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# On Optimal Control for $G-$ Fractional Stochastic Dynamical Systems with Delay

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**Abstract:**
This paper deals with a system of $G-$fractional stochastic differential equations with delay. Under some regularity conditions we study the well posedness of the system and we establish the existence of stochastic optimal control.

**Keywords**: G Brownian motion, Stochastic differential equation, Fractional derivative, Sublinear expectation, Optimal control, cost functional.

## 1. Introduction

Fractional stochastic differential equations are studied to describe different problems, for example in finance, physics and control theory. Many researchers worked out some interesting results of fractional stochastic differential equations. Among the important results, we cite the existence and uniqueness of solution, stability results, the averaging principle of fractional systems with perturbations, see [1,2,3,4,5,10,11,13,19].

The theory of sublinear expectation has quickly developed and has been of interest to many researchers due to the important potential applications in uncertainty problem. The concept of uncertainty in fluctuations was studied by Peng, see [21,22,23,24,25], who established a new stochastic process called $G-$Brownian motion as a way to incorporate the unknown volatility into financial models. Denis and Martini [15], suggested a structure based on the quasi-sure analysis from the abstract potential theory to construct a similar structure using a tight family of possibly mutually singular probability measures. At the current time, it should be noted that the problems with uncertainty based on $G-$Brownian motion have been widely studied by several authors. On the other hand, the optimal control theory is one of the most important fundamental concepts in mathematics that plays an important role in both deterministic and stochastic systems. This theory has found increasing application in the domain of applied mathematics. In the $G-$framework, Redjil et al. [26] proved the existence of relaxed optimal control, the existence of optimal control for stochastic system with controlled jumps is established [27]. We note that many authors have considered the optimal control problems and stochastic differential equations under incertainty, further, details can be found in [6,7,8,9,12,14,16,17,27,30,31,33].

Stochastic control problem is the study of dynamical systems subject to random perturbations and which can be controlled in order to optimize some performance criterion. Historically handled with Bellman's and Pontryagin's optimality principles, the research on control theory considerably developed over these last years, inspired in particular by problems emerging from mathematical finance.

Motivated by the applications of fractional SDEs, the solution properties have been widely studied, B.P. Moghaddam et al.[18], studied the sufficient conditions for existence and uniqueness of fractional stochastic delay differential equations. In [29], Saci et al. established the existence and uniqueness of solution for fractional stochastic differential equations driven by $G-$Brownian motion with delay ($G-$FSDEs in short) and the authors investigated the averaging principle. Recently, D. Kasinathan et al. studied the dynamical behaviors of fractional neutral stochastic integro-differential delay systems with impulses, for more details see [19]. In this paper, we are interested on studying, the existence of stochastic optimal control for fractional stochastic differential equations driven by $G-$Brownian motion with delay. Such equations are used to model a realistic financial modeling framework by combining memory effects (fractional), volatility uncertainty (G-Brownian motion), and decision delays. This makes them valuable especially in optimal control problems in uncertain markets. For this, we consider systems governed by the following equation

$$\begin{cases} D_t^\alpha X(t) = b(t, X_t)dt + h(t, X_t)d\langle B\rangle_t + \sigma(t, X_t)dB_t, t \in [0, T] \\ X_0 = \Psi := \{\Psi(\theta): -\tau \leq \theta \leq 0\}, \end{cases} \tag{1.1}$$

where $X_t = \{X(t+\theta): -\tau \leq \theta \leq 0\}$, $\tau \in [0, +\infty[$, $D_t^\alpha$ is the "Caputo fractional derivative", $\alpha \in \left(\frac{1}{2}, 1\right)$. The coefficients $b, h, \sigma$ are in the space $\mathcal{M}_G^2([0,T];\mathbb{R})$, $\{\langle B\rangle_t, t \geq 0\}$ is the quadratic variation process of $G-$Brownian motion. We denote by $BC([-\tau, 0];\mathbb{R})$, the family of bounded continuous $\mathbb{R}-$ valued mapping $\phi$ defined on $[-\tau, 0]$ with norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$,

the proofs of the well posedness of problem $(1.1)$ are based on the results of Saci et al.[28].

Our main result is proving the existence of an optimal control in the $G-$framework. To this end, we consider the following controlled $G-$FSDE with delay

$$\begin{cases} D_t^\alpha X_t = b(t, X_t, u_t)dt + h(t, X_t, u_t)d\langle B\rangle_t + \sigma(t, X_t)dB_t \\ X_0 = \Psi := \{\Psi(\theta): -\tau \leq \theta \leq 0\}, \end{cases} \tag{1.2}$$

where $u(.) \in A$ stands for the control variable for each $t \in [0,T]$, and $A$ is a compact polish space of $\mathbb{R}$. Let the set $A$ denote the space action and the set $U = U[0,T]$ is the set of admissible controls.

This article is organized as follows. In Section 2, we give some preliminaries required for the further of the subject. In Section 3, we study the existence and uniqueness of solution. Later, we established the existence of an optimal control of our controlled system. The last section is devoted to studying an example on stochastic optimal control for fractional SDEs in order to confirm the validity of our results.

**Basic settings**

In this section, we introduce preliminary results in the $G$ −framework which are required in the rest.

Let $C_{b,lip}(\mathbb{R}^n)$ be the space of all bounded and lipschitz continuous functions on $\mathbb{R}^n$. Let $T \in \mathbb{R}^+$ be a fixed time. Consider the space $\Omega := \{w: [0,T] \to \mathbb{R},\ w \text{ is continuous function and } w(0) = 0\ \}$, which is equipped with the following distance:

$$\rho(\omega^1, \omega^2) = \sum_{n=1}^{\infty} 2^{-n} \left( \max_{t=[0,n]} |\omega_t^1 - \omega_t^2| \wedge 1 \right), \omega^1, \omega^2 \in \Omega$$

and consider the canonical process $B_t(\omega) = \omega_t$ for $t \in [0,\infty)$ and $\omega \in \Omega$. Let

$$Lip(\Omega_t) = \{\phi(B_{t_1}, B_{t_2} - B_{t_n}): t_1, t_2, \ldots t_n \in [0,t], \phi \in C_{l,lip}(\mathbb{R}^n)\}$$

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

We have $Lip(\Omega_t) \subset Lip(\Omega_T)$ for each $t \in [0,T]$.

