Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 2s (2025)

Z-open Maps in Fermatean Fuzzy Topological Spaces and an Entropy-Based Approach to Ecological Restoration Site Selection

P.Premalatha [1] , **A. Swaminathan** [2*] and **A. Vadivel** [3]

[1] Department of Mathematics, Chettinad College of Engineering and Technology, Karur, Tamilnadu, India.

[2] PG and Research Department of Mathematics, Govt. Arts College(A), Kumbakonam-612002, Tamilnadu, India.

[3] PG and Research Department of Mathematics, Arignar Anna Govt. Arts College, Namakkal-637002, Tamilnadu, India.

[1,2,3] Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India.

* corresponding author- A. Swaminathan

[1] premalathagukesh@gmail.com

[2*] asnathanway@gmail.com

3 avmaths@gmail.com

Article History:

Received: 21-01-2025

Revised: 22-02-2025

Accepted: 02-03-2025

Abstract:

In this paper, we discuss Fermatean fuzzy Z-open, Fermatean fuzzy Z-closed functions in Fermatean fuzzy topological spaces and we discuss some of its properties. Also we give an real life application for decision making using entropy measure on Fermatean fuzzy sets.

Keywords and phrases: Fermatean fuzzy Z-open mappings, Fermatean fuzzy Z-closed mappings, entropy measure.

AMS (2000) subject classification: 03E72, 54A05, 54C05, 54A40.

1 Introduction

Fuzzy sets were introduced by Zadeh [25] in 1965. The fuzzy set concept was the basis of mathematical testing of the fuzzy concept that exists in our real world and the formation of new branches in mathematics. The fuzzy set concept corresponding to unexplained physical situations gives useful applications on many topics such as statistics, data processing and linguistics. A lot of research has been done on this subject since 1965. In 1968, Chang [6] defined the concept of fuzzy topological space and generalized some basic notions of topology such as open set, closed set, continuity and compactness to fuzzy topological spaces. The idea of intuitionistic fuzzy set was first published by Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2, 5]. Coker [7] initiated a study of intuitionistic fuzzy topological spaces. Later Yager [23] launched a non standard fuzzy set referred to as Pythagorean fuzzy set. Olgun et al., [12] defined a Pythagorean fuzzy topological spaces. Fermatean fuzzy sets proposed by Senapati and Yager in 2020 [15], can handle uncertain information more easily in the process of decision making. They defined basic operations over the Fermatean fuzzy sets. Hariwan Z.

Ibrahim defined a Fermatean fuzzy topological spaces and the continuity of a function defind among Fermatean fuzzy topological spaces. we developed the concept of some stronger and weaker forms of Fermatean fuzzy open sets in Fermatean fuzzy topological spaces and also specialized some of their basic properties with examples. S. Saha [14] defined δ-open sets in fuzzy topological spaces, topological space by Pankajam et al. [13] and neutrosophic topological space by Vadivel et al. [21]. Lellis Thivagar et.al [11] explored a new concept of neutrosophic topology, intuitionistic topology and fuzzy topology. El-Maghrabi and Al-Juhani [8] proposed the concept of M-open sets in topological spaces in 2011 and examined some of their features. Padma et al. [16] also found M-open sets in topological spaces. Vadivel et al. [17, 18, 19] discussed some open sets in fuzzy and neutrosophic topological spaces. Kalaiyarsan et al. [10] and Vadivel et al. [20] introduced M-open sets in fuzzy and neutrosophic topological spaces. In section 2, of this paper some basic definitions of fs's, IFS's pfs's and $\mathfrak{F}\mathcal{F}$s's are briefly reviewed.

Research Gap: No investigation on some stronger and weaker forms Fermatean fuzzy open maps such as Fermatean fuzzy δ open maps, Fermatean fuzzy δ-semi open maps, Fermatean fuzzy pre open maps, Fermatean fuzzy Z open maps, strongly Fermatean fuzzy Z open maps and perfectly Fermatean fuzzy Z open maps on Fermatean fuzzy topological space has been reported in the fuzzy literature.

The purpose of this paper is to discuss Fermatean fuzzy Z-O and Fermatean fuzzy Z-C in Fermatean fuzzy topological spaces. Finally, here we introduce entropy measure for Fermatean fuzzy sets and give an example for the decision making in real life problem.

2 Preliminaries

We recall some basic notions of fuzzy sets, IFS's, pfs's and $\mathfrak{F}\mathcal{F}$s's.

Definition 2.1 [25] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A: X \to [0,1]$. That is:

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if x ispartlyin X.} \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$.

Let us consider two examples:

(i) all employees of XYZ who are over $1.8\mathrm{m}$ in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over $1.8\mathrm{m}$) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2 [1] The intuitionistic fuzzy sets are defined on a non-empty sets X as objects having the form $I = \{\langle x, \mu_I(x), \lambda_I(x)\rangle : x \in X\}$, where $\mu_I(x): X \to [0,1]$ and $\lambda_I(x): X \to [0,1]$ denote the degree of memebership and the degree of non-memebership of each element $x \in X$ to the set I, respectively, and $0 \leq \mu_I(x) + \lambda_I(x) \leq 1$, for all $x \in X$.

Definition 2.3 [1, 2, 3, 4] Let a nonempty set X be fixed. An IFS A in X is an object having the form: $A = \{< x, \mu_A(x), \lambda_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\lambda_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$: $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. For each A in X: $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$ is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus: $\mu_A(x) + \lambda_A(x) + \pi_A(x) = 1$.

Example 2.1 Let $X = \{x, y, z\}$ be a fixed universe of discourse and $A = \left\{\left\langle \frac{0.6,0.1}{x} \right\rangle, \left\langle \frac{0.8,0.1}{y} \right\rangle, \left\langle \frac{0.5,0.3}{z} \right\rangle\right\}$, be the intuitionistic fuzzy set in X. The hesitation margins of the elements x, y, z to A are as follows: $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ and $\pi_A(z) = 0.2$.

Definition 2.4 [22, 23, 24] Let X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following: $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$

or $A = \left\{\left\langle \frac{\mu_A(x),\lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\lambda_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. Supposing $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ and $\pi_A(x) \in [0,1]$. In what follows, $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. We denote the set of all PFS's over X by pfs(X).

Definition 2.5 [15] Let X be a universe of discourse. A Fermatean fuzzy set ($\mathfrak{F}\mathcal{F}$s) F in X is an object having the form $F = \{< x, \mu_F(x), \lambda_F(x) >: x \in X\}$ where $\mu_F(x): X \to [0,1]$ and $\lambda_F(x): X \to [0,1]$, including the condition $0 \leq (\mu_F(x))^3 + (\lambda_F(x))^3 \leq 1$, for all $x \in X$. The numbers $\mu_F(x)$ and $\lambda_F(x)$ denote, respectively, the degree of memebership and the degree of non-memebership of the element x in the set F. For any $\mathfrak{F}\mathcal{F}$s F and $x \in X$, $\pi_F(x) = \sqrt[3]{1 - [(\mu_F(x))^3 - (\lambda_F(x))^3]}$ is identified as the degree of interminancy of x to F. In the interest of simplicity, we shall mention the symbol $F = (\mu_F, \lambda_F)$ for the $\mathfrak{F}\mathcal{F}$s $F = \{< x, \mu_F(x), \lambda_F(x): x \in X\}$.

Definition 2.6 [15] Let $F = (\mu_F, \lambda_F)$, $F_1 = (\mu_{F_1}, \lambda_{F_1})$ and $F_2 = (\mu_{F_2}, \lambda_{F_2})$, be three Fermatean fuzzy sets ($\mathfrak{F}\mathcal{F}$s's), then their operations are defined as follows:

1. $F_1 \cap F_2 = (\min\{\mu_{F_1}, \mu_{F_2}\}, \max\{\lambda_{F_1}, \lambda_{F_2}\})$.
2. $F_1 \cup F_2 = (\max\{\mu_{F_1}, \mu_{F_2}\}, \min\{\lambda_{F_1}, \lambda_{F_2}\})$.
3. $F^c = (\lambda_F, \mu_F)$.

Remark 2.1 If $\mu_{F_1} = \mu_{F_2}$ and $\lambda_{F_1} = \lambda_{F_2}$, then $F_1 = F_2$

Definition 2.7 [9] Let X be a non empty set and τ be a family of Fermatean fuzzy subsets of X. If

1. $1_F, 0_F \in \tau$
2. for any $F_1, F_2 \in \tau$, we have $F_1 \cap F_2 \in \tau$,
3. for any $\{F_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} F_i \in \tau$ where I is an arbitrary index set then τ is called a Fermatean fuzzy topology on X.

