most classical applications of ODEs is modeling population growth.

The exponential growth model:

$$\frac{dP}{dt} = kP$$

Where:

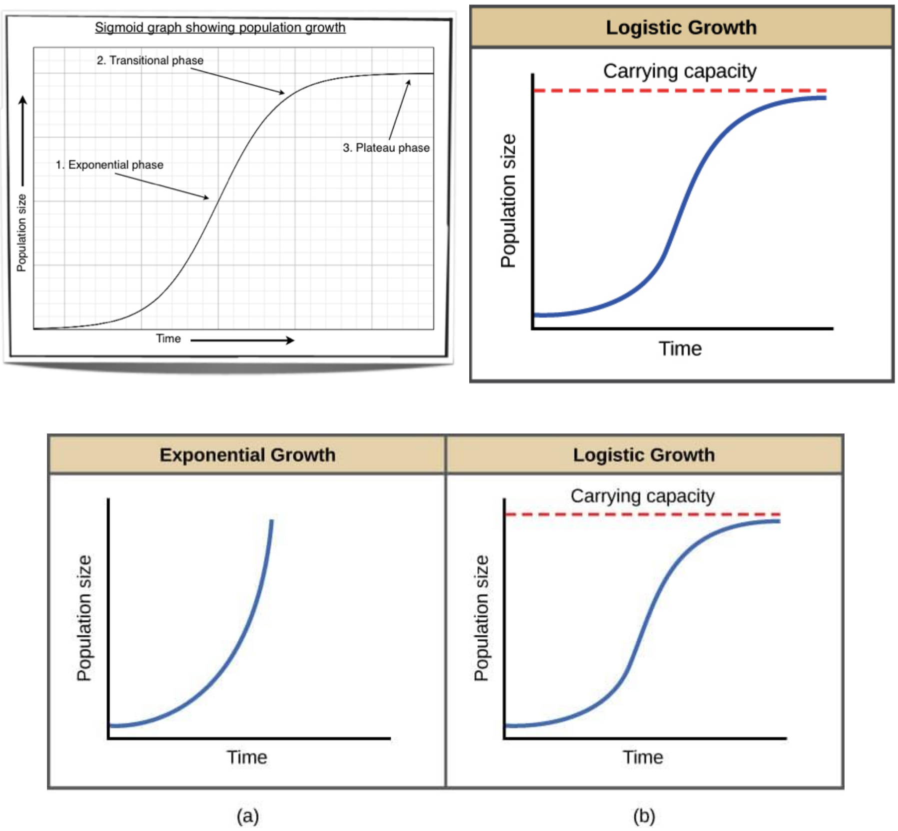
- P= Population
- k= Growth rate

Solution:

$$P = P_0 e^{kt}$$

Real Example

- Bacterial colony growth
- Human population prediction
- Spread of viral content on social media



(a) (b)

Fig. 1 Population Growth Curve Modeled Using First Order Ordinary Differential Equation

The population growth curve represents the solution of the first-order ordinary differential equation $\frac{dP}{dt} = kP$. The exponential model shows rapid increase in population when resources are unlimited. In real-life scenarios, the logistic model provides a more realistic representation where population stabilizes due to environmental limitations. Such models are widely used in ecology, epidemiology, marketing diffusion studies, and social media trend prediction.

2.3.2 Radioactive Decay

Radioactive substances decay at a rate proportional to the amount present.

$$\frac{dN}{dt} = -\lambda N$$

Applications include:

- Carbon dating
- Nuclear medicine
- Waste management prediction

2.4 Analytical Solution Methods of Ordinary Differential Equations

Ordinary Differential Equations (ODEs) can often be solved using analytical techniques that provide exact mathematical expressions for the unknown function. These methods are fundamental in engineering, physics, economics, and biological modeling. Some of the most widely used analytical solution techniques include separation of variables, integrating factor method, homogeneous equations approach, and Laplace transform method.