A functional $\hat{\mathbb{E}}: \mathcal{H} := Lip(\Omega) \to \mathbb{R}$ is a consistent sublinear expectation on the lattice $\mathcal{H}$ of real functions $i.e.$ it satisfies:

$i$ - Monotonicity: for all $X, Y \in \mathcal{H}$, $X \geq Y \Rightarrow \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

$ii$ - Constant preserving: for all $c \in \mathbb{R}$, $\hat{\mathbb{E}}[c] = c$.

$iii$ - Subadditivity: for all $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ .

$iv$ - Positive homogeneity: for all $\lambda \geq 0, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$.

The triplet $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a sublinear expectation space.

**Definition 1**. Let $Y = (Y_1, \ldots, Y_n)$ be an $n$ −dimensional random vector on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. $Y$ is said to be independent from an $m$ −dimensional random vector $X = (X_1, \ldots, X_m)$ if for each $\varphi \in C_{b,lip}(\mathbb{R}^{n+m})$, $\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}]$.

**Definition 2**. ($G$ − Brownian motion) The canonical process $(B_t)_{t\geq 0}$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a $G$ −Brownian motion if the following properties are satisfied:

1. $B_0 = 0$;

2. For each $t, s \geq 0$ the increment $B_{t+s} - B_t$ is $\mathcal{N}(0, [s\underline{\sigma}^2, s\overline{\sigma}^2])$ −distributed

3. $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ is independant of $B_t$, for $n \geq 1$ and $t_1, t_2, \ldots, t_n \in [0,t]$.

We denote by $L_G^p(\Omega_T)$ ($p \geq 1$) be the Banach space completion of $Lip(\Omega_T)$ under the natural norm $\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{1/p}$, and we consider the following type of simple process $\mathcal{M}_G^{p,0}(0,T)$: for a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0,T]$,

$$\left\{\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j\,(\omega)\mathbb{I}_{[t_j,t_{j+1})}(t), \quad \xi_j \in L_G^p\left(\Omega_{t_j}\right)\right\}$$

Let us denote the completion of $\mathcal{M}_G^{p,0}(0,T)$ by $\mathcal{M}_G^p(0,T)$ under the norm

$$\|\eta\|_{\mathcal{M}_G^p(0,T)} := \left[\int_0^T \hat{\mathbb{E}}\,[|\eta_s|^p]ds\right]^{\frac{1}{p}},\ p \geq 1.$$

for each $\eta \in \mathcal{M}_G^{2,0}(0,T)$, the related Itô intergal of $(B_t)_{t\geq 0}$ is defined by

$$I(\eta) = \int_0^T \eta\,(s)dB_s := \sum_{j=0}^{N-1} \eta_j\left(B_{t_{j+1}} - B_{t_j}\right)$$

where the mapping $I: \mathcal{M}_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is continuously extended to $\mathcal{M}_G^2(0,T)$. The quadratic variation process $\langle B\rangle_t$ of $(B_t)_{t\geq 0}$ defined by

$$\langle B\rangle_t := B_t^2 - 2\int_0^t B_s\,dB_s.$$

For each $\eta \in \mathcal{M}_G^{1,0}(0,T)$, let the mapping $J: \mathcal{M}_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ given by

$$J(\eta) = \int_0^T \eta\,(t)d\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j\left(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}\right).$$

Then $J$ can be extended continuously to $\mathcal{M}_G^1(0,T)$.

**Definition 3**. We define the capacity $\mathbb{C}$ associated with $\hat{\mathbb{E}}$ by putting

$$\mathbb{C}(A) := \sup_{p\in\mathcal{P}} P(A), A \in \mathcal{B}(\Omega).$$

We will say that a set $A \in \mathcal{B}(\Omega)$ is polar if $\mathbb{C}(A) = 0$. A property holds quasi-surely $(q.s.$ in short) if it holds outside a polar set.

**Lemma 4**. [11] *Let $X \in L_G^p(\Omega)$, then for each $\alpha > 0$*

$$\mathbb{C}(|X| > \alpha) \leq \frac{\hat{\mathbb{E}}[|X|^p]}{\alpha^p}.$$

The following two lemmas are the $G - BDG$ type inequalities with respect to the quadratic variation process $\langle B\rangle_t$ and $B_t$ respectively.

**Lemma 5**. *Let for $p \geq 1$, $\eta \in \mathcal{M}_G^p(0,T)$ and $0 \leq s \leq t \leq u \leq T$. Then we have:*

$$\hat{\mathbb{E}}\left[\sup_{s\leq t\leq u}\left|\int_s^t \eta_v\,d\langle B\rangle_v\right|^p\right] \leq C_1|u-s|^{p-1}\int_s^u \hat{\mathbb{E}}\,[|\eta_v|^p]dv,$$

*where $C_1$ is a positive constant independent of $\eta$.*

**Lemma 6**. *Let for* $p \geq 2$, $\eta \in \mathcal{M}_G^p(0,T)$ *and* $0 \leq s \leq t \leq u \leq T$. *Then we have:*

$$\hat{\mathbb{E}}\left[\sup_{s\leq t\leq u}\left|\int_s^t \eta_v \, dB_v\right|^p\right] \leq C_2|u-s|^{\frac{p}{2}-1}\int_s^u \hat{\mathbb{E}}\left[|\eta_v|^p\right]dv,$$

*where* $C_2$ *is a positive constant independent of* $\eta$.