The pair (X, τ) is said to be a Fermatean fuzzy topological space. Each member of τ is called an Fermatean fuzzy open set. The complement of an Fermatean fuzzy open set is called a Fermatean fuzzy closed set.

Remark 2.2 [9] As any Intuitionistic fuzzy subset or Pythagorean fuzzy subset of a set can be considered as Fermatean fuzzy subset, we observe that any Intuitionstic fuzzy topological space or Pythagorean fuzzy topological space is a Fermatean fuzzy topological space as well. On the other hand, it is obvious that a Fermatean fuzzy topological space need not be Intuitionistic fuzzy topological space and Pythagorean fuzzy topological space. Even an Fermatean fuzzy open set maybe neither an Intuitionistic fuzzy set nor Pythagorean fuzzy set.

Example 2.2 [9] Let $X = \{c_1, c_2\}$. Consider the following family Fermatean fuzzy subsets $\tau = \{1_F, 0_F, F_1, F2\}$ where $F_1 = \{\langle c_1, \mu_{F_1}(c_1) = 0.4, \lambda_{F_1}(c_1) = 0.6\rangle, \langle c_2, \mu_{F_1}(c_2) = 0.1, \lambda_{F_1}(c_2) = 0.3\rangle\}$ and $F_2 = \{\langle c_1, \mu_{F_2}(c_1) = 0.9, \lambda_{F_2}(c_1) = 0.6\rangle, \langle c_2, \mu_{F_2}(c_2) = 0.2, \lambda_{F_2}(c_2) = 0.3\rangle\}$. Observe that (X, τ) is a Fermatean fuzzy topological space but (X, τ) is neither Intuitionistic fuzzy topological space nor Pythagorean fuzzy topological space.

Definition 2.8 [9] Let (X, τ) be an $\mathfrak{F}\mathcal{F}$ts and $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ be an $\mathfrak{F}\mathcal{F}s$ in X. Then the Fermatean fuzzy interior and the Fermatean fuzzy closure of A are denoted by $\mathfrak{F}\mathcal{F}\text{int}(A)$ and $\mathfrak{F}\mathcal{F}\text{cl}(A)$ and are defined as follows: $\mathfrak{F}\mathcal{F}\text{int}(A) = \cup \{G | G \text{ isa } \mathfrak{F}\mathcal{F}\text{os and } G \subseteq A\}$ and $\mathfrak{F}\mathcal{F}\text{cl}(A) = \cap \{K | K \text{ isa } \mathfrak{F}\mathcal{F}\text{cs and } A \subseteq K\}$. Also, it can be established that $\mathfrak{F}\mathcal{F}\text{cl}(A)$ is an $\mathfrak{F}\mathcal{F}$cs and $\mathfrak{F}\mathcal{F}\text{int}(A)$ is an $\mathfrak{F}\mathcal{F}$os, A is an $\mathfrak{F}\mathcal{F}$cs if and only if $\mathfrak{F}\mathcal{F}\text{cl}(A) = A$ and A is an $\mathfrak{F}\mathcal{F}$os if and only if $\mathfrak{F}\mathcal{F}\text{int}(A) = A$. We say that A is $\mathfrak{F}\mathcal{F}$-dense if $\mathfrak{F}\mathcal{F}\text{cl}(A) = 1_{\mathfrak{F}}$.

Lemma 2.1 [9] For any Fermatean fuzzy set A in (X, τ), we have $1_{\mathfrak{F}} - \mathfrak{F}\mathcal{F}\text{int}(A) = \mathfrak{F}\mathcal{F}\text{cl}(1_{\mathfrak{F}} - A)$ and $1_{\mathfrak{F}} - \mathfrak{F}\mathcal{F}\text{cl}(A) = \mathfrak{F}\mathcal{F}\text{int}(1_{\mathfrak{F}} - A)$.

3 Fermatean fuzzy Z (resp. δ, δ$\mathcal{S}$ and pre)-open and closed maps

In this section, we introduce Fermatean fuzzy (resp. δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) open maps and Fermatean fuzzy (resp. δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) closed maps in $\mathfrak{F}\mathcal{F}$ts and obtain certain characterizations of these classes of maps.

Definition 3.1 Let (X, τ) be an $\mathfrak{F}\mathcal{F}$ts and A be an $\mathfrak{F}\mathcal{F}s$. Then A is said to be an Fermatean fuzzy (i) regular open set ($\mathfrak{F}\mathcal{F}$ros in short) if $A = \mathfrak{F}\mathcal{F}\text{int}(\mathfrak{F}\mathcal{F}\text{cl}(A))$. (ii) regular closed set ($\mathfrak{F}\mathcal{F}$rcs in short) if $A = \mathfrak{F}\mathcal{F}\text{cl}(\mathfrak{F}\mathcal{F}\text{int}(A))$. By Lemma 2.1, it follows that A is an $\mathfrak{F}\mathcal{F}$ros iff $\overline{A}$ is an $\mathfrak{F}\mathcal{F}$rcs.

Definition 3.2 Let (X, τ) be a $\mathfrak{F}\mathcal{F}$ts. Let S be a $\mathfrak{F}\mathcal{F}s$ of X. Then Fermatean

1. fuzzy δ interior of S (briefly, $\mathfrak{F}\mathcal{F}\delta\text{int}(S)$) is defined by $\mathfrak{F}\mathcal{F}\delta\text{int}(S) = \bigcup \{I : I \subseteq S \,\&\, I \text{isa} \mathfrak{F}\mathcal{F}\text{ro set inX}\}$.
2. fuzzy δ closure of S (briefly, $\mathfrak{F}\mathcal{F}\delta\text{cl}(S)$) is defined by $\mathfrak{F}\mathcal{F}\delta\text{cl}(S) = \bigcap \{A : S \subseteq A \,\&\, A\text{isa}\mathfrak{F}\mathcal{F}\text{rcset in X}\}$.

Definition 3.3 Let (X, τ) be a $\mathfrak{F}\mathcal{F}$ts. Then a $\mathfrak{F}\mathcal{F}s$ S in X is said to be Fermatean

1. fuzzy δ-open (briefly, $\mathfrak{F}\mathcal{F}\delta$o) set if $S = \mathfrak{F}\mathcal{F}\delta\text{int}(S)$.
2. fuzzy $\delta\alpha$-open (briefly, $\mathfrak{F}\mathcal{F}\delta\alpha$o) set if $S \subseteq \mathfrak{F}\mathcal{F}\text{int}(\mathfrak{F}\mathcal{F}\text{cl}\ (\mathfrak{F}\mathcal{F}\delta\text{int}\ (S)))$.
3. fuzzy δ-semi open (briefly, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o) set if $S \subseteq \mathfrak{F}\mathcal{F}\text{cl}(\mathfrak{F}\mathcal{F}\delta\text{int}(S))$.

4. fuzzy pre open (briefly, $\mathfrak{F}\mathcal{F}\mathcal{P}o$) set if $S \subseteq \mathfrak{F}\mathcal{F}int(\mathfrak{F}\mathcal{F}cl(S))$.

The complement of an $\mathfrak{F}\mathcal{F}\delta o$ (resp. $\mathfrak{F}\mathcal{F}\delta\alpha o$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}o$ & $\mathfrak{F}\mathcal{F}\mathcal{P}o$) set is called a Fermatean fuzzy δ (resp. Fermatean fuzzy $\delta\alpha$, Fermatean fuzzy δ-semi & Fermatean fuzzy pre) closed (briefly, $\mathfrak{F}\mathcal{F}\delta c$ (resp. $\mathfrak{F}\mathcal{F}\delta\alpha c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ & $\mathfrak{F}\mathcal{F}\mathcal{P}c$)) in X.

Definition 3.4 Let (X,τ) be a $\mathfrak{F}\mathcal{F}ts$. Let S be a $\mathfrak{F}\mathcal{F}s$ of X. Then Fermatean fuzzy

1. δ semi interior of S (briefly, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}int(S)$) is defined by $\mathfrak{F}\mathcal{F}\delta\mathcal{S}int(S) = \cup\{I: I \subseteq S \,\&\, I is a \mathfrak{F}\mathcal{F}\delta\mathcal{S}o \text{ set in} X\}$.

2. δ semi closure of S (briefly, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}cl(S)$) is defined by $\mathfrak{F}\mathcal{F}\delta\mathcal{S}cl(S) = \cap\{A: S \subseteq A \,\&\, A is a \mathfrak{F}\mathcal{F}\delta\mathcal{S}c set \text{ in } X\}$.

