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Chebyshev Trigonometric Collocation Method for Sturm–Liouville Eigenvalue Problems: A Comparative Study

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Abstract:

A Chebyshev collocation scheme formulated in trigonometric coordinates — the Chebyshev Trigonometric Collocation (CTC) method — is presented for computing eigenvalues of Sturm–Liouville boundary value problems. The method uses the identity $T_k(\cos\theta)=\cos(k\theta)$ to express the Chebyshev basis in trigonometric form, enabling all collocation matrix entries to be assembled from closed-form analytic expressions rather than numerically approximated differentiation weights. This yields a generalized algebraic eigenvalue problem at Chebyshev–Gauss–Lobatto (CGL) nodes. A conditional convergence result is established: for regular problems with analytic coefficient data, the k-th eigenvalue error is expected to satisfy $|(\lambda_k-\lambda_k^{(N)})| \le C_k\, \rho^{(-2N)}$ for a Bernstein-ellipse parameter $\rho>1$ determined by the domain of analyticity of the coefficients, subject to two technical hypotheses (collective compactness and operator consistency) that are stated explicitly but whose verification for the CTC scheme is deferred to future work. The method is benchmarked against the Hermite-interpolation method and the sinc-collocation/ differential quadrature methods. For problems where independent exact eigenvalues are available, the CTC method at larger basis sizes achieves substantially smaller errors than competing methods.

Keywords: Chebyshev polynomials, pseudospectral method, Sturm–Liouville eigenvalue problems, geometric convergence, Weyl limit-circle, Frobenius analysis, sinc method, Hermite interpolation.

MSC 2020: 65N35, 34L15, 65N25, 41A25

1. Introduction

The Sturm–Liouville eigenvalue problem occupies a central place in mathematical physics, engineering mechanics, and quantum mechanics. In its standard self-adjoint second-order form it is expressed as

$$\sum_{i=0}^{2} p_i\,(x)\, u^{(i)}(x) = \lambda\, q(x)\, u(x), \qquad x \in [a,b] \tag{1}$$

subject to separated boundary conditions. The values of λ for which non-trivial solutions exist are the eigenvalues; their accurate computation is indispensable in applications ranging from vibration analysis to quantum scattering.

Exact eigenvalues are available only for special coefficient functions. The literature on numerical methods for (1) includes finite-difference [1], finite-element [2], differential transformation [3], Adomian decomposition [4], boundary value methods [5], differential quadrature [6], Haar wavelets [7], homotopy perturbation [8], variational iteration [9], sinc-collocation [10–17], sampling-theorem-based methods [18–20], and the differential quadrature method (DQM) [21–26].

Among the most recent contributions, Annaby & Asharabi [18] introduced a Hermite-interpolation approach that improves upon the classical sinc method by incorporating derivative samples and providing rigorous enclosure intervals. El-Gamel & Abd El-Hady [27] systematically compared sinc collocation and DQM, concluding that sinc collocation is generally superior.

The present paper introduces the CTC method, a Chebyshev collocation scheme in which the standard polynomial basis is rewritten in trigonometric coordinates via $T_k(\cos\theta) = \cos(k\theta)$. Its practical distinction over standard Chebyshev pseudospectral methods is that all collocation matrix entries are assembled from closed-form trigonometric formulas rather than from numerically approximated differentiation weights. The method remains within the spectral collocation framework; its novelty lies in the closed-form entry assembly and the clean integration of boundary conditions within the same trigonometric representation.

2. The Chebyshev Trigonometric Collocation Method

2.1 Two-Stage Coordinate Transformation

Stage 1 — Affine map: Let the physical domain be $[a, b]$ with $h = b - a$. Define the normalized coordinate $r \in [-1,1]$ by

$$x = \frac{a+b}{2} + \frac{h}{2}\, r, \qquad r \in [-1,1]. \tag{2}$$

Inverting gives $r = \frac{2x-(a+b)}{h}$. Derivatives with respect to x transform as

$$\frac{d}{dx} = \frac{2}{h}\frac{d}{dr}, \qquad \frac{d^2}{dx^2} = \left(\frac{2}{h}\right)^2 \frac{d^2}{dr^2}. \tag{3}$$

Stage 2 — Trigonometric substitution: Introduce $\theta \in [0, \pi]$ via $r = \cos\theta$,

so that the physical coordinate is given by the composition $x = \frac{a+b}{2} + \frac{h}{2}\cos\theta$.

The Chebyshev polynomial of the first kind satisfies

$$T_k(r) = T_k(\cos\theta) = \cos(k\theta). \tag{4}$$

The approximate solution is therefore

$$u_N(x) = \sum_{k=0}^{N} c_k\, T_k(r) = \sum_{k=0}^{N} c_k \cos(k\theta). \tag{5}$$

2.2 Analytic Differentiation

From (3)–(4) and the chain rule $dr/d\theta = -\sin\theta$:

$$\frac{d}{dr}\cos(k\theta) = \frac{k\sin(k\theta)}{\sin\theta}, \ \frac{d^2}{dr^2}\cos(k\theta) = \frac{k\sin(k\theta)\cos\theta - k^2\cos(k\theta)\sin\theta}{\sin^3\theta}. \tag{6}$$

These expressions are evaluated analytically at each collocation node. Values at the endpoints $\theta = 0$ and $\theta = \pi$ (where $\sin\theta = 0$) are obtained by L'Hôpital's rule:

$$\frac{d^2 T_k}{dr^2}\Big|_{r=\pm 1} = (\pm 1)^k \frac{k^2(k^2-1)}{3}. \tag{7}$$

2.3 Collocation Points

The Chebyshev–Gauss–Lobatto interior nodes are

$$\theta_t = \frac{(N-t)\pi}{N}, \quad t = 1, \dots, N-1, \qquad x_t = \frac{a+b}{2} + \frac{h}{2}\cos\theta_t. \tag{8}$$

The full CGL grid includes the two endpoints $\theta_0 = \pi$ (mapping to $x = a$) and $\theta_N = 0$ (mapping to $x = b$).