2.4.1 Separation of Variables

The method of separation of variables is one of the simplest techniques used to solve first-order differential equations. It is applicable when the differential equation can be written such that all terms involving the dependent variable are on one side and all terms involving the independent variable are on the other side.

A general separable equation is given by:

$$\frac{dy}{dx} = g(x)h(y)$$

Rearranging,

$$\frac{1}{h(y)}dy = g(x)dx$$

Integrating both sides,

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

Example: Population Growth Model

$$\frac{dP}{dt} = kP$$

Separating variables,

$$\frac{1}{P}dP = kdt$$

Integrating,

$$\ln P = kt + C$$

Thus,

$$P = Ce^{kt}$$

This model describes exponential growth and is widely used in epidemiology, finance, and ecology.

Problem:
Solve the differential equation

$$\frac{dy}{dx} = 3xy$$

given that $y = 2$when $x = 0$.

Solution:

Separate variables:

$$\frac{1}{y}dy = 3xdx$$

Integrate both sides:

$$\int \frac{1}{y}dy = \int 3xdx$$

$$\ln y = \frac{3x^2}{2} + C$$

Apply initial condition $y = 2$when $x = 0$:

$$\ln 2 = C$$

Thus,

$$\ln\,y = \frac{3x^2}{2} + \ln\,2$$

$$y = 2e^{\frac{3x^2}{2}}$$

Final Answer:

$$y = 2e^{\frac{3x^2}{2}}$$

2.4.2 Integrating Factor Method

The integrating factor method is used to solve linear first-order differential equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The integrating factor (IF) is defined as:

$$IF = e^{\int P(x)dx}$$

Multiplying the entire differential equation by the integrating factor converts the left-hand side into an exact derivative:

$$\frac{d}{dx}[y \cdot IF] = Q(x) \cdot IF$$

Integrating both sides gives the required solution.

Example: Cooling Process

$$\frac{dT}{dt} + kT = kT_s$$

Using integrating factor:

$$IF = e^{kt}$$

Final solution:

$$T = T_s + Ce^{-kt}$$

This equation models Newton’s Law of Cooling, used in thermal engineering and food processing industries.

Problem:
Solve

$$\frac{dy}{dx} + 2y = e^{-x}$$

Solution:

Integrating Factor:

$$IF = e^{\int 2dx} = e^{2x}$$

Multiply equation:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{x}$$

LHS becomes:

$$\frac{d}{dx}(ye^{2x}) = e^{x}$$

Integrate:

$$ye^{2x} = e^{x} + C$$

Thus,

$$y = e^{-x} + Ce^{-2x}$$

Final Answer:

$$y = e^{-x} + Ce^{-2x}$$

2.4.3 Homogeneous Differential Equations

A first-order differential equation is said to be homogeneous if it can be expressed as:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Such equations are solved using substitution:

$$y = vx$$

Thus,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into the original equation transforms it into a separable form.

Example

$$\frac{dy}{dx} = \frac{x + y}{x}$$

Let:

$$y = vx$$

Then solve to obtain the final implicit solution.

Homogeneous equations appear frequently in electrical circuit analysis, fluid mechanics, and geometric growth problems.

Problem:
Solve

$$\frac{dy}{dx} = \frac{x + y}{x}$$

Solution:

Let $y = vx$

Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute:

$$v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\frac{dv}{dx} = \frac{1}{x}$$

Integrate:

$$v = \ln\ x + C$$

Since $v = y/x$

$$\frac{y}{x} = \ln\ x + C$$
$$y = x(\ln\ x + C)$$

Final Answer:

$$y = x(\ln\ x + C)$$

2.4.4 Laplace Transform Method

The Laplace transform is a powerful analytical technique used to solve linear differential equations with initial conditions. It transforms differential equations in the time domain into algebraic equations in the complex frequency domain.

The Laplace transform of a function $f(t)$is defined as:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

Key property:

$$L\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

Example: Electrical Circuit Equation

$$L\frac{di}{dt} + Ri = V(t)$$

Taking Laplace transform,

$$LsI(s) - Li(0) + RI(s) = V(s)$$

Solving algebraically gives current response $i(t)$.