3. pre interior of S (briefly, $\mathfrak{F}\mathcal{F}\mathcal{P}int(S)$) is defined by $\mathfrak{F}\mathcal{F}\mathcal{P}int(S) = \cup\{I: I \subseteq S \,\&\, I is a \mathfrak{F}\mathcal{F}\mathcal{P}o \text{ set in} X\}$.

4. pre closure of S (briefly, $\mathfrak{F}\mathcal{F}\mathcal{P}cl(S)$) is defined by $\mathfrak{F}\mathcal{F}\mathcal{P}cl(S) = \cap\{A: S \subseteq A \,\&\, A is a \mathfrak{F}\mathcal{F}\mathcal{P}c set \text{ in } X\}$.

Definition 3.5 Let (X,τ) be a $\mathfrak{F}\mathcal{F}ts$. Then a $\mathfrak{F}\mathcal{F}s$ S in X is said to be a Fermatean fuzzy

1. Z-open (briefly, $\mathfrak{F}\mathcal{F}Zo$) set if $S \subseteq \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}\delta int(S)) \cap \mathfrak{F}\mathcal{F}int(\mathfrak{F}\mathcal{F}cl(S))$,

2. Z-closed (briefly, $\mathfrak{F}\mathcal{F}Zc$) set if $\mathfrak{F}\mathcal{F}int(\mathfrak{F}\mathcal{F}\delta cl(S)) \cap \mathfrak{F}\mathcal{F}cl(\mathfrak{F}\mathcal{F}int(S)) \subseteq S$.

The family of all $\mathfrak{F}\mathcal{F}Zo$ (resp. $\mathfrak{F}\mathcal{F}Zc$) sets of a space (X,τ) will be as always denoted by $\mathfrak{F}\mathcal{F}ZO(X)$ (resp. $\mathfrak{F}\mathcal{F}ZC(X)$).

Definition 3.6 Let (X,τ) be a $\mathfrak{F}\mathcal{F}ts$. Let K be a $\mathfrak{F}\mathcal{F}s$ of X, then the Fermatean

1. fuzzy Z-interior of K is the union of all $\mathfrak{F}\mathcal{F}Zo$ sets contained in K and denoted by $\mathfrak{F}\mathcal{F}Zint(K)$.

2. fuzzy Z-closure of K is the intersection of all $\mathfrak{F}\mathcal{F}Zc$ sets containing K and denoted by $\mathfrak{F}\mathcal{F}Zcl(K)$.

Theorem 3.1 Let K be a Fermatean fuzzy subset of a space (X,τ) Then

1. K is a $\mathfrak{F}\mathcal{F}Zo$ set iff $K = \mathfrak{F}\mathcal{F}Zint(K)$,

2. K is a $\mathfrak{F}\mathcal{F}Zc$ set iff $K = \mathfrak{F}\mathcal{F}Zcl(K)$.

Definition 3.7 A function $h_{\mathfrak{F}}: (X_1,\tau_1) \to (X_2,\tau_2)$ is said to be Fermatean fuzzy

1. continuous (briefly, $\mathfrak{F}\mathcal{F}Cts$), if for each $\mathfrak{F}\mathcal{F}o$ set M of X_2, the set $h_{\mathfrak{F}}^{-1}(M)$ is $\mathfrak{F}\mathcal{F}o$ set of X_1.

2. δ continuous (briefly, $\mathfrak{F}\mathcal{F}\delta Cts$), if for each $\mathfrak{F}\mathcal{F}o$ set M of X_2, the set $h_{\mathfrak{F}}^{-1}(M)$ is

$\mathfrak{F}\mathcal{F}\delta$o set of X_1.

3. δ semi continuous (briefly, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$Cts), if for each $\mathfrak{F}\mathcal{F}$o set M of X_2, the set $h_{\mathfrak{F}}^{-1}(M)$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o set of X_1.

4. pre continuous (briefly, $\mathfrak{F}\mathcal{F}\mathcal{P}$Cts), if for each $\mathfrak{F}\mathcal{F}$o set M of X_2, the set $h_{\mathfrak{F}}^{-1}(M)$ is $\mathfrak{F}\mathcal{F}\mathcal{P}$o set of X_1.

5. Z continuous (briefly, $\mathfrak{F}\mathcal{F}$ZCts), if for each $\mathfrak{F}\mathcal{F}$o set M of X_2, the set $h_{\mathfrak{F}}^{-1}(M)$ is $\mathfrak{F}\mathcal{F}$Zo set of X_1.

Lemma 3.1 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be a function. Then the following statements hold.

1. If A and B are Fermatean fuzzy subsets of X_1 such that $A \subseteq B$, then $h_{\mathfrak{F}}(A) \subseteq h_{\mathfrak{F}}(B)$.
2. If A and B are Fermatean fuzzy subsets of X_2 such that $A \subseteq B$, then $h_{\mathfrak{F}}^{-1}(A) \subseteq h_{\mathfrak{F}}^{-1}(B)$.

Lemma 3.2 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be a function. If A is a Fermatean fuzzy subset of X_1 and B is a Fermatean fuzzy subset of X_2. Then

1. $h_{\mathfrak{F}}(h_{\mathfrak{F}}^{-1}(A)) \subseteq A$
2. $h_{\mathfrak{F}}(h_{\mathfrak{F}}^{-1}(A)) = A \Leftrightarrow h_{\mathfrak{F}}$ is surjective.
3. $h_{\mathfrak{F}}^{-1}(h_{\mathfrak{F}}(A)) \supseteq A$
4. $h_{\mathfrak{F}}^{-1}(h_{\mathfrak{F}}(A)) = A$ whenever $h_{\mathfrak{F}}$ is injective.

Theorem 3.2 A function $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is $\mathfrak{F}\mathcal{F}$ZCts iff the inverse image of every $\mathfrak{F}\mathcal{F}$c set in X_2 is $\mathfrak{F}\mathcal{F}$Zc in X_1.

Definition 3.8 Let (X_1, τ_1) and (X_2, τ_2) be two $\mathfrak{F}\mathcal{F}$ts. A function $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be Fermatean fuzzy (resp. δ, δ$\mathcal{S}$, $\mathcal{P}$ and Z) open map (briefly, $\mathfrak{F}\mathcal{F}$O (resp. $\mathfrak{F}\mathcal{F}\delta$O, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$O, $\mathfrak{F}\mathcal{F}\mathcal{P}$O and $\mathfrak{F}\mathcal{F}$ZO)) if the image of each $\mathfrak{F}\mathcal{F}$o set in X_1 is $\mathfrak{F}\mathcal{F}$o (resp. $\mathfrak{F}\mathcal{F}\delta$o, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o, $\mathfrak{F}\mathcal{F}\mathcal{P}$o and $\mathfrak{F}\mathcal{F}$Zo)-set in X_2.

Definition 3.9 Let (X_1, τ_1) and (X_2, τ_2) be two $\mathfrak{F}\mathcal{F}$ts. A function $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is said to be Fermatean fuzzy (resp. δ, δ$\mathcal{S}$, $\mathcal{P}$ and Z) closed map (briefly, $\mathfrak{F}\mathcal{F}$C (resp. $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C, $\mathfrak{F}\mathcal{F}\mathcal{P}$C and $\mathfrak{F}\mathcal{F}$ZC)) if the image of each $\mathfrak{F}\mathcal{F}$c set in X_1 is $\mathfrak{F}\mathcal{F}$c (resp. $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c, $\mathfrak{F}\mathcal{F}\mathcal{P}$c and $\mathfrak{F}\mathcal{F}$Zc)-set in X_2.

Theorem 3.3 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be a mapping. Then every

1. $\mathfrak{F}\mathcal{F}\delta$O is $\mathfrak{F}\mathcal{F}$O.

2. $\mathfrak{F}\mathcal{F}O$ is $\mathfrak{F}\mathcal{F}\mathcal{P}O$.
3. $\mathfrak{F}\mathcal{F}\delta O$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$.
4. $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ is $\mathfrak{F}\mathcal{F}ZO$.
5. $\mathfrak{F}\mathcal{F}\mathcal{P}O$ is $\mathfrak{F}\mathcal{F}ZO$.

Proof. (i) Let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be $\mathfrak{F}\mathcal{F}\delta O$ and L is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Then $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\delta o$ set in X_2. Since every $\mathfrak{F}\mathcal{F}\delta os$ is $\mathfrak{F}\mathcal{F}os$, $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}o$ set in X_2. Therefore $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}O$.

(ii) Let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be $\mathfrak{F}\mathcal{F}O$ and L is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Then $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}o$ set in X_2. Since every $\mathfrak{F}\mathcal{F}os$ is $\mathfrak{F}\mathcal{F}\mathcal{P}os$, $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\mathcal{P}o$ set in X_2. Therefore $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}\mathcal{P}O$.