We will need the generalized Gronwall Lemma:

**Lemma 7**. [31] *Suppose* $\beta > 0, a(.) \geq 0$ *is a nonnegative function locally integrable on* $0 \leq t \leq T$ (some $T \leq +\infty$) *and* $g(.)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \leq t \leq T$, $g(t) \leq M$(constant), *and suppose* $u(.)$ *is nonnegative and locally integrable on* $0 \leq t \leq T$ *with*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds.$$

*Then,*

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\beta)\right)^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds \text{ for each } 0 \leq t < T.$$

**Corollary 8**. [31] *Under the hypothesis of Lemma* 7, *we have*

$$u(t) \leq a(t)E_\beta\left(g(t)\Gamma(\beta)t^\beta\right),$$

*where* $E_\beta$ *is the Mittag-Leffler function defined by* $E_\beta(z) = \sum_{k=0}^{\infty}\frac{(z)^k}{\Gamma(k\beta+1)}$.

**Existence and uniqueness**

The objective of this section is to prove existence and uniqueness of the solution of equation (1.1), the proof is based on the results of Saci et al. [29],where the functions $b(.,x), h(.,x)$ and $\sigma(.,x) \in \mathcal{M}_G^2([0,T])$ for each $x \in \mathbb{R}$. We consider the following assumptions:

$(H_1)$ $J = b, h, \sigma: [0,T] \times BC([-\tau,0];\mathbb{R}) \times \Omega \to \mathbb{R}$, satisfy the Lipschitz continuous condition with respect to $x$, uniformly with respect to $t$, that is for any $x, y \in BC([-\tau,0];\mathbb{R})$

$$|J(t,x) - J(t,y)|^2 \leq D\|x-y\|^2, q.s.$$

And

$(H_2)$ $\quad |J(t,0)|^2 \leq P, q.s.$

uniformly with respect to $t$, where $D$ and $P$ are positive constants.

**Definition 9**. The process $X \in \mathcal{M}_G^2([-\tau,T])$ is called a solution of the equation (1.1) with initial condition $\Psi$, if for all $t \in [0,T]$;

$$\begin{aligned} X(t) &= \Psi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, b(s,X_s)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, h(s,X_s)d\langle B\rangle_s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s)dB_s \qquad (3.1) \end{aligned}$$

and $X(t)=\Psi(t)$ for all $t\in[-\tau,0]$.

**Theorem 10**. *Under the assumptions* $(H_1)$ *and* $(H_2)$*, the* $G-FSDE$ $(1.1)$ *has a unique solution in* $\mathcal{M}_G^2([-\tau,T])$.

*Proof.* **Uniqueness**

Let $X$ and $Z\in\mathcal{M}_G^2([-\tau,T])$ represent two solutions of the $G$ −FSDE $(1.1)$ with the same initial condition $\Psi$. For all $0\le s\le T$,

$$\|X_s-Z_s\|\le \sup_{u\in[-\tau,s]}|X(u)-Z(u)|\le \sup_{u\in[0,s]}|X(u)-Z(u)|.$$

It is clear that for $0\le t\le T$

$$\begin{aligned} X(t)-Z(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[b(s,X_s)-b(s,Z_s)]ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[h(s,X_s)-h(s,Z_s)]d\langle B\rangle_s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[\sigma(s,X_s)-\sigma(s,Z_s)]dB_s, \end{aligned}$$

so that

$$\begin{aligned} \sup_{t\in[0,T]}|X(t)-Z(t)|^2 &\le \frac{3}{\Gamma(\alpha)^2}\sup_{t\in[0,T]}\left|\int_0^t (t-s)^{\alpha-1}\,[b(s,X_s)-b(s,Z_s)]ds\right|^2 \\ &+\frac{3}{\Gamma(\alpha)^2}\sup_{t\in[0,T]}\left|\int_0^t (t-s)^{\alpha-1}\,[h(s,X_s)-b(s,Z_s)]d\langle B\rangle_s\right|^2 \\ &+\frac{3}{\Gamma(\alpha)^2}\sup_{t\in[0,T]}\left|\int_0^t (t-s)^{\alpha-1}\,[\sigma(s,X_s)-\sigma(s,Z_s)]dB_s\right|^2. \end{aligned}$$

Under the hypothesis $(H_1)$, we apply Cauchy-Schwarz, Holder and $G-BDG$ inequalities, we obtain the following estimate:

$$\hat{\mathbb{E}}\sup_{t\in[0,T]}|X(t)-Z(t)|^2\le r_1\int_0^T \hat{\mathbb{E}}\left[\sup_{u\in[0,s]}|X(u)-Z(u)|^2\right]ds,$$

where $r_1=\frac{3DT^{2\alpha-2}(T+TC_1+C_2)}{\Gamma(\alpha)^2}$. By classical Gronwell's lemma, we obtain

$$\hat{\mathbb{E}}\left[\sup_{s\in[0,T]}|X(s)-Z(s)|^2\right]=0,$$

which implies that $X(s)=Z(s)\ \ q.s.$ for any $s\in[0,T]$ and then $X(s)=Z(s)\ \ q.s.$ for all $s\in[-\tau,T]$.

**Existence**

*Let* $X^0(t)=0$ for any $t\in[-\tau,T]$. Define the following Picard sequence: For each $n\geq 1$, we set $X_0^n=\Psi$ and

$$\begin{aligned} X^n(t) \quad &= \Psi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, b(s,X_s^{n-1})ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, h(s,X_s^{n-1})d\langle B\rangle_s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s^{n-1})dB_s. \end{aligned} \tag{3.2}$$

The existence will be proved in 3 steps

*Step 1: First, we prove that* $X^n(t)\in\ L_G^2(\Omega)$ *for all* $t\in[0,T]$. We claim that $X^n\in M_G^2([-\tau,T])$. Indeed, we have:

$$\begin{aligned} |X^n(t)|^2 \quad &\leq 4|\Psi(0)|^2+\frac{4}{\Gamma(\alpha)^2}\left|\int_0^t (t-s)^{\alpha-1}\, b(s,X_s^{n-1})ds\right|^2 \\ &+\frac{4}{\Gamma(\alpha)^2}\left|\int_0^t (t-s)^{\alpha-1}\, h(s,X_s^{n-1})d\langle B\rangle_s\right|^2 \\ &+\frac{4}{\Gamma(\alpha)^2}\left|\int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s^{n-1})dB_s\right|^2. \end{aligned}$$