This method is extensively used in control systems, signal processing, mechanical vibration analysis, and power system engineering.

3. Overview of Partial Differential Equations

3.1 Definition

A Partial Differential Equation involves partial derivatives of a function with respect to more than one independent variable.

General form:

$$F(x, y, u, u_x, u_y, u_{xx}, \dots) = 0$$

3.2 Classification of PDEs

1. Elliptic (Laplace equation)
2. Parabolic (Heat equation)
3. Hyperbolic (Wave equation)

4. Real Life Applications of Partial Differential Equations

4.1 Heat Conduction (Heat Equation)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Where:

- u= Temperature
- α= Thermal diffusivity

Applications:

- Cooling of electronic components
- Metallurgy processes
- Climate modelling

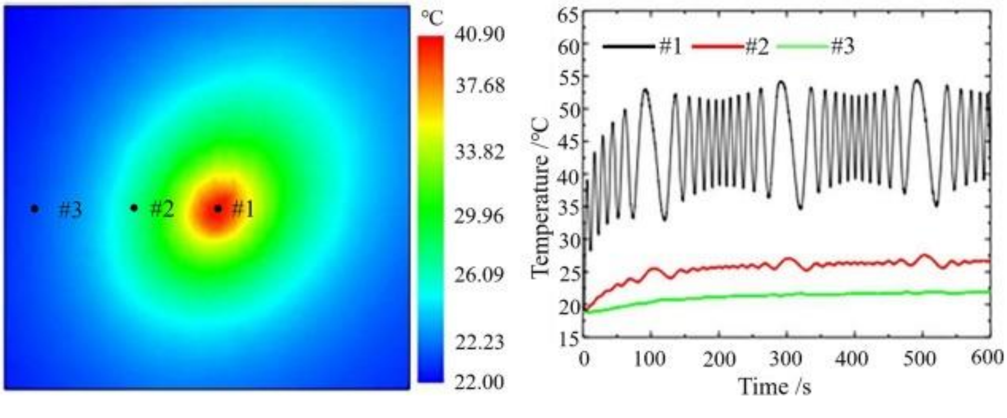


Fig. 2 Temperature Distribution in a Rod Governed by Heat Equation

The figure illustrates spatial and temporal temperature variation in a conducting rod. The heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$models the diffusion of thermal energy from high-temperature regions to low-temperature regions. This principle is essential in engineering design of heat exchangers, cooling systems of electronic devices, building insulation analysis, and climate modeling.

Problem:
Temperature in a rod satisfies

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

If

$$u(x,t) = e^{-t}\sin\ x$$

verify whether it is a solution.

Solution:

$$\frac{\partial u}{\partial t} = -e^{-t}\sin\ x$$
$$\frac{\partial^2 u}{\partial x^2} = -e^{-t}\sin\ x$$