(iii) Let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be $\mathfrak{F}\mathcal{F}\delta O$ and L is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Then $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\delta o$ set in X_2. Since every $\mathfrak{F}\mathcal{F}\delta os$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}os$, $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}o$ set in X_2. Therefore $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$.

(iv) Let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and L is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Then $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}o$ set in X_2. Since every $\mathfrak{F}\mathcal{F}\delta\mathcal{S}os$ is $\mathfrak{F}\mathcal{F}Zos$, $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}Zo$ set in X_2. Therefore $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$.

(iv) Let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be $\mathfrak{F}\mathcal{F}\mathcal{P}O$ and L is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Then $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}\mathcal{P}o$ set in X_2. Since every $\mathfrak{F}\mathcal{F}\mathcal{P}os$ is $\mathfrak{F}\mathcal{F}Zos$, $h_{\mathfrak{F}}(L)$ is $\mathfrak{F}\mathcal{F}Zo$ set in X_2. Therefore $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$.

The converse of the Theorem 3.3 need not be true.

Example 3.1 Let $X_1=X_2=X=\{a,b\}$ and the $\mathfrak{F}\mathcal{F}$s's A_1 and A_2 are defined as

$\mu_{A_1}(a)=0.4,\lambda_{A_1}(a)=0.1,\mu_{A_1}(b)=0.6,\lambda_{A_1}(b)=0.3;\mu_{A_2}(a)=0.9,\lambda_{A_2}(a)=0.2,\mu_{A_2}(b)=0.6,\lambda_{A_2}(b)=0.3;$ Let $\tau_1=\tau_2=\tau=\{0_{\mathfrak{F}},1_{\mathfrak{F}},A_1,A_2\}$ be a $\mathfrak{F}\mathcal{F}ts$ on X and let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be an identity function, Then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}O$ but not $\mathfrak{F}\mathcal{F}\delta O$. Since, A_2 is a $\mathfrak{F}\mathcal{F}o$ set in X_1 but $h_{\mathfrak{F}}(A_2)=A_2$ is not $\mathfrak{F}\mathcal{F}\delta o$ set in X_2.

Example 3.2 Let $X_1=X_2=X=\{a,b\}$ and the $\mathfrak{F}\mathcal{F}$s's A_1, A_2 and A_3 are defined as

$\mu_{A_1}(a)=0.2,\lambda_{A_1}(a)=0.8,\mu_{A_1}(b)=0.3,\lambda_{A_1}(b)=0.7;\mu_{A_2}(a)=0.1,\lambda_{A_2}(a)=0.9,\mu_{A_2}(b)=0.1,\lambda_{A_2}(b)=0.9;\mu_{A_3}(a)=0.2,\lambda_{A_3}(a)=0.8,\mu_{A_3}(b)=0.4,\lambda_{A_3}(b)=0.6;$ Let $\tau_1=\tau_2=\tau=\{0_{\mathfrak{F}},1_{\mathfrak{F}},A_1,A_2,A_3\}$ be a $\mathfrak{F}\mathcal{F}ts$ on X and let $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ be an identity function, Then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ but not $\mathfrak{F}\mathcal{F}\delta O$. Since, A_1 is a $\mathfrak{F}\mathcal{F}o$ set in X_1 but $h_{\mathfrak{F}}(A_1)=A_1$ is not $\mathfrak{F}\mathcal{F}\delta o$ set in X_2.

Example 3.3 Let $X_1 = X_2 = X = \{a, b\}$ and the $\mathfrak{F}\mathcal{F}$s's A_1, A_2, B_1 and B_2 are defined as

$\mu_{A_1}(a) = 0.2, \lambda_{A_1}(a) = 0.7, \mu_{A_1}(b) = 0.1, \lambda_{A_1}(b) = 0.8; \mu_{A_2}(a) = 0.3, \lambda_{A_2}(a) = 0.6, \mu_{A_2}(b) = 0.4, \lambda_{A_2}(b) = 0.5; \mu_{B_1}(a) = 0.1, \lambda_{B_1}(a) = 0.9, \mu_{B_1}(b) = 0.2, \lambda_{B_1}(b) = 0.9; \mu_{B_2}(a) = 0.2, \lambda_{B_2}(a) = 0.3, \mu_{B_2}(b) = 0.4, \lambda_{B_2}(b) = 0.7;$ Let $\tau_1 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2\}$ and $\tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, B_1, B_2\}$ are $\mathfrak{F}\mathcal{F}$ts's on X and let $h_{\mathfrak{F}}: (X_2, \tau_2) \to (X_1, \tau_1)$ be an identity function, Then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}\mathcal{P}O$ but not $\mathfrak{F}\mathcal{F}O$. Since, B_2 is a $\mathfrak{F}\mathcal{F}$o set in X_2 but $h_{\mathfrak{F}}(B_2) = B_2$ is not $\mathfrak{F}\mathcal{F}$o set in X_1.

Example 3.4 Let $X_1 = X_2 = X = \{a, b\}$ and the $\mathfrak{F}\mathcal{F}$s's A_1, A_2, A_3, A_4, and A_5 are defined as

$\mu_{A_1}(a) = 0.2, \lambda_{A_1}(a) = 0.8, \mu_{A_1}(b) = 0.4, \lambda_{A_1}(b) = 0.6; \mu_{A_2}(a) = 0.1, \lambda_{A_2}(a) = 0.9, \mu_{A_2}(b) = 0.3, \lambda_{A_2}(b) = 0.7; \mu_{A_3}(a) = 0.9, \lambda_{A_3}(a) = 0.1, \mu_{A_3}(b) = 0.7, \lambda_{A_3}(b) = 0.3; \mu_{A_4}(a) = 0.2, \lambda_{A_4}(a) = 0.8, \mu_{A_4}(b) = 0.3, \lambda_{A_4}(b) = 0.7.$ Let $\tau_1 = \tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3, A_4\}$ be $\mathfrak{F}\mathcal{F}$ts's on X and let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be an identity function, Then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$ but not $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$. Since, A_4 is $\mathfrak{F}\mathcal{F}$o set in X_1 but $h_{\mathfrak{F}}(A_4) = A_4$ is not $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o set in X_2.

Example 3.5 Let $X_1 = X_2 = X = \{a, b\}$ and the $\mathfrak{F}\mathcal{F}$s's A_1, A_2, A_3, A_4, B_1, B_2, B_3 and B_4 are defined as

$\mu_{A_1}(a) = 0.2, \lambda_{A_1}(a) = 0.8, \mu_{A_1}(b) = 0.4, \lambda_{A_1}(b) = 0.6; \mu_{A_2}(a) = 0.1, \lambda_{A_2}(a) = 0.9, \mu_{A_2}(b) = 0.3, \lambda_{A_2}(b) = 0.7; \mu_{A_3}(a) = 0.9, \lambda_{A_3}(a) = 0.1, \mu_{A_3}(b) = 0.7, \lambda_{A_3}(b) = 0.3; \mu_{A_4}(a) = 0.2, \lambda_{A_4}(a) = 0.8, \mu_{A_4}(b) = 0.3, \lambda_{A_4}(b) = 0.7; \mu_{B_1}(a) = 0.4, \lambda_{B_1}(a) = 0.6, \mu_{B_1}(b) = 0.5, \lambda_{B_1}(b) = 0.5; \mu_{B_2}(a) = 0.6, \lambda_{B_2}(a) = 0.4, \mu_{B_2}(b) = 0.6, \lambda_{B_2}(b) = 0.4; \mu_{B_3}(a) = 0.7, \lambda_{B_3}(a) = 0.3, \mu_{B_3}(b) = 0.6, \lambda_{B_3}(b) = 0.4; \mu_{B_4}(a) = 0.4, \lambda_{B_4}(a) = 0.6, \mu_{B_4}(b) = 0.4, \lambda_{B_4}(b) = 0.6;$ Let $\tau_1 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, B_1, B_2, B_3, B_4\}$ are $\mathfrak{F}\mathcal{F}$ts's on X and let $h_{\mathfrak{F}}: (X_2, \tau_2) \to (X_1, \tau_1)$ be an identity function, Then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$ but not $\mathfrak{F}\mathcal{F}\mathcal{P}O$. Since, B_4 is a $\mathfrak{F}\mathcal{F}$o set in X_2 but $h_{\mathfrak{F}}(B_4) = B_4$ is not $\mathfrak{F}\mathcal{F}\mathcal{P}$o set in X_1.