Applying Cauchy-Schwarz, Holder and $G-BDG$ inequalities, we get the following estimation:

$$\begin{aligned} \hat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^n(t)|^2\right) \quad &\leq 4|\Psi(0)|^2 \\ &+\frac{4T}{\Gamma(\alpha)^2}\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}\int_0^t (t-s)^{2\alpha-2}\,|b(s,X_s^{n-1})|^2 ds\right) \\ &+\frac{4TC_1}{\Gamma(\alpha)}\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}\int_0^t (t-s)^{2\alpha-2}\,|h(s,X_s^{n-1})|^2 ds\right) \\ &+\frac{4C_2}{\Gamma(\alpha)}\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}\int_0^t (t-s)^{2\alpha-2}\,\left|\sigma\left(s,X_s^{n-1}\right)\right|^2 ds\right). \end{aligned}$$

From $(H_1)$ and $(H_2)$, we derive

$$|J(s,x)|^2\leq 2|J(s,x)-J(s,0)|^2+2|J(s,0)|^2\leq 2D\|x\|^2+2P.$$

It follows that

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[\sup_{t\in[0,T]}|X^n(t)|^2\right] \leq 4\|\Psi\|^2 \\
&+\frac{8T}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\left(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P\right)ds \\
&+\frac{8TC_1}{\Gamma(\alpha)}\int_0^T (t-s)^{2\alpha-2}\left(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P\right)ds \\
&+\frac{8C_2}{\Gamma(\alpha)}\int_0^T (t-s)^{2\alpha-2}\left(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P\right)ds \\
&\leq 4\|\Psi\|^2+\frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2} \\
&+\frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]ds.
\end{aligned}$$

Noting that the fact

$$\|X_s^{n-1}\|^2 \leq \sup_{u\in[-\tau,s]}|X^{n-1}(u)|^2 \leq \|\Psi\|^2+\sup_{u\in[0,s]}|X^{n-1}(u)|^2,$$

so, it follows that

$$\begin{aligned}
&\widehat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^n(t)|^2\right) \\
&\leq 4\|\Psi\|^2+\frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2} \\
&+\frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,\widehat{\mathbb{E}}\left[\|\Psi\|^2+\sup_{u\in[0,s]}|X^{n-1}(u)|^2\right]ds \\
&\leq r_2+r_3\int_0^T (t-s)^{(2\alpha-1)-1}\,\widehat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n-1}(u)|^2\right)ds,
\end{aligned}$$

where

$$r_2 = 4\|\Psi\|^2\left(1+\frac{2DT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2}\right)+\frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2},$$

and

$$r_3=\frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}.$$

On the other hand, for any $k \geq n$, we have

$$\max_{1\leq n\leq k}\widehat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^n(t)|^2\right) \leq r_2+r_3\int_0^T (t-s)^{2\alpha-2}\max_{1\leq n\leq k}\widehat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n-1}(u)|^2\right)ds.$$

Moreover,

$$\begin{aligned}\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n-1}(u)|^2\right) &\le \max\left\{\hat{\mathbb{E}}\|\Psi\|^2, \max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n}(u)|^2\right)\right\}\\ &\le \|\Psi\|^2+\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n}(u)|^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^{n}(t)|^2\right)\\ &\le r_2+r_3\int_0^T(t-s)^{2\alpha-2}\left(\|\Psi\|^2+\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n}(u)|^2\right)\right)ds\\ &\le r_4+r_3\int_0^T(t-s)^{2\alpha-2}\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{u\in[0,s]}|X^{n}(u)|^2\right)ds,\end{aligned}$$

where $r_4=r_2+r_3\frac{T^{2\alpha-1}}{(2\alpha-1)}\|\Psi\|^2$. Now by corollary 8, we get:

$$\max_{1\le n\le k}\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^{n}(t)|^2\right)\le r_4E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}).$$

We deduce that:

$$\hat{\mathbb{E}}\left(\sup_{t\in[0,T]}|X^{n}(t)|^2\right)\le r_4E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}). \tag{3.3}$$

*Step 2: Second, we prove that $(X^n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{M}_G^2([0,T])$.*

Let the space

$$H_T:=\left\{X\in\mathcal{M}_G^2([0,T]):\hat{\mathbb{E}}\left[\sup_{s\in[0,T]}|X(s)|^2\right]<\infty\right\},$$

equipped with the norm

$$N(X)=\left(\hat{\mathbb{E}}\left[\sup_{s\in[0,T]}|X(s)|^2\right]\right)^{\frac{1}{2}}.$$

It follows from (3.2) that:

$$\begin{aligned}X^1(t)-X^0(t) &=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,b(s,0)ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,h(s,0)d\langle B\rangle_s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,\sigma(s,0)dB_s.\end{aligned}$$

Similary to the proof of uniqueness, we have:

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}|X^1(t)-X^0(t)|^2\right] &\leq \frac{3}{\Gamma(\alpha)^2}\sup_{t\in[0,T]}\left|\int_0^t (t-s)^{\alpha-1}\, b(s,0)ds\right|^2 \\ &+\frac{3}{\Gamma(\alpha)^2}\left|\sup_{t\in[0,T]}\int_0^t (t-s)^{\alpha-1}\, h(s,0)d\langle B\rangle_s\right|^2 \\ &+\frac{3}{\Gamma(\alpha)^2}\sup_{t\in[0,T]}\left|\int_0^t (t-s)^{\alpha-1}\, \sigma(s,0)dB_s\right|^2 \\ &\leq \frac{3T}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,|b(s,0)|^2 ds \\ &+\frac{3TC_1}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,|h(s,0)|^2 ds \\ &+\frac{3C_2}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,|\sigma(s,0)|^2 ds.\end{aligned}$$

Then we get from $(H_2)$

$$\begin{aligned}&\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}|X^1(t)-X^0(t)|^2\right] \\ &\leq \frac{3TP}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,ds+\frac{3TC_1P}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,ds+\frac{3C_2P}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\,ds \\ &\leq \frac{3(T+TC_1+C_2)PT^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}.\end{aligned}$$