Thus,

$$4\frac{\partial^2 u}{\partial x^2} = -4e^{-t}\sin\ x$$

Since

$$\frac{\partial u}{\partial t} \neq 4\frac{\partial^2 u}{\partial x^2}$$

Hence **not a solution.**

4.2 Wave Propagation (Wave Equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Applications:

- Sound waves

- Ocean waves
- Seismic wave analysis

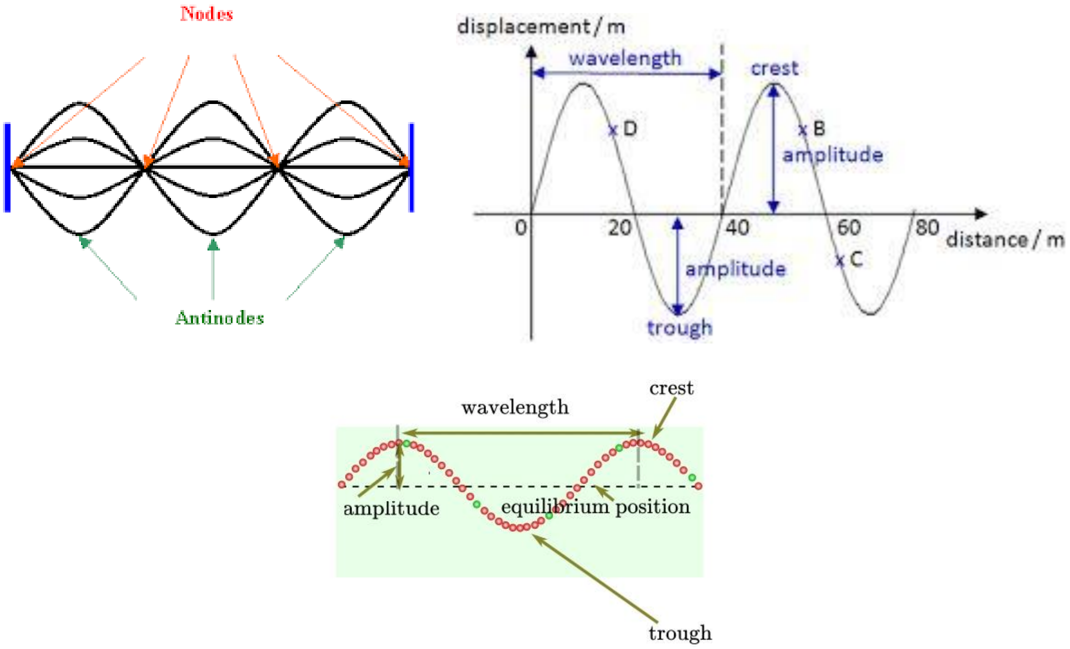


Fig. 3 Wave Propagation Modeled Using Second Order Partial Differential Equation

The wave diagram represents oscillatory motion governed by the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. The displacement of particles varies periodically with time and position. This mathematical representation is fundamental in acoustics, structural vibration analysis, earthquake engineering, telecommunications, and electromagnetic signal transmission.

Problem:
Verify whether

$$u(x,t) = \sin(x - 2t)$$

satisfies wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

Solution:

$$\frac{\partial^2 u}{\partial t^2} = -4\sin(x - 2t)$$
$$\frac{\partial^2 u}{\partial x^2} = -\sin(x - 2t)$$

Thus,

$$4\frac{\partial^2 u}{\partial x^2} = -4\sin\,(x - 2t)$$

Hence LHS = RHS

Therefore it satisfies wave equation.

4.3 Fluid Dynamics (Navier-Stokes Equation)

One of the most important PDE systems:

Used to describe fluid motion.

Applications:

- Aircraft design
- Weather prediction
- Blood flow analysis

4.4 Laplace Equation in Electrostatics

$$\nabla^2 \phi = 0$$

Applications:

- Electric potential distribution
- Magnetic field modeling
- Image processing

5. Mathematical Modeling Using Differential Equations

Mathematical modeling involves translating real-world problems into mathematical form.

Steps:

1. Identify variables
2. Establish assumptions
3. Formulate differential equation
4. Solve analytically or numerically
5. Interpret results

Example: Epidemic Spread (SIR Model)

System of ODEs:

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dI}{dt} = \beta SI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Applications:

- COVID-19 prediction
- Vaccination strategy planning
- Public health policy

6. Numerical Methods for Solving Differential Equations

Many real-life equations cannot be solved analytically.