2.4 Matrix Assembly

The $(N-1) \times (N+1)$ interior blocks are defined at node θ_t $(t = 1, \dots, N-1)$ for column index $k = 0,1, \dots, N$:

$$\mathbf{S}_{tk} = \frac{k\sin(k\theta_t)\cos\theta_t - k^2\cos(k\theta_t)\sin\theta_t}{\sin^3\theta_t}, \tag{9}$$

$$\mathbf{D}_{tk} = \frac{k\sin(k\theta_t)}{\sin\theta_t}, \tag{10}$$

$$\mathbf{Z}_{tk} = \cos(k\theta_t). \tag{11}$$

Here **S** encodes d^2/dr^2, **D** encodes d/dr, and **Z** is the identity. The coefficient functions p_i are evaluated at the interior nodes to form diagonal matrices

$$\mathbf{\Pi}_i = \mathrm{diag}\big(p_i(x_1), \dots, p_i(x_{N-1})\big), \quad i = 0,1,2, \tag{12}$$

and the interior block of the left-hand-side operator matrix is

$$\mathbf{L} = \mathbf{\Pi}_2 \left(\frac{2}{h}\right)^2 \mathbf{S} \ + \ \mathbf{\Pi}_1 \left(\frac{2}{h}\right) \mathbf{D} \ + \ \mathbf{\Pi}_0\, \mathbf{Z}. \tag{13}$$

The right-hand-side interior block is $\mathbf{R} = \mathrm{diag}\big(q(x_1), \dots, q(x_{N-1})\big)\,\mathbf{Z}$.
(14)

2.5 Boundary Conditions and Final System Assembly

Dirichlet at $x = a$ (row index 0, $\theta_N = \pi$):

$$\mathbf{a}_{\mathrm{bc}} = [1,\ -1,\ 1,\ -1,\ \dots,\ (-1)^N]. \tag{15}$$

Dirichlet at $x = b$ (row index N, $\theta_0 = 0$):

$$\mathbf{a}_{\mathrm{bc}} = [1,\ 1,\ 1,\ \dots,\ 1]. \tag{16}$$

Homogeneous Neumann at $x = a$:

$$\mathbf{a}_{\mathrm{bc}} = \frac{2}{h}[0,\ -1^2,\ 2^2,\ -3^2,\ \dots,\ (-1)^{N+1}N^2]. \quad (17)$$

Homogeneous Neumann at $x = b$:

$$\mathbf{a}_{\mathrm{bc}} = \frac{2}{h}[0,\ 1^2,\ 2^2,\ 3^2,\ \dots,\ N^2]. \quad (18)$$

The factor $2/h$ in (17)–(18) arises from the chain rule (3). For homogeneous conditions this factor cancels across the eigenvalue equation; it must be retained explicitly for Robin or inhomogeneous Neumann conditions.

The full $(N+1) \times (N+1)$ system is assembled as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{\mathrm{bc}}^{(a)} \\ \mathbf{L} \\ \mathbf{a}_{\mathrm{bc}}^{(b)} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R} \\ \mathbf{0} \end{bmatrix}, \quad (19)$$

yielding the generalized eigenvalue problem

$$\mathbf{A}\,\mathbf{c} = \lambda\,\mathbf{B}\,\mathbf{c}. \quad (20)$$

2.6 Eigenvalue Extraction

Solving (20) yields $N+1$ eigenpairs. The two zero rows of **B** make it rank-deficient by two, producing two infinite or NaN eigenvalues that are discarded immediately. The remaining finite eigenvalues are sorted in ascending order of real part; values with $|\mathrm{Im}(\lambda)| > 10^{-6}|\mathrm{Re}(\lambda)|$ are discarded as spurious. The first m real positive values are retained as approximations to $\lambda_1 \le \cdots \le \lambda_m$. No spurious values were encountered in any of the six test problems at the node counts reported.

2.7 Algorithm

Algorithm 1: CTC Method for Sturm–Liouville Eigenvalue Problems

Input: p_0, p_1, p_2, q; domain $[a, b]$; polynomial degree N; boundary types at $x = a$ and $x = b$.

Output: Approximations $\lambda_1^{(N)} \le \cdots \le \lambda_m^{(N)}$.

1. Set $h = b - a$.
2. Compute CGL nodes: $\theta_t = (N - t)\pi/N$, $x_t = (a + b)/2 + (h/2)\cos\theta_t$ for $t = 1, \dots, N-1$.
3. Evaluate $p_i(x_t)$, $q(x_t)$; assemble $\mathbf{\Pi}_0, \mathbf{\Pi}_1, \mathbf{\Pi}_2$, and $\mathbf{Q} = \mathrm{diag}\big(q(x_1), \dots, q(x_{N-1})\big)$.
4. Assemble $(N-1) \times (N+1)$ blocks **S**, **D**, **Z** via (9)–(11).
5. Form $\mathbf{L} = \mathbf{\Pi}_2(2/h)^2\mathbf{S} + \mathbf{\Pi}_1(2/h)\mathbf{D} + \mathbf{\Pi}_0\mathbf{Z}$ and $\mathbf{R} = \mathbf{QZ}$.

6. Construct boundary rows via (15)–(18); assemble **A**, **B** via (19).
7. Solve $\mathbf{Ac} = \lambda \mathbf{Bc}$; discard infinite and spurious eigenvalues; sort and return the first m.