It follows that

$$N(X^1-X^0)\leq K, \qquad (3.4)$$

where $K=\left(\frac{3(T+TC_1+C_2)PT^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\right)^{\frac{1}{2}}$. Now we have for any arbitrary $n\geq 1$ and $t\in[0,T]$:

$$\begin{aligned}X^{n+1}(t)-X^n(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[b(s,X_s^n)-b(s,X_s^{n-1})]ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[h(s,X_s^n)-h(s,X_s^{n-1})]d\langle B\rangle_s \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,[\sigma(s,X_s^n)-\sigma(s,X_s^{n-1})]dB_s.\end{aligned}$$

With similarity to the proof of uniqueness, we get:

$$\begin{aligned}&\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}|X^{n+1}(t)-X^n(t)|^2\right] \\ &\leq \frac{3D(T+TC_1+C_2)T^{2\alpha-2}}{\Gamma(\alpha)^2}\int_0^T \hat{\mathbb{E}}\left(\sup_{0\leq t_2\leq t_1}|X^n(t_2)-X^{n-1}(t_2)|^2\right)dt_1.\end{aligned}$$

By seeting $\gamma=\frac{3D(T+TC_1+C_2)T^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}$,we can write

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{s\in[0,T]}|X^{n+1}(s)-X^n(s)|^2\right] &\le \gamma\int_0^T \hat{\mathbb{E}}\left(\sup_{0\le t_2\le t_1}|X^n(t_2)-X^{n-1}(t_2)|^2\right)dt_1\\ &\le \gamma^2\int_0^T\int_0^{t_1}\hat{\mathbb{E}}\left(\sup_{0\le t_3\le t_2}|X^{n-1}(t_3)-X^{n-2}(t_3)|^2\right)dt_1dt_2\\ &\le \gamma^n K\int_0^T\int_0^{t_1}\dots\int_0^{t_{n-1}}d\,t_1\dots dt_n\\ &\le K\frac{(\gamma T)^n}{n!}.\end{aligned}$$

It follows that

$$N(X^{n+1}-X^n)\le\left(K\frac{(\gamma T)^n}{n!}\right)^{\frac{1}{2}}. \qquad (3.5)$$

Let $m>n$,

$$\begin{aligned}N(X^m-X^n) &= N\left(\sum_{i=n+1}^{m}\left(X^i-X^{i-1}\right)\right)\\ &\le \sum_{i=n+1}^{m} N\left(X^i-X^{i-1}\right)\\ &\le \sum_{i>n} N\left(X^i-X^{i-1}\right)\\ &\le \sum_{i>n}\sqrt{K\frac{(\gamma T)^i}{i!}}.\end{aligned}$$

This implies that $(X^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_T$ and also in $\mathcal{M}_G^2([0,T])$. Let $X$ be the limit of this sequence.

*Step 3: We prove that for all $t\in[0,T]$, $X_t$ is the solution of the $G$ −*FSDE (1.1).

By using the continuity of the norm $N$, we deduce from equation (3.3) that

$$\hat{\mathbb{E}}\left(\sup_{s\in[0,T]}|X(s)|^2\right)\le r_4E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}), \qquad (3.6)$$

which implies that

$$\begin{aligned}\hat{\mathbb{E}}\left(\int_{-\tau}^T|X(s)|^2\,ds\right) &= \hat{\mathbb{E}}\left(\int_{-\tau}^0|X(s)|^2\,ds\right)+\hat{\mathbb{E}}\left(\int_0^T|X(s)|^2\,ds\right)\\ &\le \tau\|\Psi\|^2+r_4TE_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1})<\infty.\end{aligned}$$

Therefore, $X\in\mathcal{M}_G^2([-\tau,T])$.

According to the unicity of the limit, it suffices to prove that for each $t \in [0,T]$, the sequence of random variables $\left(K_n(t)\right)_n$ (resp. $\left(T_n(t)\right)_n, \left(V_n(t)\right)_n$) converges in $L_G^2(\Omega)$ to the random variable $\int_0^t (t-s)^{\alpha-1}\, b(s,X_s)ds$

(resp. $\int_0^t (t-s)^{\alpha-1}\, h(s,X_s)d\langle B\rangle_s, \int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s)dB_s$), where

$$K_n(t) = \int_0^t (t-s)^{\alpha-1}\, b(s,X_s^{n-1})ds$$

$$T_n(t) = \int_0^t (t-s)^{\alpha-1}\, h(s,X_s^{n-1})d\langle B\rangle_s,$$

and

$$V_n(t) = \int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s^{n-1})dB_s.$$

Indeed, we have by using Holder's inequality,

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\int_0^t (t-s)^{\alpha-1}\, b(s,X_s^n)ds - \int_0^t (t-s)^{\alpha-1}\, b(s,X_s)ds\right]^2 \\
&\leq T\int_0^T (t-s)^{2\alpha-2}\, \hat{\mathbb{E}}|b(s,X_s^n) - b(s,X_s)|^2 ds \\
&\leq DT^{2\alpha-1}\int_0^t \hat{\mathbb{E}}\left(\sup_{r\in[0,T]} |X_r^n - X_r|^2\right)ds,
\end{aligned}$$

then

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, b(s,X_s^n)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, b(s,X_s)ds\right]^2 \\
&\leq \frac{DT^{2\alpha}}{\Gamma(\alpha)^2}\left(N(X^n - X)\right)^2,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_0^t (t-s)^{\alpha-1}\, b(s,X_s^n)ds = \int_0^t (t-s)^{\alpha-1}\, b(s,X_s)ds \text{ in } L_G^2(\Omega).$$

Similarly, by using $G-BDG$ inequalities, we can write

$$\lim_{n\to\infty}\int_0^t (t-s)^{\alpha-1}\, h(s,X_s^n)d\langle B\rangle_s = \int_0^t (t-s)^{\alpha-1}\, h(s,X_s)d\langle B\rangle_s \text{ in } L_G^2(\Omega),$$

and

$$\lim_{n\to\infty}\int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s^n)dB_s = \int_0^t (t-s)^{\alpha-1}\, \sigma(s,X_s)dB_s \text{ in } L_G^2(\Omega).$$

The proof is complete.