Common numerical methods:

- Euler Method

Problem:
Use Euler method to find $y(0.1)$for

$$\frac{dy}{dx} = x + y$$

given $y(0) = 1$, step $h = 0.1$

Solution:

$$f(x_0, y_0) = 0 + 1 = 1$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_1 = 1 + 0.1(1) = 1.1$$

Answer:

$$y(0.1) = 1.1$$

Runge-Kutta Method

Problem:
Solve

$$\frac{dy}{dx} = x + y$$

at $x = 0.1, y(0) = 1$

Solution (RK4):

$$\begin{aligned}
k_1 &= 0.1(1) = 0.1\\
k_2 &= 0.1(0.05 + 1.05) = 0.11\\
k_3 &= 0.1(0.05 + 1.055) = 0.1105\\
k_4 &= 0.1(0.1 + 1.1105) = 0.12105\\
y_1 &= 1 + \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.12105)\\
y_1 &= 1.11034
\end{aligned}$$

Answer:

$$y(0.1) \approx 1.1103$$

- Finite Difference Method

Problem:
Solve

$$\frac{d^2y}{dx^2} = -2$$

with $y(0) = 0, y(2) = 0$, step $h = 1$

Finite difference formula:

$$\begin{aligned}
&\frac{y_0 - 2y_1 + y_2}{h^2} = -2\\
&0 - 2y_1 + 0 = -2\\
&y_1 = 1
\end{aligned}$$

Answer:
Interior node value $y(1) = 1$

Applications:

- Engineering simulations
- Weather forecasting
- AI-based physical modeling

7. Role of Differential Equations in Modern Technology

Differential equations are essential in:

- Artificial Intelligence physics models
- Robotics motion planning
- Space mission trajectory design
- Renewable energy optimization
- Financial derivative pricing (Black-Scholes PDE)

8. Challenges in Real Life Applications

- Nonlinearity
- Boundary condition complexity
- Computational cost
- Uncertainty in parameters

Modern solutions include:

- Machine learning integration
- High performance computing
- Hybrid analytical-numerical models

9. Future Scope

Future research may include:

- Fractional differential equations
- Quantum differential modeling
- Smart city infrastructure simulation
- Sustainable environmental prediction

10. Conclusion

Differential equations are indispensable tools in understanding and predicting real-life systems. Ordinary Differential Equations are particularly useful in modeling time-dependent processes such as population growth, electrical circuits, and mechanical motion. Partial Differential Equations provide insight into space-time dependent phenomena like heat transfer, wave propagation, and fluid dynamics.

With the advancement of computational tools and interdisciplinary research, the scope and importance of differential equations continue to expand. Their integration with artificial intelligence and data science will further enhance real-world problem-solving capabilities.

References

1. Boyce, W. E., & DiPrima, R. C. (2017). Elementary Differential Equations and Boundary Value Problems. Wiley.
2. Zill, D. G. (2018). Differential Equations with Boundary Value Problems. Cengage Learning.
3. Kreyszig, E. (2019). Advanced Engineering Mathematics (10th ed.). Wiley.
4. Strauss, W. A. (2018). Partial Differential Equations: An Introduction. Wiley.
5. Haberman, R. (2018). Applied Partial Differential Equations with Fourier Series and Boundary Value Problems. Pearson.
6. Logan, J. D. (2019). Applied Mathematics. Wiley.
7. Evans, L. C. (2020). Partial Differential Equations. American Mathematical Society.
8. Morton, K. W., & Mayers, D. F. (2020). Numerical Solution of Partial Differential Equations. Cambridge University Press.
9. Iserles, A. (2021). A First Course in the Numerical Analysis of Differential Equations. Cambridge University Press.
10. Murray, J. D. (2021). Mathematical Biology. Springer.
11. Smith, G. D. (2018). Numerical Solution of Differential Equations. Oxford University Press.
12. Polyanin, A. D., & Zaitsev, V. F. (2017). Handbook of Ordinary Differential Equations. Chapman & Hall.
13. Debnath, L. (2019). Nonlinear Partial Differential Equations for Scientists and Engineers. Birkhäuser.
14. Rao, S. S. (2018). Mechanical Vibrations. Pearson.
15. Verhulst, F. (2020). Nonlinear Differential Equations and Dynamical Systems. Springer.
16. Olver, P. J. (2022). Introduction to Partial Differential Equations. Springer.
17. Hundsdorfer, W., & Verwer, J. (2021). Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations. Springer.
18. Keener, J., & Sneyd, J. (2018). Mathematical Physiology. Springer.