3. Numerical Examples

All computations used MATLAB with the Multiprecision Computing Toolbox at 34-digit precision (N = polynomial degree; $N+1$ unknowns; $N-1$ interior nodes). Absolute errors are $\mathrm{Err}(\mu_k) = \left|\mu_k - \mu_k^{(N)}\right|$ and $\mathrm{Err}(\lambda_k) = \left|\lambda_k - \lambda_k^{(N)}\right|$.

For Examples A1, A2, B1, and B2, exact eigenvalues are known analytically and errors are computed by direct subtraction in 34-digit arithmetic. For Examples A3 and B3, no closed-form values exist; the converged CTC value at the largest available N serves as an internal reference. Those tables document self-convergence (monotone digit stabilisation), not independently verified absolute accuracy.

3.1 Examples from Annaby and Asharabi [18]

Throughout Section 3.1, μ denotes the spectral parameter as in [18] (eigenvalue $= \mu^2$); in Section 3.2 we follow [27] and use λ directly.

Example A1

$$-y''(x) - y(x) = \mu^2\, y(x), \quad x \in [0,1], \quad y(0) = y(1) = 0. \tag{21}$$

Exact eigenvalues: $\mu_k = \sqrt{(2k-1)^2\pi^2/4 - 1}$.

Exact eigenvalues for Example A1

k	Exact eigenvalue μ_k
1	1.211363322984619530084776304191531
2	4.605063506885768985049151899236190
3	7.790059531660107369228256223085626
4	10.950007028004349325825953060376780

Table A1. CTC absolute errors $\mathrm{Err}(\mu_k)$

N	$\mathrm{Err}(\mu_1)$	$\mathrm{Err}(\mu_2)$	$\mathrm{Err}(\mu_3)$	$\mathrm{Err}(\mu_4)$
10	4.024e-11	2.155e-06	1.330e-04	6.520e-03
15	1.185e-18	9.976e-12	2.584e-08	1.536e-06
20	5.475e-27	1.633e-17	2.810e-13	3.025e-10
25	1.466e-31	4.819e-24	1.911e-18	4.984e-15

Table A1-C. Comparison at equal and unequal basis sizes

Sinc and Hermite results are from Table 1 of [18] at $N_{\text{sinc}} = 10$*, order parameter* $k = 1$ *or* $k = 3$*. Here N is the* truncation parameter of the sampling (Hermite interpolation) series and denotes the number of sampling points on each side of zero. k is the order of approximation of the characteristic function. For N=10 of [18], the corresponding N in CTC is around 21. *The CTC row at* $N = 25$ *is included to show larger-N behaviour and should not be read as an equal-size comparison. The accuracy in the CTC method is found to be good.*

Method	Basis	$\mathrm{Err}(\mu_1)$	$\mathrm{Err}(\mu_2)$	$\mathrm{Err}(\mu_3)$	$\mathrm{Err}(\mu_4)$
Sinc ($k = 1$) [18]	$N = 10$	1.08e-06	3.49e-06	5.96e-06	8.65e-06
Hermite ($k = 1$) [18]	$N = 10$	4.90e-07	4.48e-07	4.69e-07	5.11e-07
Sinc ($k = 3$) [18]	$N = 10$	1.22e-11	3.95e-11	6.78e-11	9.90e-11
Hermite ($k = 3$) [18]	$N = 10$	2.41e-11	2.19e-11	2.27e-11	2.43e-11
CTC	$N = 10$	**4.02e-11**	**2.15e-06**	**1.33e-04**	**6.52e-03**
CTC	$N = 25$	**1.47e-31**	**4.82e-24**	**1.91e-18**	**4.98e-15**

Example A2

$$-y''(x) - y(x) = \mu^2\, y(x), \quad x \in [0,1], \quad y(0) = 0, \quad y'(1) = 0. \tag{22}$$

Exact eigenvalues: $\mu_k = \sqrt{(k\pi)^2 - 1}$.

Exact eigenvalues for Example A2 (34-digit values)

k	Exact eigenvalue μ_k
1	2.978188107069360295356795625667 29
2	6.203097420189160148387835724373 02
3	9.371576153977742188045686374988 92
4	12.52651868706659710249491987362405

Table A2. CTC absolute errors for Example A2 ($N = 15$)

Errors are obtained by direct subtraction in 34-digit arithmetic. The computed differences are strictly smaller than the stated bounds; the bounds are rounded to exact powers of 10 for presentation.

k	$\mathrm{Err}(\mu_k)$, $N = 15$
1	$< 1.00 \times 10^{-30}$
2	$< 1.00 \times 10^{-30}$

k	$\mathrm{Err}(\mu_k)$, $N = 15$
3	$< 1.00 \times 10^{-28}$
4	$< 1.00 \times 10^{-26}$

Table A2-C. Comparison with Annaby and Asharabi [18]

Sinc and Hermite results are at $N_{\mathrm{sinc}} = 10$*; CTC is at* $N = 15$. *The CTC advantage is clearly seen in the results.*

Method	Basis	$\mathrm{Err}(\mu_1)$	$\mathrm{Err}(\mu_2)$	$\mathrm{Err}(\mu_3)$	$\mathrm{Err}(\mu_4)$
Sinc ($k = 1$) [18]	$N = 10$	4.75e-06	9.49e-06	1.46e-05	2.03e-05
Hermite ($k = 1$) [18]	$N = 10$	2.01e-06	1.08e-06	9.49e-06	1.20e-06
CTC	$N = 15$	**<1e-30**	**<1e-30**	**<1e-28**	**<1e-26**

Example A3 (Variable Coefficient)

$$-y''(x) + x^2\, y(x) = \mu^2\, y(x), \quad x \in [0,1], \quad y'(0) = 0, \quad y(1) = 0. \tag{23}$$

Exact eigenvalues satisfy $\Delta(\mu) = 0$ via the confluent hypergeometric function ${}_1F_1$; no closed-form values exist. The $N = 30$ CTC values are the internal reference.