**Corollary 11**. *Let X be the solution of* $G$ $-FSDE$ $(1.1)$. *Then we have*

$$\hat{\mathbb{E}}\left[\sup_{t\in[-\tau,T]}|X(t)|^2\right] \leq \|\Psi\|^2 + r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}).$$

□

*Proof.* Follows from the formula $(3.6)$ and the fact that $\sup_{t\in[-\tau,0]}|X(t)| = \|\Psi\|$. □

**Existence of stochastic optimal control**

We study stochastic control problems where the control domain need not be convex, the system is governed by fractional stochastic differential equation driven by $G-$ Brownian motion with delay. A control process $u^*$ that solves $\inf_{u\in U} J(u) = J(u^*)$ is called optimal, where the infimum is taken over all admissible controls. The exepected cost to be minimized over the class of admissible controls has the form

$$J(u) = \hat{\mathbb{E}}\left[\int_0^T \zeta\,(t, X_t^u, u_t)dt\right].$$

For each fixed control $u \in U$,we study the existence and uniqueness of solution of the system $(1.2)$.

**Definition 12**. We say that the process $(X_t^u)_{t\geq 0} \in \mathcal{M}_G^2([0,T])$ is called a solution of the equation $(1.2)$ for each fixed control $u \in U$ if $X_t^u$ satisfies for $t \in [0,T]$;

$$\begin{aligned} X_t^u \quad &= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, b(s, X_s, u_s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, h(s, X_s, u_s)d\langle B\rangle_s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, \sigma(s, X_s)dB_s \end{aligned} \qquad (4.1)$$

We consider the following assumptions:

$(H_3)$ Lipschitz condition:

- The functions $Q = b, h: [0,T] \times \mathbb{R} \times U \to \mathbb{R}$ are Lipschitz continuous with respect to $x$ uniformly with respect to $(t,u)$, $i.e.$

$$|Q(t,x,u) - Q(t,y,u)| \leq l|x-y| \text{ for each } x \in \mathbb{R},$$

- The function $\sigma: [0,T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with respect to $x$ uniformly with respect to $t$, $i.e.$

$$|\sigma(t,x) - \sigma(t,y)| \leq l|x-y| \text{ for each } x \in \mathbb{R},$$

$(H_4)$ Linear growth condition:

- The functions $Q$ are linearly growing with respect to $x$ uniformly with respect to $(t,\omega)$, $i.e.$

$$|Q(t,x)| \leq g(1+|x|^2) \text{ for each } x \in \mathbb{R},$$

-The functions $\sigma$ is linearly growing with respect to $x$ uniformly with respect to $(t,\omega)$, $i.e.$

$$|\sigma(t,x)| \leq g(1+|x|^2) \text{ for each } x \in \mathbb{R},$$

where $l$ and $g$ are positive constants

**Theorem 13**. *Let the assumption* $(H_3)$ *be satisfied. Then for each* $u \in U$, *the system* $(1.2)$ *admits a unique solution. The process* $(X_t^u)_{t\in[0,T]}$ *is the solution of the controlled system.*

*Proof.* Using the same reasoning as Theorem $(3.2)$, we can conclude that the system $(1.2)$ admits a unique solution. □

**Lemma 14**. *Assume that the hypothesis* $(H_4)$ *is satisfied. Let* $(X_t^u)_{t\geq 0}$ *be the solution of the* $G$ $-FSDE$ $(1.2)$ *with initial condition* $x_0 \in \mathbb{R}$. *Then we have,* $\sup_{0\leq t\leq T} \hat{\mathbb{E}}[|X_t^u|^2] \leq M$,

*Proof.* Using the same reasoning as Corollary $(3.3)$ we can easily obtain the boundedness of $X_t^u$. □

Now, we prove the existence of fractional optimal control. Note that, the problem of the existence of fractional optimal control for semilinear stochastic system has been studied in [19]. We consider the real cost functional $J$ defined over $U$ the set of admissible controls by $J(u) = \hat{\mathbb{E}}\left[\int_0^T \zeta\,(t, X_t, u_t)dt\right]$, we have the following hypothesis $(H_5)$:

- The integrand $\zeta$ is a real function, $\mathcal{F}_t -$ measurable, for each $t \in [0,T]$.

- $\zeta$ is continous in $X$ and $u$ for almost all $t \in [0,T]$.

- $\zeta$ is convex on the space of admissible controls $U$ for each $X$ and $t \in [0,T]$.

- There exists a constants $a, b \geq 0$, $\theta \in L_G^1([0,T],\mathbb{R})$ such that

$$\theta(t) + a\hat{\mathbb{E}}|X|^2 + b|u|_u^p \leq \zeta(t,X,u)$$

**Theorem 15**. *Let us assume that the hypothesis* $(H_3),(H_4)$ *and* $(H_5)$ *hold, then there esists an optimal control* $u^*$: $J(u^*) = \inf_{u\in U} J(u)$

*Proof.* We turn to the proof of existence of optimal control.

If $\inf\{J(u), u \in U\} = +\infty$, then there is nothing to prove. We assume that the

$\inf\{\,J(u), u \in U\} = \varepsilon < \infty$.

We have by the definition of the infimum, there is a minimizing sequence $(X^m, u^m)$ such that $J(X^m, u^m)$ converges to $\varepsilon$ as $m \to +\infty$, with $(X^m)$ is the sequence of solution of the controlled system $(1.2)$ corresponding to the sequence of admissible controls $(u^m)$.

Since $(u^m)_{m\geq 1} \in U$, is a bounded subset of the separable reflexive Banach space $L_G^p([0,T],A)$, there exists a subsequence, relabeled as $(u^m)$ and $u^* \in L_G^p([0,T],A)$ such that:

$u^m \to u^*$ weakly as $m \to +\infty$ in $L_G^p([0,T],A)$

Since $U$ is closed and convex, the Mazur lemma forces us to conclude that $u^* \in U$.