Remark 3.1 We obtain the following diagram from the results are discussed above.

$$\begin{tikzcd}
\mathfrak{F}\mathcal{F}\delta O \arrow[r] \arrow[d] & \mathfrak{F}\mathcal{F}\delta\mathcal{S}O \arrow[d, leftrightarrow, "/" marking] \arrow[dr] & \\
\mathfrak{F}\mathcal{F}O \arrow[r] & \mathfrak{F}\mathcal{F}\mathcal{P}O \arrow[r] & \mathfrak{F}\mathcal{F}ZO
\end{tikzcd}$$

Note: $K \to L$ denotes K implies L, but not conversely.

Theorem 3.4 A function $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) mapping if and only if $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$ (resp. $\mathfrak{F}\mathcal{F}cl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$, $\mathfrak{F}\mathcal{F}\delta cl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}cl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$ and $\mathfrak{F}\mathcal{F}\mathcal{P}cl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$) for every Fermatean fuzzy set A of X_1.

Proof. Suppose $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ is a $\mathfrak{F}\mathcal{F}ZC$ function and A is any $\mathfrak{F}\mathcal{F}s$ in X_1. Then $\mathfrak{F}\mathcal{F}cl(A)$ is a $\mathfrak{F}\mathcal{F}c$ set in X_1. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$ is a $\mathfrak{F}\mathcal{F}Zc$ set in X_2. Then by Theorem 3.1 (ii), $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A)))=h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$. Therefore $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))\subseteq \mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A)))=h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$. Hence $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl\ (A))$.

Conversely, Let A be a $\mathfrak{F}\mathcal{F}c$ set in X_1 . Then $\mathfrak{F}\mathcal{F}cl(A)=A$ and so $h_{\mathfrak{F}}(A)=h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}cl(A))$. By our assumption $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))\subseteq h_{\mathfrak{F}}(A)$. But $h_{\mathfrak{F}}(A)\subseteq \mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))$. Hence $\mathfrak{F}\mathcal{F}Zcl(h_{\mathfrak{F}}(A))=h_{\mathfrak{F}}(A)$ and therefore by Theorem 3.1 (ii), $h_{\mathfrak{F}}(A)$ is $\mathfrak{F}\mathcal{F}Zcs$ in X_2. Thus $h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}ZC$ map.

Other cases are similar.

Theorem 3.5 A map $h_{\mathfrak{F}}:(X_1,\tau_1)\to(X_2,\tau_2)$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) mapping if and only if for each Fermatean fuzzy set S of X_2 and for each $\mathfrak{F}\mathcal{F}o$ set U of X_1 containing $h_{\mathfrak{F}}^{-1}(S)$ there exists a $\mathfrak{F}\mathcal{F}Zo$ (resp. $\mathfrak{F}\mathcal{F}o$, $\mathfrak{F}\mathcal{F}\delta o$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}o$ and $\mathfrak{F}\mathcal{F}\mathcal{P}o$) set V of X_2 such that $S\subseteq V$ and $h_{\mathfrak{F}}^{-1}(V)\subseteq U$.

Proof. Suppose $h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}ZC$ map. Let S be any Fermatean fuzzy set in X_2 and U be a $\mathfrak{F}\mathcal{F}Zo$ set of X_1 such that $h_{\mathfrak{F}}^{-1}(S)\subseteq U$. Then $V=(h_{\mathfrak{F}}(U^c))^c$ is $\mathfrak{F}\mathcal{F}Zo$ set containing S such that $h_{\mathfrak{F}}^{-1}(V)\subseteq U$. Conversely, Let S be a $\mathfrak{F}\mathcal{F}c$ set of X_1. Then $h_{\mathfrak{F}}^{-1}((h_{\mathfrak{F}}(S))^c)\subseteq S^c$ and S^c is $\mathfrak{F}\mathcal{F}os$ in X_1.

By assumption, there exists a $\mathfrak{F}\mathcal{F}Zo$ set V of X_2 such that $(h_{\mathfrak{F}}(S))^c\subseteq V$ and $h_{\mathfrak{F}}^{-1}(V)\subseteq S^c$ and so $S\subseteq(h_{\mathfrak{F}}^{-1}(V))^c$. Hence $V^c\subseteq h_{\mathfrak{F}}(S)\subseteq h_{\mathfrak{F}}((h_{\mathfrak{F}}^{-1}(V))^c)\subseteq V^c$, which implies $h_{\mathfrak{F}}(S)=V^c$. Since V^c is $\mathfrak{F}\mathcal{F}Zcs$, $h_{\mathfrak{F}}(S)$ is $\mathfrak{F}\mathcal{F}Zcs$ and $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ map.

Other cases are similar.

Remark 3.2 The composition of two $\mathfrak{F}\mathcal{F}ZO$ (resp. $\mathfrak{F}\mathcal{F}O$, $\mathfrak{F}\mathcal{F}\delta O$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and $\mathfrak{F}\mathcal{F}\mathcal{P}O$) maps need not be a $\mathfrak{F}\mathcal{F}ZO$ (resp. $\mathfrak{F}\mathcal{F}O$, $\mathfrak{F}\mathcal{F}\delta O$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and $\mathfrak{F}\mathcal{F}\mathcal{P}O$) map, which is shown in the following example.

Example 3.6 Let $X_1=X_2=X_3=X=\{a,b\}$ and the $\mathfrak{F}\mathcal{F}s$'s A_1, A_2, B_1, B_2 and C_1 are defined as

$\mu_{A_1}(a)=0.2,\lambda_{A_1}(a)=0.7,\mu_{A_1}(b)=0.1,\lambda_{A_1}(b)=0.8;\mu_{A_2}(a)=0.3,\lambda_{A_2}(a)=0.6,\mu_{A_2}(b)=0.4,\lambda_{A_2}(b)=0.5;\mu_{B_1}(a)=0.1,\lambda_{B_1}(a)=0.9,\mu_{B_1}(b)=0.2,\lambda_{B_1}(b)=0.9;\mu_{B_2}(a)=0.2,\lambda_{B_2}(a)=0.3,\mu_{B_2}(b)=0.4,\lambda_{B_2}(b)=0.7;\mu_{C_1}(a)=0.3,\lambda_{C_1}(a)=$

$0.2, \mu_{C_1}(b) = 0.7, \lambda_{C_1}(b) = 0.4$; Let $\tau_1 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, A_1, A_2\}$, $\tau_2 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, B_1, B_2\}$ and $\tau_3 = \{0_{\mathfrak{F}}, 1_{\mathfrak{F}}, C_1\}$ are $\mathfrak{F}\mathcal{F}$ts's on X and

let $h_{\mathfrak{F}}: (X_3, \tau_3) \to (X_2, \tau_2)$ and $g_{\mathfrak{F}}: (X_2, \tau_2) \to (X_1, \tau_1)$ be an identity function, Then $h_{\mathfrak{F}}$ and $g_{\mathfrak{F}}$ are $\mathfrak{F}\mathcal{F}ZO$ but $(g_{\mathfrak{F}} \circ h_{\mathfrak{F}})$ is not $\mathfrak{F}\mathcal{F}ZO$. Since, C_1 is a $\mathfrak{F}\mathcal{F}o$ set in X_3 but $(g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(C_1) = C_1$ is not $\mathfrak{F}\mathcal{F}Zo$ set in X_1.

Theorem 3.6 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be a $\mathfrak{F}\mathcal{F}C$ map and $g_{\mathfrak{F}}: (X_2, \tau_2) \to (X_3, \tau_3)$ be a $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map. Then their composition $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_3, \tau_3)$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$).

Proof. Let F be a $\mathfrak{F}\mathcal{F}c$ set in X_1. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}C$, $h_{\mathfrak{F}}(F)$ is $\mathfrak{F}\mathcal{F}cs$ in X_2. Since $g_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$), $g_{\mathfrak{F}}(h_{\mathfrak{F}}(F)) = (g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(F)$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$)-set in X_3. Hence $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

Theorem 3.7 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ and $g_{\mathfrak{F}}: (X_2, \tau_2) \to (X_3, \tau_3)$ be two mappings such that their composition $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_3, \tau_3)$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map. Then the followings are true.