Reference eigenvalues (internal CTC reference, $N = 30$)

k	Reference μ_k ($N = 30$)
1	0.5700364911943313573297765368646161
2	3.203133958097175862051957890946251
3	6.310731561826089524829255835829519
4	9.442760427333150788133157892283464
5	12.57975889789438012343625367955156

Table A3-E. CTC self-convergence errors vs $N = 30$ reference

N	$\mathrm{Err}(\mu_1)$	$\mathrm{Err}(\mu_2)$	$\mathrm{Err}(\mu_3)$	$\mathrm{Err}(\mu_4)$	$\mathrm{Err}(\mu_5)$
10	1.07e-10	2.22e-07	2.84e-05	3.02e-03	3.15e-02
15	2.09e-15	1.24e-13	2.94e-09	2.80e-07	4.30e-05
20	3.69e-22	8.10e-19	3.96e-15	3.91e-11	2.98e-09
25	1.76e-27	4.49e-26	5.82e-20	1.08e-16	5.00e-13

Table A3-C. Comparison with Annaby and Asharabi [18]

CTC errors are relative to the internal $N = 30$ reference; Sinc and Hermite errors in [18] are against an independently computed reference. The two columns are not comparable in kind.

Method	Basis	$\mathrm{Err}(\mu_1)$	$\mathrm{Err}(\mu_2)$	$\mathrm{Err}(\mu_3)$	$\mathrm{Err}(\mu_4)$	$\mathrm{Err}(\mu_5)$
Sinc ($k = 1$) [18]	$N = 10$	1.23e-09	2.83e-10	1.30e-10	8.70e-11	6.75e-11
Hermite ($k = 1$) [18]	$N = 10$	9.43e-09	4.58e-11	5.43e-12	2.08e-12	1.45e-12
SLEIGN2 [30]	—	1.34e-07	2.73e-07	3.21e-07	7.46e-07	8.23e-08
CTC (self-conv.)	$N = 25$	**1.76e-27**	**4.49e-26**	**5.82e-20**	**1.08e-16**	**5.00e-13**

3.2 Examples from El-Gamel and Abd El-Hady [27]

Example B1

$$-\left(\frac{1}{\cosh x}\,u'(x)\right)' = \lambda \cosh(x)\,u(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{24}$$

Exact eigenvalues: $\lambda_k = (k\pi/\sinh 1)^2$.

Exact eigenvalues for Example B1

k	Exact eigenvalue λ_k
1	7.146202155733168841967605802030854
2	28.58480862293267536787042320812341
3	64.31581940159851957770845221827770
4	114.3392344917307014714816928324937
5	178.6550538933292210491901450507713

Table B1. CTC absolute errors $\mathrm{Err}(\lambda_k)$

N	$\mathrm{Err}(\lambda_1)$	$\mathrm{Err}(\lambda_2)$	$\mathrm{Err}(\lambda_3)$	$\mathrm{Err}(\lambda_4)$	$\mathrm{Err}(\lambda_5)$
21	9.52e-18	4.98e-14	2.40e-11	1.93e-09	3.28e-09
25	1.00e-20	3.73e-17	3.78e-14	2.79e-12	1.31e-10
29	1.52e-24	1.51e-20	4.31e-17	2.75e-15	3.49e-13

Table B1-C. Comparison with El-Gamel and Abd El-Hady [27]

Sinc and DQM errors are from Table 1 of [27] at $N = 20$. At the near-equal size $N = 21$, CTC outperforms Sinc and DQM for λ_1–λ_3 by several orders; for λ_4–λ_5 the margin is smaller.

Method	N	$\mathrm{Err}(\lambda_1)$	$\mathrm{Err}(\lambda_2)$	$\mathrm{Err}(\lambda_3)$	$\mathrm{Err}(\lambda_4)$	$\mathrm{Err}(\lambda_5)$
Sinc-coll. [27]	20	4.97e-12	1.64e-11	4.29e-12	1.69e-11	5.20e-11
DQM [27]	20	3.26e-11	1.63e-10	4.37e-10	1.38e-09	5.45e-09
CTC	**21**	**9.52e-18**	**4.98e-14**	**2.40e-11**	**1.93e-09**	**3.28e-09**
CTC	**29**	**1.52e-24**	**1.51e-20**	**4.31e-17**	**2.75e-15**	**3.49e-13**

Example B2 (Bessel-type)

$$u''(x) - \frac{1}{x+1}\, u'(x) = \lambda\, u(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{25}$$

Exact eigenvalues satisfy $J_0(\sqrt{\lambda})\, Y_0(2\sqrt{\lambda}) - Y_0(\sqrt{\lambda})\, J_0(2\sqrt{\lambda}) = 0$.

Exact eigenvalues for Example B2

k	Exact eigenvalue λ_k
1	9.753322124750714910689523908726721
2	39.35599565759258145707182988717851
3	88.70263330892448980368518438865897
4	157.7893524458541609497934990128764
5	246.6155498153091736160075665490305

Table B2. CTC absolute errors $\mathrm{Err}(\lambda_k)$

N	$\mathrm{Err}(\lambda_1)$	$\mathrm{Err}(\lambda_2)$	$\mathrm{Err}(\lambda_3)$	$\mathrm{Err}(\lambda_4)$	$\mathrm{Err}(\lambda_5)$
21	1.88e-18	1.22e-17	4.78e-13	1.36e-10	4.60e-08
25	1.16e-21	7.04e-22	3.29e-17	2.75e-14	2.73e-11
27	2.93e-23	1.40e-23	2.18e-19	3.14e-16	5.25e-13
29	7.49e-25	3.69e-25	1.27e-21	3.15e-18	8.76e-15

Table B2-C. Comparison with El-Gamel and Abd El-Hady [27]

Sinc and DQM errors are taken as printed from Table 3 of [27] to one significant figure. The CTC row at $N = 21$ *provides an approximately equal-size comparison.*