Let $(X^m)$ be the sequence of solutions of the system $(1.2)$ corresponding to $(u^m)$, by Lemma (4.3), it is easy to see that there exists $\delta \geq 0$ such that:

$$\widehat{\mathbb{E}}|X^m|^2 \leq \delta, m \geq 0.$$

We denote by $X^*$ the solution of the system $(1.2)$ corresponding to the control $u^* \in U$.

For all $t \in [0, T]$, using conditions $(H_3)$, $(H_4)$, Cauchy-Schwartz and Holder inequalities, we obtain that:

$$\widehat{\mathbb{E}}|X^m(t) - X^*(t)|^2$$

is bounded.

By the well-known singular version of Gronwall inequality, there exists a constant $K^*(\alpha)$ independent of $u, m$ and $t$ such that

$$\widehat{\mathbb{E}}|X^m(t) - X^*(t)|^2 \leq C\big(K^*(\alpha)\big). \qquad \text{(a1)}$$

Since $b$ is continuous, we get

$$\widehat{\mathbb{E}}|b(t, X^m, u^m) - b(t, X^*, u^*)|^2_{L^p_G([0,T],A)} \to 0 \; as \; m \to +\infty. \qquad \text{(a2)}$$

From $(a1)$ and $(a2)$ we conclude that

$$\widehat{\mathbb{E}}|X^m(t) - X^*(t)|^2 \to 0 \; as \; m \to +\infty$$

This implies that

$$\widehat{\mathbb{E}}|X^m(t) - X^*(t)|^2 \to 0 \; in \; C\big([0, T], L^2_G(\Omega, \mathbb{R})\big) \; as \; m \to +\infty$$

We can conclude that

$$(X, u) \to \widehat{\mathbb{E}}\left[\int_0^T \zeta\,(t, X^u_t, u_t)dt\right]$$

is sequentially continuous in the strong topology of $L^1([0, T], \mathbb{R})$ and weak topology of $L^p_G([0, T], A) \subset L^1_G([0, T], A)$. Hence, $J$ is weakly continuous on $L^p_G([0, T], A)$, and since by $J > -\infty$, $J$ attaints its infimum at $u^* \in U$, that is,

$$\varepsilon := \lim_{m \to +\infty} \widehat{\mathbb{E}}\left[\int_0^T \zeta\,(t, X^m_t, u^m_t)dt\right] \geq \widehat{\mathbb{E}}\left[\int_0^T \zeta\,(t, X^*_t, u^*_t)dt\right] = J(X^*, u^*) \geq \varepsilon.$$

This completes the proof. □

**Simulation**

In this section we would provide an example with simulation strengthen the impact of the theoretical findings by illustrating how the proposed fractional stochastic optimal control model behaves numerically.

**Example 17**. Consider the following $G$-FSDE:

$$D^\alpha_t X(t) = b(t, X_t)dt + h(t, X_t)d\langle B\rangle_t + \sigma(t, X_t)dB_t$$

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where

$$b(t,x) = \frac{8}{10}.\sin\left(\frac{x}{2}\right),\ \ h(t,x) = \frac{2}{10}\arctan\left(\frac{3x}{10}\right), \text{and } \sigma(t,x) = \frac{4x}{10(1+|x|)}$$

with the initial condition

$$\Psi(\theta) = 1 + \frac{3}{10}\sin(2\pi\theta),\ \ \theta \in [-\tau, 0]$$

In the simulation we set the parameters as follows:

$$\alpha = 0.7, \tau = 0.2, T = 2, \sigma \in \{0.3, 0.5, 0.7\}$$

By using python we obtain the following figures

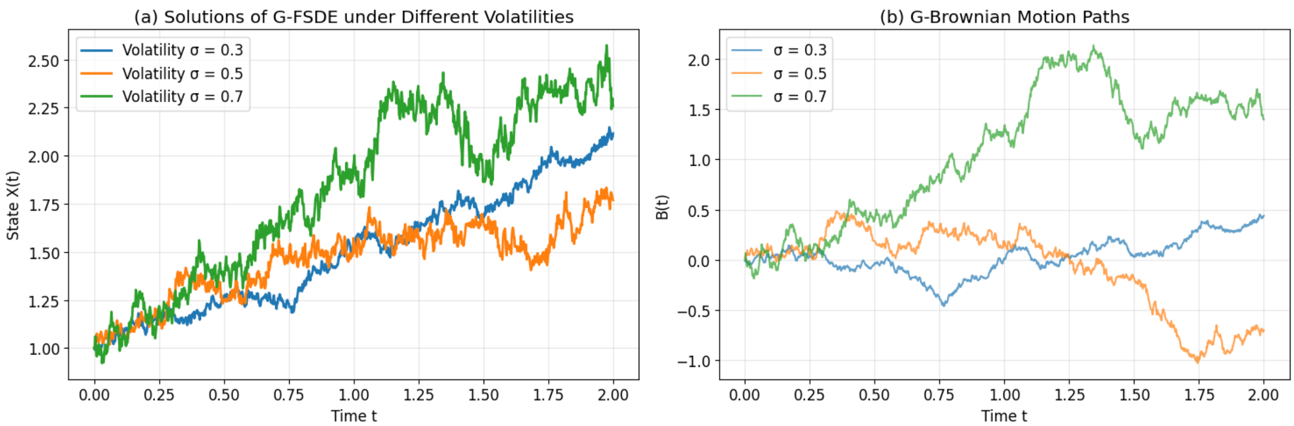


Figure 1: Solution of the G-FSDE under different volatilities

**Control Strategies**

Now, we simulate the controlled equation, we set the controlled coefficients as follows:

$$b(t,x,u) = \frac{8}{10}\sin\left(\frac{x}{2}\right) + \frac{3}{10}u,\ \ h(t,x,u) = \frac{2}{10}\arctan\left(\frac{3x}{10}\right) + \frac{1}{10}u$$

and the diffusion coefficient $\sigma$ is unchanged. The coefficients are carefully chosen so that they satisfy all theoretical assumptions. The cost functional is