1. If $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}Cts$ and surjective, then $g_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.
2. If $g_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZIrr$ (resp. $\mathfrak{F}\mathcal{F}Irr$, $\mathfrak{F}\mathcal{F}\delta Irr$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Irr$ and $\mathfrak{F}\mathcal{F}\mathcal{P}Irr$) and injective, then $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

Proof. (i) Let A be a $\mathfrak{F}\mathcal{F}c$ set of X_2. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}Cts$ map, $h_{\mathfrak{F}}^{-1}(A)$ is $\mathfrak{F}\mathcal{F}c$ in X_1. Since $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map, $(g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(h_{\mathfrak{F}}^{-1}(A))$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$) set in Z. Since $h_{\mathfrak{F}}$ is surjective, $g_{\mathfrak{F}}(A)$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$) set in X_3. Hence $g_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

(ii) Let B be any $\mathfrak{F}\mathcal{F}c$ set of X_1. Since $g_{\mathfrak{F}} \circ h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map, $(g_{\mathfrak{F}} \circ h_{\mathfrak{F}})(B)$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$)-set in X_3. Since $g_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZIrr$ (resp. $\mathfrak{F}\mathcal{F}Irr$, $\mathfrak{F}\mathcal{F}\delta Irr$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Irr$ and $\mathfrak{F}\mathcal{F}\mathcal{P}Irr$), $g_{\mathfrak{F}}^{-1}(g_{\mathfrak{F}} \circ h_{\mathfrak{F}}(B))$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$)-set in X_2. Since $g_{\mathfrak{F}}$ is injective, $h_{\mathfrak{F}}(B)$ is $\mathfrak{F}\mathcal{F}Zc$ (resp. $\mathfrak{F}\mathcal{F}c$, $\mathfrak{F}\mathcal{F}\delta c$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}c$ and $\mathfrak{F}\mathcal{F}\mathcal{P}c$)-set in X_2. Hence $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

Theorem 3.8 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

1. If A is $\mathfrak{F}\mathcal{F}$c set of X_1, then the restriction $h_{\mathfrak{F}_A}: (X_{1_A}, \tau_{1_A}) \to (X_2, \tau_2)$ is $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C) map.
2. If $A = h_{\mathfrak{F}}^{-1}(B)$ for some $\mathfrak{F}\mathcal{F}$c set B of X_2, then the restriction $h_{\mathfrak{F}_A}: (X_{1_A}, \tau_{1_A}) \to (X_2, \tau_2)$ is $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C) map.

Proof. (i) Let B be any $\mathfrak{F}\mathcal{F}$c set of A. Then $B = A \cap L$ for some $\mathfrak{F}\mathcal{F}$c set L of X_1 and so B is $\mathfrak{F}\mathcal{F}$cs in X_1. By hypothesis, $h_{\mathfrak{F}}(B)$ is $\mathfrak{F}\mathcal{F}$Zc (resp. $\mathfrak{F}\mathcal{F}$c, $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c and $\mathfrak{F}\mathcal{F}\mathcal{P}$c)-set in X_2. But $h_{\mathfrak{F}}(B) = h_{\mathfrak{F}_A}(B)$, therefore $h_{\mathfrak{F}_A}$ is a $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C) map.

(ii) Let D be a $\mathfrak{F}\mathcal{F}$c set of A. Then $D = A \cap H$, for some $\mathfrak{F}\mathcal{F}$c set H in X_1. Now, $h_{\mathfrak{F}_A}(D) = h_{\mathfrak{F}}(D) = h_{\mathfrak{F}}(A \cap H) = h_{\mathfrak{F}}(h_{\mathfrak{F}}^{-1}(B) \cap H) = B \cap h_{\mathfrak{F}}(H)$. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C), $h_{\mathfrak{F}}(H)$ is $\mathfrak{F}\mathcal{F}$Zc (resp. $\mathfrak{F}\mathcal{F}$c, $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c and $\mathfrak{F}\mathcal{F}\mathcal{P}$c)-set in X_2. Hence $h_{\mathfrak{F}_A}$ is a $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C) map.

Theorem 3.9 A function $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is $\mathfrak{F}\mathcal{F}$ZO (resp. $\mathfrak{F}\mathcal{F}$O, $\mathfrak{F}\mathcal{F}\delta$O, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$O and $\mathfrak{F}\mathcal{F}\mathcal{P}$O) map if and only if $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(A))$ (resp. $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\mathrm{int}(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\delta\mathrm{int}(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\delta\mathcal{S}\mathrm{int}(h_{\mathfrak{F}}(A))$ and $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\mathcal{P}\mathrm{int}(h_{\mathfrak{F}}(A))$), for every Fermatean fuzzy set A of X_1.

Proof. Suppose $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is a $\mathfrak{F}\mathcal{F}$ZO function and A is $\mathfrak{F}\mathcal{F}$s in X_1. Then $\mathfrak{F}\mathcal{F}\mathrm{int}(A)$ is a $\mathfrak{F}\mathcal{F}$o set in X_1. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}$ZO, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A))$ is a $\mathfrak{F}\mathcal{F}$Zo set. Since $\mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A))) \subseteq \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(A))$.

Conversely, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\mathrm{int}(A)) \subseteq \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(A))$ for every Fermatean fuzzy set A in X_1. Let U be a $\mathfrak{F}\mathcal{F}$o set in X_1. Then $\mathfrak{F}\mathcal{F}\mathrm{int}(U) = U$ and by hypothesis, $h_{\mathfrak{F}}(U) \subseteq \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(U))$. But $\mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(U)) \subseteq h_{\mathfrak{F}}(U)$. Therefore, $h_{\mathfrak{F}}(U) = \mathfrak{F}\mathcal{F}\mathrm{Zint}(h_{\mathfrak{F}}(U))$. Then by Theorem 3.1 (i), $h_{\mathfrak{F}}(U)$ is $\mathfrak{F}\mathcal{F}$Zos. Hence $h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}$ZO map.

Other cases are similar.

Definition 3.10 A Fermatean fuzzy set A in a $\mathfrak{F}\mathcal{F}$ts (X, τ) is called a Fermatean fuzzy (δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) q-neighborhood of a Fermatean fuzzy point x_r if there exists a $\mathfrak{F}\mathcal{F}$o (resp. $\mathfrak{F}\mathcal{F}\delta$o, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o, $\mathfrak{F}\mathcal{F}\mathcal{P}$o and $\mathfrak{F}\mathcal{F}$Zo) set V in (X, τ) such that $x_r qV \subseteq A$.

Definition 3.11 Let A and B be any two $\mathfrak{F}\mathcal{F}$s's of a $\mathfrak{F}\mathcal{F}$ts's. Then A is Fermatean fuzzy (δ, $\delta\mathcal{S}$, $\mathcal{P}$ and Z) q-neighbourhood (briefly, $\mathfrak{F}\mathcal{F}$q-nbhd (resp. $\mathfrak{F}\mathcal{F}\delta$q-nbhd, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$q-nbhd, $\mathfrak{F}\mathcal{F}\mathcal{P}$q-nbhd and $\mathfrak{F}\mathcal{F}$Zq-nbhd)) with B if there exists a $\mathfrak{F}\mathcal{F}$o (resp. $\mathfrak{F}\mathcal{F}\delta$o, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o, $\mathfrak{F}\mathcal{F}\mathcal{P}$o and $\mathfrak{F}\mathcal{F}$Zo) set O with $AqO \subseteq B$.

Theorem 3.10 Let $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ be a mapping. Then the following statements are equivalent.

1. $h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}ZO$ (resp. $\mathfrak{F}\mathcal{F}O$, $\mathfrak{F}\mathcal{F}\delta O$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and $\mathfrak{F}\mathcal{F}\mathcal{P}O$) mapping,
2. For a subset A of X_1, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A)) \subseteq \mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(A))$ (resp. $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A)) \subseteq \mathfrak{F}\mathcal{F}int(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}\ int(A)) \subseteq \mathfrak{F}\mathcal{F}\delta int(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A)) \subseteq \mathfrak{F}\mathcal{F}\delta\mathcal{S}int(h_{\mathfrak{F}}(A))$ and $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A)) \subseteq \mathfrak{F}\mathcal{F}\mathcal{P}int(h_{\mathfrak{F}}(A))$.
3. For each $x_\alpha \in X_1$ and for each $\mathfrak{F}\mathcal{F}$q-nbhd U of x_α in X_1, there exists A $\mathfrak{F}\mathcal{F}Z$q-nbhd (resp. $\mathfrak{F}\mathcal{F}$q-nbhd, $\mathfrak{F}\mathcal{F}\delta$q-nbhd, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$q-nbhd and $\mathfrak{F}\mathcal{F}\mathcal{P}$q-nbhd) W of $h_{\mathfrak{F}}(x_\alpha)$ in X_2 such that $W \subseteq h_{\mathfrak{F}}(U)$.

Proof.

(i) $\Rightarrow$ (ii): Suppose $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ is a $\mathfrak{F}\mathcal{F}ZO$ function and $A \subseteq X_1$. Then $\mathfrak{F}\mathcal{F}int(A)$ is a $\mathfrak{F}\mathcal{F}o$ set in X_1. Since $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$ map, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A))$ is a $\mathfrak{F}\mathcal{F}Zo$ set. Since $\mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A))) \subseteq \mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(A))$, $h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(A)) \subseteq \mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(A))$. This proves (ii).