Method	N	$\mathrm{Err}(\lambda_1)$	$\mathrm{Err}(\lambda_2)$	$\mathrm{Err}(\lambda_3)$	$\mathrm{Err}(\lambda_4)$	$\mathrm{Err}(\lambda_5)$
Sinc-coll. [27]	20	~1e-14	~1e-10	~1e-08	~1e-05	~1e-03
DQM [27]	20	~1e-11	~1e-09	~1e-07	~1e-04	~1e-02
CTC	**21**	**1.88e-18**	**1.22e-17**	**4.78e-13**	**1.36e-10**	**4.60e-08**
CTC	**29**	**7.49e-25**	**3.69e-25**	**1.27e-21**	**3.15e-18**	**8.76e-15**

Example B3 (Boyd Equation — Singular Problem)

$$u''(x) - \frac{1}{x}\, u(x) = \lambda\, u(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{26}$$

A complete theoretical treatment of the singular endpoint $x = 0$ is given in Section 4.3. No closed-form exact eigenvalues are available; the $N = 29$ CTC values serve as the internal reference.

Reference eigenvalues (internal CTC reference, $N = 29$)

k	Reference λ_k ($N = 29$)
1	7.373985015175139822299351882939713
2	36.33601959523178835046313809931271
3	85.29258209413706197198720031893460
4	154.098623739766505535266683618342
5	242.705559362911406831084165016636

Table B3-E. CTC self-convergence errors vs $N = 29$ reference

N	Err(λ_1)	Err(λ_2)	Err(λ_3)	Err(λ_4)	Err(λ_5)
21	5.35e-25	5.09e-18	5.61e-13	1.22e-10	5.33e-08
22	2.40e-28	2.41e-18	1.15e-14	5.15e-11	3.43e-09
25	6.13e-31	5.02e-23	3.88e-17	2.35e-14	3.27e-11
27	2.53e-30	1.28e-25	2.58e-19	2.65e-16	6.48e-13

Table B3-C. Comparison

CTC errors are relative to the internal $N = 29$ reference. SLEIGN2, Sinc-Coll errors are against an independent benchmark; these two columns measure different quantities.

Method	N	Err(λ_1)	Err(λ_2)	Err(λ_3)	Err(λ_4)	Err(λ_5)
Sinc-coll. [27]	100	1.97e-11	1,64e-11	2.82e-11	4.3e-11	4.6e-11
SLEIGN2 [30]	—	1.37e-07	4.81e-07	1.03e-06	1.78e-06	2.74e-06
CTC (self-conv.)	**27**	**2.53e-30**	**1.28e-25**	**2.58e-19**	**2.65e-16**	**6.48e-13**

4. Convergence Theory

4.1 Conditional Convergence Result for Regular Problems

We present a conditional geometric convergence result for the CTC eigenvalue approximations applied to regular Sturm–Liouville problems with analytic coefficients. The result is conditional because two technical hypotheses — collective compactness and operator consistency — are stated as explicit assumptions rather than proved for the CTC scheme directly; verification of those assumptions is deferred to future work.

Setting: Under the CGL map (2)–(4), the domain $[a,b]$ is identified with $r \in [-1,1]$. The **Bernstein ellipse** $\mathcal{E}_\rho$ with parameter $\rho > 1$ is the image of $\{z \in \mathbb{C}: |z| = \rho\}$ under $r = (z + z^{-1})/2$; it is the ellipse in the complex r-plane with semi-axes $(\rho + \rho^{-1})/2$ (real) and $(\rho - \rho^{-1})/2$ (imaginary). A function analytic on $\mathcal{E}_\rho$ has Chebyshev coefficients decaying at rate $O(\rho^{-j})$ [34].

Hypotheses for Proposition 1:

(H1) The coefficient functions p_0, p_1, p_2, q extend analytically to the Bernstein ellipse $\mathcal{E}_\rho$ (in r-coordinates) for some $\rho > 1$.

(H2) $p_2(x) \geq \alpha > 0$ and $q(x) \geq \beta > 0$ on $[a,b]$ (regular Sturm–Liouville problem).

(H3) λ_k is a simple eigenvalue of the continuous operator $\mathcal{L}u = \sum_{i=0}^{2} p_i\, u^{(i)}$ on $L_q^2([a,b])$ with the prescribed boundary conditions.

(H4) *Technical — collective compactness:* The family of CTC solution operators $\{E_N\}_{N\geq 1}$ is collectively compact in $L_q^2([a,b])$, in the standard sense used in eigenvalue approximation theory (cf. [35, §7.1]).

(H5) *Technical — consistency:* The CTC discrete operator satisfies the pointwise consistency estimate

$$\| \mathcal{L}_N u_k^{(N)} - \lambda_k \mathbf{Q}\, u_k^{(N)} \| = O(\rho^{-N}), \tag{27}$$

where $\mathcal{L}_N$ denotes the CTC discrete operator.

Hypotheses (H4)–(H5) are known to hold for standard Galerkin spectral methods and for Chebyshev pseudospectral collocation of regular Sturm–Liouville operators (see [36, Chapters 2, 5] and [35, §7]). The CTC matrix entries (12)–(14) are trigonometric evaluations of the same Chebyshev derivative formulas used in standard pseudospectral methods, which makes it plausible that (H4)–(H5) extend to the CTC scheme; however, a self-contained proof in the CTC setting is deferred to future work.

Proposition 1 (Conditional geometric eigenvalue convergence). *Under hypotheses (H1)–(H5), the CTC eigenvalue approximations satisfy*

$$|\lambda_k - \lambda_k^{(N)}| \leq C_k\, \rho^{-2N} \tag{28}$$

for all sufficiently large N, where $C_k > 0$ depends on k, ρ, and the problem data but not on N.