$$\zeta(t,x,u) = \left(x - \frac{1}{2}\right)^2 + \frac{u^2}{10}$$

To demonstrate the existence of an optimal control, we evaluate several admissible control strategies $u(\cdot) \in U$. The following controls are considered:

$$\text{Strategy 1 (Zero Control):} \quad u_1(t) = 0 \tag{5.1}$$
$$\text{Strategy 2 :} \quad u_2(t) = 0.3 \tag{5.2}$$
$$\text{Strategy 3 (Constant Large) :} \quad u_3(t) = 0.8 \tag{5.3}$$
$$\text{Strategy 4 :} \quad u_4(t) = A\sin(\omega t), \quad A = 0.5,\ \omega = 2\pi \tag{5.4}$$
$$\text{Strategy 5 (Linear):} \quad u_5(t) = k(t - t_0), \quad k = 0.4,\ t_0 = 1.0 \tag{5.5}$$
$$\text{Strategy 6 :} \quad u_6(t) = u_0 e^{-\beta t}, \quad u_0 = 0.6,\ \beta = 2.0 \tag{5.6}$$

These strategies satisfy the admissibility conditions $u(\cdot) \in U$. We obtain the following results:

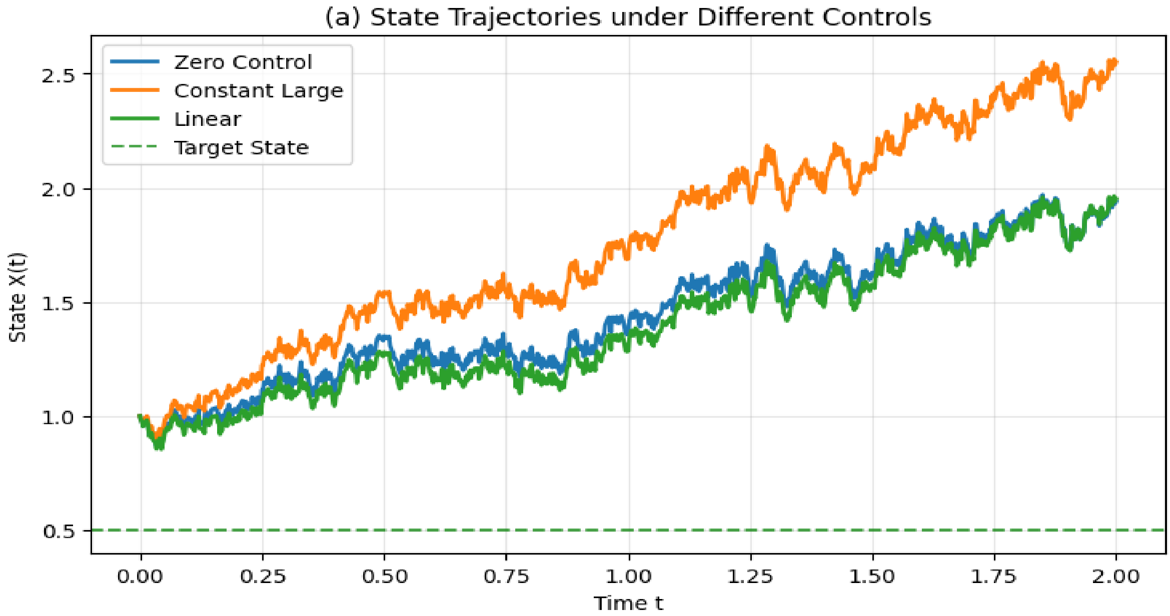


Figure 2.

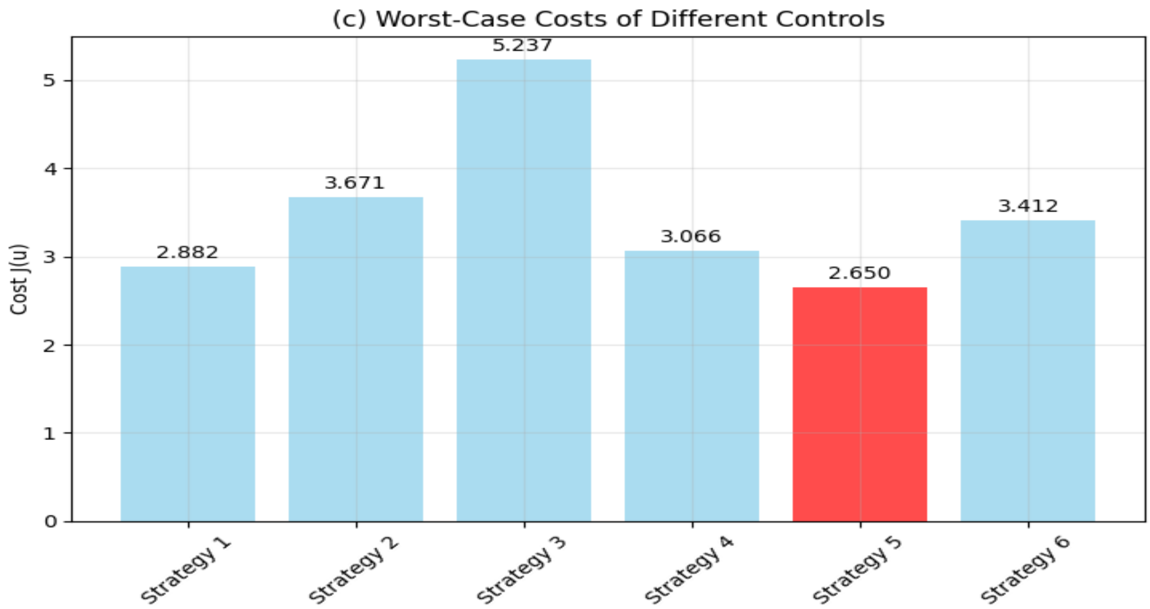


Figure 3.

*Remark 16*. The optimal control (red bar) achieves the minimum worst case cost of $J(u^*) = 2.650$ and one can see that the zero control strategy (first bar) does not depend on the highest cost which indicates that some control strategies can act worse than no control. The best strategy control here is the linear control.

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