(ii) $\Rightarrow$ (iii): Let $x_\alpha \in X_1$ and U be any arbitrary $\mathfrak{F}\mathcal{F}$q-nbhd of x_α in X_1. Then there exists a $\mathfrak{F}\mathcal{F}o$ set G such that $x_\alpha \in G \subseteq U$. By (ii), $h_{\mathfrak{F}}(G) = h_{\mathfrak{F}}(\mathfrak{F}\mathcal{F}int(G)) \subseteq \mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(G))$. But, $\mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(G)) \subseteq h_{\mathfrak{F}}(G)$. Therefore, $\mathfrak{F}\mathcal{F}Zint(h_{\mathfrak{F}}(G)) = h_{\mathfrak{F}}(G)$ and hence $h_{\mathfrak{F}}(G)$ is $\mathfrak{F}\mathcal{F}Zo$ in X_2. Since $x_\alpha \in G \subseteq U$, $h_{\mathfrak{F}}(x_\alpha) \in h_{\mathfrak{F}}(G) \subseteq h_{\mathfrak{F}}(U)$ and so (iii) holds, by taking $W = h_{\mathfrak{F}}(G)$.

(iii) $\Rightarrow$ (i): Let U be any $\mathfrak{F}\mathcal{F}o$ set in X_1. Let $x_\alpha \in U$ and $h_{\mathfrak{F}}(x_\alpha) = y_\beta$. Then for each $x_\alpha \in U$, $y_\beta \in h_{\mathfrak{F}}(U)$, by assumption there exists a $\mathfrak{F}\mathcal{F}qZ$-nbhd $W(y_\beta)$ of y_β in X_2 such that $W(y_\beta) \subseteq h_{\mathfrak{F}}(U)$. Since $W(y_\beta)$ is a $\mathfrak{F}\mathcal{F}qZ$-nbhd of y_β, there exists a $\mathfrak{F}\mathcal{F}Zo$ set $V(y_\beta)$ in X_2 such that $y_\beta \in V(y_\beta) \subseteq W(y_\beta)$. Therefore, $h_{\mathfrak{F}}(U) = \cup\{V(y_\beta) | y_\beta \in h_{\mathfrak{F}}(U)\}$. Since the union of $\mathfrak{F}\mathcal{F}Zo$ sets is $\mathfrak{F}\mathcal{F}Zo$, $h_{\mathfrak{F}}(U)$ is a $\mathfrak{F}\mathcal{F}Zo$ set in X_2. Thus, $h_{\mathfrak{F}}$ is a $\mathfrak{F}\mathcal{F}ZO$ map.

Other cases are similar.

Theorem 3.11 For any bijective map $h_{\mathfrak{F}}: (X_1, \tau_1) \to (X_2, \tau_2)$ the following statements are equivalent:

1. $h_{\mathfrak{F}}^{-1}: (X_2, \tau_2) \to (X_1, \tau_1)$ is $\mathfrak{F}\mathcal{F}ZCts$ (resp. $\mathfrak{F}\mathcal{F}Cts$, $\mathfrak{F}\mathcal{F}\delta Cts$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}Cts$ and $\mathfrak{F}\mathcal{F}\mathcal{P}Cts$).
2. $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$ (resp. $\mathfrak{F}\mathcal{F}O$, $\mathfrak{F}\mathcal{F}\delta O$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and $\mathfrak{F}\mathcal{F}\mathcal{P}O$) map.
3. $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZC$ (resp. $\mathfrak{F}\mathcal{F}C$, $\mathfrak{F}\mathcal{F}\delta C$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}C$ and $\mathfrak{F}\mathcal{F}\mathcal{P}C$) map.

Proof. (i) $\to$ (ii): Let U be a $\mathfrak{F}\mathcal{F}o$ set in X_1. By assumption, $(h_{\mathfrak{F}}^{-1})^{-1}(U) = h_{\mathfrak{F}}(U)$ is $\mathfrak{F}\mathcal{F}Zo$ (resp. $\mathfrak{F}\mathcal{F}o$, $\mathfrak{F}\mathcal{F}\delta o$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}o$ and $\mathfrak{F}\mathcal{F}\mathcal{P}o$)-set in X_2 and so $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}ZO$ (resp. $\mathfrak{F}\mathcal{F}O$, $\mathfrak{F}\mathcal{F}\delta O$, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}O$ and $\mathfrak{F}\mathcal{F}\mathcal{P}O$) map.

(ii) → (iii): Let F be a $\mathfrak{F}\mathcal{F}$c set of X_1. Then F^c is a $\mathfrak{F}\mathcal{F}$o set in X_1. By assumption $h_{\mathfrak{F}}(F^c)$ is $\mathfrak{F}\mathcal{F}$Zo (resp. $\mathfrak{F}\mathcal{F}$o, $\mathfrak{F}\mathcal{F}\delta$o, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$o and $\mathfrak{F}\mathcal{F}\mathcal{P}$o) set in X_2. But $h_{\mathfrak{F}}(F^c) = (h_{\mathfrak{F}}(F))^c$. Therefore $h_{\mathfrak{F}}(F)$ is $\mathfrak{F}\mathcal{F}$Zc (resp. $\mathfrak{F}\mathcal{F}$c, $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c and $\mathfrak{F}\mathcal{F}\mathcal{P}$c) set in X_2. Hence, $h_{\mathfrak{F}}$ is $\mathfrak{F}\mathcal{F}$ZC (resp. $\mathfrak{F}\mathcal{F}$C, $\mathfrak{F}\mathcal{F}\delta$C, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$C and $\mathfrak{F}\mathcal{F}\mathcal{P}$C) map.

(iii) ⇒ (i): Let F be a $\mathfrak{F}\mathcal{F}$Zc (resp. $\mathfrak{F}\mathcal{F}$c, $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c and $\mathfrak{F}\mathcal{F}\mathcal{P}$c) set of X_1. By assumption, $h_{\mathfrak{F}}(F)$ is $\mathfrak{F}\mathcal{F}$Zc (resp. $\mathfrak{F}\mathcal{F}$c, $\mathfrak{F}\mathcal{F}\delta$c, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$c and $\mathfrak{F}\mathcal{F}\mathcal{P}$c) set in X_2. But $h_{\mathfrak{F}}(F) = (h_{\mathfrak{F}}^{-1})^{-1}(F)$ and therefore by Theorem 3.2, $h_{\mathfrak{F}}^{-1}$ is $\mathfrak{F}\mathcal{F}$ZCts (resp. $\mathfrak{F}\mathcal{F}$Cts, $\mathfrak{F}\mathcal{F}\delta$Cts, $\mathfrak{F}\mathcal{F}\delta\mathcal{S}$Cts and $\mathfrak{F}\mathcal{F}\mathcal{P}$Cts).

4 Application

Entropy as a measure of fuzziness was first proposed by Zadeh [26]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for $\mathfrak{F}\mathcal{F}s$ that connects the degree of membership and non-membership. As an example, we have applied the proposed entropy measure in environmental decision making.

Definition 4.1

Let $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$ be a $\mathfrak{F}\mathcal{F}s$ in X. The new entropy measure for A denoted by $\varepsilon_{\mathfrak{F}\mathcal{F}s}(A)$, is a function, $\varepsilon_{\mathfrak{F}\mathcal{F}s}: \tau_{\mathfrak{F}\mathcal{F}s}(X) \to [0,1]$ and is defined as $\varepsilon_{\mathfrak{F}\mathcal{F}s}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n} (\mu_A - \lambda_A)^2$; forevery $x_i \in A$, where $\tau_{\mathfrak{F}\mathcal{F}s}(X)$ denote the family of all $\mathfrak{F}\mathcal{F}$s's on X.

Example 4.1 A regional environmental authority is tasked with identifying the most suitable water body among four candidate lakes (L_1, L_2, L_3, L_4) in a river basin for prioritising ecological restoration and sustainable water resource management. The evaluation is carried out with respect to five key environmental parameters:

f_1 : pH level suitability

f_2 : Dissolved oxygen (DO) content

f_3 : Biochemical oxygen demand (BOD) (lower BOD ⇒ higher suitability)

f_4 : Heavy metal concentration (lower concentration ⇒ higher suitability)

f_5 : Turbidity level (lower turbidity ⇒ higher suitability)

A panel of three environmental experts independently examined each lake with respect to every parameter f_j, $j = 1,2,\dots,5$, and expressed their assessments as Fermatean fuzzy values $\langle \mu, \lambda \rangle$, where μ represents the degree of suitability and λ the degree of unsuitability for ecological restoration, subject to the constraint $0 \le \mu^3 + \lambda^3 \le 1$. The consolidated expert ratings are recorded in Table 1.