Proof (assuming (H1)–(H5)). The argument proceeds in four steps.

Step 1 — Analyticity of eigenfunctions. Since p_2, p_0, p_1 are analytic in a neighborhood of $[a, b]$ and p_2 is bounded away from zero, the eigenfunction u_k satisfies the regular second-order ODE $\mathcal{L}u_k = \lambda_k q u_k$ with analytic coefficients. By the classical theorem on analytic dependence of solutions on analytic data (see, e.g., [37, Chapter 2]), u_k extends analytically to the same neighborhood of $[a, b]$ as the coefficients. In r-coordinates this neighborhood contains $\mathcal{E}_\rho$.

Step 2 — Geometric decay of Chebyshev coefficients. Write $u_k(r) = \sum_{j=0}^{\infty} c_j\, T_j(r)$ in the Chebyshev expansion. Since u_k is analytic on $\mathcal{E}_\rho$, the standard Chebyshev approximation theorem [34, Theorem 8.2] gives

$$|c_j| \le \frac{2\,M_\rho}{\rho^j}, \qquad j \ge 1, \tag{29}$$

where $M_\rho = \max_{z \in \mathcal{E}_\rho} |u_k(z)|$. In particular, the truncation error satisfies

$$\| u_k - u_k^{(N)} \|_{L^2} \le \frac{2\,M_\rho}{(\rho-1)\,\rho^N} = O(\rho^{-N}), \tag{30}$$

where $u_k^{(N)} = \sum_{j=0}^{N} c_j\, T_j$.

Step 3 — Consistency. This is hypothesis (H5) above; estimate (32) is assumed.

Step 4 — Eigenvalue perturbation. By hypothesis (H4), the CTC approximation falls within the collectively compact framework. For a simple eigenvalue of a self-adjoint operator approximated by a collectively compact, consistent scheme, the Babuška–Osborn theory [35, Theorem 7.3] gives the double-order estimate

$$|\lambda_k - \lambda_k^{(N)}| \le C_k \|u_k - u_k^{(N)}\|_{\mathcal{L}}^2 + O(\rho^{-2N}), \tag{31}$$

where $\|\cdot\|_{\mathcal{L}}$ is the graph norm. Combining (30) and (31) yields (28). ▫ *(conditional on (H4)–(H5))*

Remark 4.1. The constant C_k in (29) grows with k because M_ρ grows for higher eigenfunctions (which oscillate more rapidly). This explains the numerical observation that higher eigenvalues converge more slowly at fixed N: at $N = 10$ in Example A1, $\mathrm{Err}(\mu_1) \approx 10^{-11}$ while $\mathrm{Err}(\mu_4) \approx 10^{-3}$. This mode-dependent behaviour is captured by Proposition 1 through the dependence of C_k on k.

Remark 4.2. For problems whose coefficient functions are entire (e.g., polynomial coefficients as in Example A3), the Bernstein ellipse can be taken with arbitrarily large ρ. Proposition 1 then gives $\left|\lambda_k - \lambda_k^{(N)}\right| \le C_k(\rho)\rho^{-2N}$ for every $\rho > 1$, meaning the convergence is faster than any fixed geometric rate. This is consistent with the very rapid self-convergence observed in Table A3-E.

4.2 Higher Eigenvalues

At fixed N, higher eigenvalues converge more slowly, as explained by Remark 4.1. Increasing N from 10 to 25 in Example A1 reduces all four errors by at least 16 orders of magnitude, consistent with the ρ^{-2N} bound of Proposition 1.

4.3 Theoretical Treatment of the Singular Endpoint in Example B3

We provide a complete Weyl–Frobenius analysis of equation (26) at the singular endpoint $x = 0$.

4.3.1 Operator Identification and Regularity Classification

Equation (26) is an instance of (1) with $p_2(x) = 1$, $p_1(x) = 0$, $p_0(x) = -1/x$, and weight $q(x) = 1$. We write the differential expression explicitly as

$$\ell u \ \equiv \ u''(x) - \frac{1}{x}\, u(x), \qquad (32)$$

so that (26) reads $\ell u = \lambda u$ with eigenparameter λ as in (1). The endpoint $x = 1$ is **regular** (all coefficients are analytic there) with the Dirichlet condition $u(1) = 0$. The endpoint $x = 0$ is **singular** because $p_0(x) = -1/x \to -\infty$; the standard existence and uniqueness theory for (1) does not apply at $x = 0$ without further analysis.

4.3.2 Frobenius Analysis at $x = 0$

We seek local solutions of (26), rewritten as $xu'' - u - \lambda xu = 0$ after multiplying through by x, via the Frobenius substitution $u = \sum_{n=0}^{\infty} a_n\, x^{n+\sigma}$. Inserting and collecting powers:

- Lowest-order term ($x^{\sigma-1}$): indicial equation $\sigma(\sigma - 1) = 0$, giving $\sigma = 0$ or $\sigma = 1$.

Since the roots $\sigma_1 = 1$ and $\sigma_2 = 0$ differ by a positive integer, there may be a logarithmic term in the second solution.

First Frobenius solution ($\sigma = 1$). The recurrence for a_n ($n \geq 1$) is

$$(n + 1)n\, a_n \ = \ a_{n-1} + \lambda\, a_{n-2}, \qquad a_{-1} = 0. \qquad (33)$$

Setting $a_0 = 1$, the first few coefficients are

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1+2\lambda}{12}, \qquad a_3 = \frac{1+8\lambda}{144}, \quad \ldots \qquad (34)$$

The ratio $|a_n/a_{n-1}| \leq (|a_{n-1}| + |\lambda||a_{n-2}|)/(n(n + 1)|a_{n-1}|) \to 0$ as $n \to \infty$ for any fixed λ. Hence the first Frobenius series

$$y_1(x; \lambda) \ = \ x \sum_{n=0}^{\infty} a_n\,(\lambda)\, x^n \qquad (35)$$

has infinite radius of convergence. In particular, y_1 is a genuine power series in x, analytic on $[0, \infty)$ with $y_1(0; \lambda) = 0$.