Table 1. Expert assessments of water bodies with respect to environmental parameters.

	f_1	f_2	f_3	f_4	f_5
L_1	$\langle 0.70, 0.40\rangle$	$\langle 0.65, 0.35\rangle$	$\langle 0.50, 0.60\rangle$	$\langle 0.45, 0.55\rangle$	$\langle 0.60, 0.30\rangle$
L_2	$\langle 0.80, 0.30\rangle$	$\langle 0.40, 0.70\rangle$	$\langle 0.60, 0.50\rangle$	$\langle 0.55, 0.45\rangle$	$\langle 0.35, 0.65\rangle$
L_3	$\langle 0.55, 0.65\rangle$	$\langle 0.75, 0.25\rangle$	$\langle 0.90, 0.20\rangle$	$\langle 0.30, 0.80\rangle$	$\langle 0.45, 0.55\rangle$
L_4	$\langle 0.60, 0.50\rangle$	$\langle 0.55, 0.45\rangle$	$\langle 0.35, 0.75\rangle$	$\langle 0.70, 0.30\rangle$	$\langle 0.80, 0.20\rangle$

One can verify that all entries satisfy the Fermatean fuzzy condition $\mu^3 + \lambda^3 \leq 1$. For instance, for L_3 under f_3: $(0.90)^3 + (0.20)^3 = 0.729 + 0.008 = 0.737 \leq 1$. We now compute $\varepsilon_{\mathfrak{F}\mathcal{F}s}$ for each assessment. Since each table entry is a single Fermatean fuzzy value (n = 1), the entropy formula reduces to

$$\varepsilon_{\mathfrak{F}\mathcal{F}s}(\langle\mu, \lambda\rangle) = 1 - (\mu - \lambda)^2.$$

The individual entropy values are presented in Table 2, where the minimum entropy in each column is highlighted in bold.

Table 2. Entropy measure of each water body with respect to the environmental parameters.

	f_1	f_2	f_3	f_4	f_5
L_1	0.91	0.91	0.99	0.99	0.91
L_2	**0.75**	0.91	0.99	0.99	0.91
L_3	0.99	**0.75**	**0.51**	**0.75**	0.99
L_4	0.99	0.99	0.84	0.84	**0.64**

The aggregate entropy of each lake is obtained by summing the individual parameter entropies, as shown in Table 3.

Table 3. Aggregate entropy measure of each water body.

	$\varepsilon_{\mathfrak{F}\mathcal{F}s}(L_i)$
L_1	0.94
L_2	0.91
L_3	**0.80**

L_4	0.86

From Table 3, we obtain the ranking

$$\varepsilon_{\mathfrak{F}\mathcal{F}s}(\mathrm{L}_3) < \varepsilon_{\mathfrak{F}\mathcal{F}s}(\mathrm{L}_4) < \varepsilon_{\mathfrak{F}\mathcal{F}s}(\mathrm{L}_2) < \varepsilon_{\mathfrak{F}\mathcal{F}s}(\mathrm{L}_1).$$

A lower aggregate entropy indicates that the experts' collective assessment of a water body exhibits a more decisive and unambiguous differentiation between suitability and unsuitability across the environmental parameters, reflecting a clearer ecological profile. Consequently, L_3 attains the minimum entropy value of 3.99, signifying that its environmental status is most distinctly characterised and thus best identifiable for targeted restoration intervention. Therefore, the environmental authority should prioritise Lake L_3 for the ecological restoration programme.

Conclusion

In this paper, we have discussed Fermatean fuzzy Z-open map, Fermatean fuzzy Z-closed map in Fermatean fuzzy topological spaces. In future, this can be extended to Fermatean fuzzy Z homeomorphism functions. We present a measure of entropy and one application related to it. This measure is consistent with similar considerations for other sets like fuzzy sets and Fermatean fuzzy sets etc. Hence the proposed entropy measure can be used to measure the uncertainty factor in real life world.

References

[1] K. T. Atanassov (1983), Intuitionistic fuzzy sets, VII ITKRâ€™s Session, Sofia.

[2] K. T. Atanassov (1986), Intuitionistic fuzzy sets, Fuzzy Sets Syst. **20**, 87-96.

[3] K. T. Atanassov (1999), Intuitionistic fuzzy sets: theory and applications, Physica, Heidelberg.

[4] K. T. Atanassov (2012), On intuitionistic fuzzy sets theory, Springer, Berlin.

[5] K. Atanassov (2016), Review and new results on intuitionistic fuzzy sets, International Journal Bioautomation. **20**, S17-S26.

[6] C. L Chang, Fuzzy topological spaces, J. Math. Anal. Appl., **24** (1968), 182-190.

[7] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and systems, **88** (1997), 81-89.

[8] A. I. EL-Magharabi and A. M. Mubarki, Z-open sets and Z-continuity in topological spaces, International Journal of Mathematical Archive, **2** (10) (2011), 1819â€“1827.

[9] Hariwan Z. Ibrahim(2022), Fermatean fuzzy Topological Spaces, J. Appl. Math. and Informatics. **40**, 85-98.

[10] V. Kalaiyarasan, S. Tamilselvan, A. Vadivel and C. John Sundar, Normal spaces associated with fuzzy Z-open sets and its application, Journal of Mathematics and Computer Science, **29** (2) (2023), 156-166.

[11] M. Lellis Thivagar, S. Jafari, V. Sutha Devi and V. Antonysamy, A novel approach to topology via neutrosophic sets, Neutrosophic Sets and Systems, **20** (2018), 86-94.

[12] Murat Olgun, Mehmet Unver and Seyhmus Yardimci (2019), Pythagorean fuzzy topological spaces, Complex & Intelligent Systems. https://doi.org/10.1007/s40747-019-0095-2.

[13] V. Pankajam and K. Kavitha, δ-open sets and δ- continuity in δ- topological spaces, International Journal of Innovative Science and Research Technology, **2** (12) (2017), 110-118.

[14] S. Saha, Fuzzy δ-continuous mappings, Journal of Mathematical Analysis and Applications, **126** (1987), 130-142.

[15] T.Senapati and R.R.Yager(2020), Fermatean Fuzzy Sets, Journal of Ambient Intelligence and Humanized Computing **11**, 663-674.

[16] A. Padma, M. Saraswathi, A. Vadivel and G. Saravanakumar, New Notions of M-open Sets, Malaya Journal of Matematik, **S** (1) (2019), 656â€“660.

[17] R. Thangammal, M. Saraswathi, A. Vadivel and C. John Sundar, Fuzzy Z-open Sets in Fuzzy Topological Spaces, Journal of Linear and Topological Algebra Vol. 11, No. 01, 2022, 27- 38.

[18] R. Thangammal, M. Saraswathi, A. Vadivel, Samad Noeiaghdam, C. John Sundar, V. Govindan, Aiyared Iampan, Fuzzy Z-locally closed sets, extremally disconnected spaces, normal spaces, and their application, Advances in Fuzzy Systems, (2022), 3364170.

[19] A. Vadivel, C. John Sundar, K. Kirubadevi and S. Tamilselvan, More on Neutrosophic Open Sets, International Journal of Neutrosophic Science (IJNS), **18** (4) (2022), 204-222.

[20] A. Vadivel, C. John Sundar, K. Saraswathi and S. Tamilselvan, Neutrosophic M Open Sets, International Journal of Neutrosophic Science, **19** (1) (2022), 132-147.

[21] A. Vadivel, M. Seenivasan and C. John Sundar, An Introduction to δ-open sets in a Neutrosophic Topological Spaces, Journal of Physics: Conference Series, 1724 (2021), 012011.

[22] R. R. Yager (2013), Pythagorean membership grades in multicriteria decision making, In: Technical report MII-3301. Machine Intelligence Institute, Iona College, New Rochelle.

[23] R. R. Yager (2013), Pythagorean fuzzy subsets, In: Proceedings of the joint IFSA world congress NAFIPS annual meeting, 57-61.

[24] R. R. Yager (2014), Pythagorean membership grades in multicriteria decision making, IEEE Trans Fuzzy Syst. **22** (4), 958-965.

[25] L. A. Zadeh (1965), Fuzzy sets, Inf. Control, **8**, 338-353.

[26] L. Zadeh (1965), Fuzzy Sets and Systems, in Proc. Symp. on Systems Theory, Polytechnic Institute of Brooklyn, New York.