Second Frobenius solution ($\sigma = 0$). Substituting $\sigma = 0$ into the x^{σ} equation gives $a_1 \cdot 1 \cdot 0 = a_0$, i.e., $0 = a_0 \neq 0$. This inconsistency signals a logarithmic correction:

$$y_2(x;\lambda) \;=\; \gamma(\lambda)\, y_1(x;\lambda)\ln x \;+\; \sum_{n=0}^{\infty} b_n\,(\lambda)\, x^n, \tag{36}$$

where $\gamma(\lambda) \neq 0$ and $b_0 \neq 0$. As $x \to 0^+$, since $y_1(x;\lambda) \sim a_0 x$, the logarithmic term satisfies $\gamma y_1(x;\lambda)\ln x \sim \gamma a_0 x\ln x \to 0$ (because $x\ln x \to 0$). The dominant contribution near $x = 0$ is therefore the constant term b_0: $y_2(x;\lambda) \to b_0 \neq 0$. Thus y_2 is **bounded but nonzero** at $x = 0$, in direct contrast to y_1 which satisfies $y_1(0;\lambda) = 0$.

4.3.3 Weyl Endpoint Classification

Lemma 4.1 (Both solutions are L^2). *For any $\delta > 0$ and any fixed λ, both $y_1(\cdot;\lambda)$ and $y_2(\cdot;\lambda)$ belong to $L^2(0,\delta)$.*

Proof. From (35), $y_1(x;\lambda) \sim a_0 x$ as $x \to 0^+$, so $\int_0^{\delta} |y_1|^2\, dx \lesssim \int_0^{\delta} x^2\, dx = \delta^3/3 < \infty$. For y_2: as established above, $y_2(x;\lambda) \to b_0$ as $x \to 0^+$. More precisely, $y_2(x;\lambda) = b_0 + O(x\ln x)$ near $x = 0$, so $|y_2(x;\lambda)| \leq C$ uniformly on $[0,\delta]$ for some constant C. Hence $\int_0^{\delta} |y_2|^2\, dx \leq C^2\delta < \infty$.

Corollary 4.1 (Weyl limit-circle). *The endpoint $x = 0$ is in the* **Weyl limit-circle (LC)** *case for the operator (32).*

In the LC case, all solutions of $\mathcal{T}u = \mu u$ are in $L^2(0,\delta)$, and a boundary condition must be imposed at $x = 0$ to obtain a self-adjoint operator. This contrasts with the limit-point (LP) case, where only one L^2 solution exists and no boundary condition at the singular endpoint is needed.

4.3.4 Admissibility of the Dirichlet Condition and Self-Adjointness

The Dirichlet condition $u(0) = 0$ selects y_1 and excludes y_2 because $y_2(0;\lambda) = b_0 \neq 0$: any function with nonzero value at $x = 0$ fails the Dirichlet condition directly. Equivalently, for any $u \in D_{\max}$ (the maximal operator domain), the condition $u(0) = 0$ can be characterised via Green's formula as

$$\lim_{x \to 0^+} W(u,\,\phi)(x) \;=\; 0, \tag{37}$$

where $\phi(x) = y_2(x;\mu_0)$ for any fixed regular value μ_0, and $W(u,\phi) = u'\phi - u\phi'$ is the Wronskian.

Theorem 2 (Self-adjointness). *The operator*

$$T: D(T) \to L^2(0,1), \qquad Tu = u'' - \frac{1}{x}\,u \;(= \ell u), \tag{38}$$

on the domain

$$D(T) = \{u \in L^2(0,1): u,\ u' \in AC_{\text{loc}}(0,1],\ Tu \in L^2(0,1),\ u(0) = 0,\ u(1) = 0\}$$
(39)

is self-adjoint. Its spectrum is real and discrete.

Proof. By Lemma 4.1 and Corollary 4.1, the endpoint $x = 0$ is LC and $x = 1$ is regular (R). This places the problem in the LC–R class. The boundary conditions $u(0) = 0$ and $u(1) = 0$ are separated and self-adjoint in the sense of Zettl [32, Theorem 10.4.4]; see also Weidmann [33, §8.4]. Self-adjoint operators in the LC–R class have purely discrete, real spectrum.

Remark 4.3. Theorem 2 confirms that the eigenvalue problem (26) is well-posed and has a countably infinite sequence of real eigenvalues. The numerical self-convergence in Table B3-E, showing $\lambda_1 < \lambda_2 < \cdots$ growing monotonically, is consistent with this structure; the sign and lower-boundedness of the spectrum depend on the detailed spectral theory of the differential expression (32), which is beyond the scope of the present paper. The key conclusion for the CTC analysis is that the computed values are approximations to a genuine self-adjoint discrete spectrum.

4.3.5 Analyticity of Eigenfunctions and Super-Geometric CTC Convergence

Lemma 4.2 (Eigenfunctions are entire in x). *The k-th eigenfunction u_k of the operator (38) satisfies $u_k(x) = x\,\hat{u}_k(x)$, where $\hat{u}_k$ is an entire function of x (i.e., it extends to an analytic function on all of $\mathbb{C}$).*

Proof. The eigenfunction u_k satisfies $u_k(0) = 0$, so it is a multiple of the first Frobenius solution $y_1(x; \lambda_k)$. By (35), $y_1(x; \lambda_k) = x \sum_{n=0}^{\infty} a_n(\lambda_k) x^n$ where the series has infinite radius of convergence. Hence $\hat{u}_k(x) = \sum_{n=0}^{\infty} a_n(\lambda_k) x^n$ is an entire function.

Theorem 3 (Super-geometric Chebyshev convergence for Example B3): *For every $\rho > 1$, the Chebyshev coefficients of u_k on $[0,1]$ satisfy*

$$|c_j| \le \frac{2\,M_\rho(\lambda_k)}{\rho^j}, \qquad j \ge 1,$$
(40)

where $M_\rho(\lambda_k) = \max_{z\in\mathcal{E}_\rho} |\hat{u}_k(z)| \cdot \max_{z\in\mathcal{E}_\rho} |(1+z)/2|$ is finite for every $\rho > 1$.
Consequently, the CTC eigenvalue errors satisfy

$$|\lambda_k - \lambda_k^{(N)}| = O(\rho^{-2N}) \quad \text{for every } \rho > 1.$$
(41)

Proof. Under the CGL map $x = (1 + r)/2$ (with $a = 0$, $b = 1$, $r \in [-1,1]$), the eigenfunction becomes

$$u_k\left(\frac{1+r}{2}\right) = \frac{1+r}{2}\,\hat{u}_k\left(\frac{1+r}{2}\right).$$
(42)

Since $\hat{u}_k$ is entire, the function $(1 + r)\hat{u}_k\big((1 + r)/2\big)/2$ is analytic on the entire complex r-plane, and in particular analytic on the Bernstein ellipse $\mathcal{E}_\rho$ for every $\rho > 1$. The bound (40) then follows from the standard Chebyshev approximation theorem [34, Theorem 8.2]. Since (40) holds for every $\rho > 1$, Proposition 1 (applied conditionally under (H1)–(H5) with any fixed ρ) gives $\left|\lambda_k - \lambda_k^{(N)}\right| \leq C_k(\rho)\rho^{-2N}$; since ρ can be taken arbitrarily large, estimate (41) follows. *(conditional on (H4)–(H5))*

Remark 4.4 (Interpretation of super-geometric convergence). Equation (41) states that the eigenvalue errors for Example B3 decay faster than any fixed geometric rate as N increases: for each $\rho > 1$ (no matter how large), $\rho^{-2N} \to 0$ and the error is eventually smaller than this. This explains why the self-convergence table (Table B3-E) shows extremely rapid digit stabilisation — for instance, $\mathrm{Err}(\lambda_1)$ drops from 5.35×10^{-25} at $N = 21$ to 2.53×10^{-30} at $N = 27$, a factor of more than 10^5 over 6 additional nodes.

Remark 4.5 (Distinction from the regular case). For regular problems whose coefficients have a finite analytic radius, the Bernstein ellipse $\mathcal{E}_\rho$ has a specific ρ imposed by the coefficients, and the bound in Proposition 1 holds for that fixed ρ. For Example B3, the selected eigenfunction is entire (Lemma 4.2), so no such bound constrains ρ and the convergence is super-geometric. The singular nature of the equation at $x = 0$ does not impede approximation accuracy; rather, the LC structure — combined with the Dirichlet condition selecting the analytic Frobenius solution — is precisely what ensures entire eigenfunctions.

4.4 Variable Coefficients

Examples A3 and B2 confirm that variable coefficients are handled by pointwise evaluation of $\mathbf{\Pi}_i$ at CGL nodes (12)–(13), with no additional quadrature required.

4.5 Boundary Condition Scope

The present paper demonstrates the CTC method for Dirichlet and homogeneous Neumann boundary conditions only. Extension to Robin or inhomogeneous Neumann conditions requires retaining the factor $2/h$ in the boundary rows (17)–(18) and modifying the right-hand side accordingly; no numerical example of those cases is provided here.

5. Conclusions

The **CTC method** is a Chebyshev spectral collocation scheme formulated in trigonometric coordinates, assembling all matrix entries from closed-form expressions via $T_k(\cos\theta) = \cos(k\theta)$. Two theoretical contributions were presented. First, **Proposition 1** establishes — conditionally on hypotheses (H4)–(H5) concerning collective compactness and operator consistency, which hold for standard Chebyshev collocation and are plausible for CTC but not yet proved — that the CTC eigenvalue errors satisfy $\left|\lambda_k - \lambda_k^{(N)}\right| \leq C_k\rho^{-2N}$, where $\rho > 1$ is the Bernstein-ellipse radius of the coefficient data. The proof combines analyticity of eigenfunctions (Steps 1–2, unconditional) with the Babuška–Osborn eigenvalue perturbation estimate (Step 4, conditional on (H4)–(H5)). Second, for Example B3 (Boyd equation), a complete Weyl–Frobenius analysis was carried out: the endpoint $x = 0$ is **limit-circle** (**Lemma 4.1**, unconditional), the Dirichlet condition $u(0) = 0$ is admissible and determines a self-adjoint operator (**Theorem 2**, unconditional), and the selected eigenfunction is an entire function of x (**Lemma 4.2**, unconditional). This entails **super-geometric** Chebyshev convergence (**Theorem 3**, conditional on (H4)–(H5)): the eigenvalue errors decay faster than ρ^{-2N} for every $\rho > 1$.

Benchmarking against six test problems from Annaby & Asharabi [18] and El-Gamel & Abd El-Hady [27] shows:

1. **Rapid geometric convergence** for problems with analytic data, consistent with Proposition 1.
2. **Super-geometric convergence for Example B3**, consistent with Theorem 3, explained by the entirety of the Dirichlet eigenfunction.
3. **Substantial accuracy improvements at larger** N for the four problems with independent exact values; at equal or near-equal basis sizes the advantage is problem- and mode-dependent.
4. **Numerically stable self-convergence** for Examples A3 and B3; independent verification of absolute accuracy remains outstanding.
5. **Dirichlet and homogeneous Neumann boundary conditions** within the same framework; Robin and inhomogeneous cases require additional work.

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