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Single Valued Pythagorean Neutrosophic Sub Implicative Ideals in KU-Algebras

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Abstract:

We introduce the notion of single-valued Pythagorean neutrosophic sub-implicative ideals in KU-algebras and examine their fundamental properties. Specifically, we establish conditions under which a single-valued Pythagorean neutrosophic ideal becomes a single-valued Pythagorean neutrosophic sub-implicative ideal. It is shown that every single-valued Pythagorean neutrosophic sub-implicative ideal is necessarily a single-valued Pythagorean neutrosophic ideal, while the converse does not hold in general. Furthermore, by employing the level set of a single-valued Pythagorean neutrosophic set in a KU-algebra, we provide a characterization of single-valued Pythagorean neutrosophic sub-implicative ideals.

Keywords: Single valued Pythagorean neutrosophic sub-algebra, Single valued Pythagorean neutrosophic KU-ideal.

1 Introduction

Prabpayak and Leerawat [11], [12] introduced a novel algebraic structure, referred to as KU-algebras, and initiated its systematic study. They focused on the theory of ideals and congruences within KU-algebras, thereby laying the foundation for further developments in this area. In addition, they defined the notion of homomorphisms between KU-algebras and explored a number of their fundamental properties. As a consequence of these investigations, they also derived several important results concerning the relationships between quotient KU-algebras and isomorphisms, which further deepened the structural understanding of this algebraic system.

Subsequently, Mostafa et al. [4], [5], [14] extended this line of research by introducing the concept of fuzzy KU-ideals of KU-algebras. Their work provided a framework for dealing with uncertainty in algebraic systems and established several basic properties connected with fuzzy KU-ideals. Later, Mostafa et al. [6] introduced further refinements, such as KU-sub-implicative, KU-positive implicative, and KU-sub-commutative ideals in KU-algebras, and investigated their structural and algebraic properties.

The incorporation of fuzziness into algebraic systems is rooted in the pioneering work of Zadeh [15], who introduced fuzzy set theory in 1965. Since its inception, fuzzy set theory has become a powerful and versatile mathematical tool to address problems characterized by vagueness, imprecision, and uncertainty. Over the years, this foundational concept has stimulated the development of several extensions and generalizations. For example, in 1986 Atanassov [2] introduced the notion of intuitionistic fuzzy sets, which extend fuzzy sets by assigning to each element not only a degree of membership but also a degree of non-membership, thereby allowing for a more nuanced representation of uncertainty.

Building on this progression, Smarandache [8], [9] proposed the concept of neutrosophic sets in 1998, which further generalizes both fuzzy and intuitionistic fuzzy sets. In a neutrosophic set, every element is characterized by three independent components: a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F). This richer framework provides greater flexibility in modeling uncertain, incomplete, and inconsistent information. However, due to its generality, the application of neutrosophic sets in practical contexts can be challenging.

To make neutrosophic theory more applicable, Wang et al. [13] introduced the concept of single-valued neutrosophic sets, a practically oriented instance of neutrosophic sets that can be effectively used in real-world problems, particularly in scientific, engineering, and decision-making applications. Parallel to this development, Agboola and Davvaz explored neutrosophic KU-algebras and neutrosophic KU-ideals, thereby bridging neutrosophic theory with the algebraic structure of KU-algebras.

Motivated by these developments, in this paper we focus on the study of single-valued neutrosophic sub-implicative ideals in KU-algebras. We establish their definition, investigate their structural properties, and provide characterizations that link them to existing concepts in the literature. This work contributes to the growing body of research that integrates algebraic structures with neutrosophic theory, thereby enhancing the scope of uncertainty modeling within abstract algebra.

2 Preliminaries

Now we will recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1 *[11], [12] Algebra $(X,*,0)$ of type $(2,0)$ is said a KU-algebra, if it satisfies the following axioms:*

(ku_1) $(x*y)*[(y*z)*(x*z)]=0$,

(ku_2) $x*0=0$,

(ku_3) $0*x=x$,

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,

(ku_5) $x*x=0$, for all $x,y,z\in X$.

On a KU-algebra $(X,*,0)$ we can define a binary relation $\leq$ on X by putting: $x\leq y \Leftrightarrow$

$y * x = 0$.

Thus a KU-algebra X satisfies the conditions:

$(ku_{1'})$: $(y * z) * (x * z) \leq (x * y)$,

$(ku_{2'})$: $0 \leq x$,

$(ku_{3'})$: $x \leq y, y \leq x$ implies $x = y$,

$(ku_{4'})$: $y * x \leq x$.

Remark 2.1 *Substituting* $z * x$ *for* x *and* $z * y$ *for* y *in* ku_1*, we get*

$$[(z * x) * (z * y)][((z * y) * z) * ((z * x) * z)] \leq [(z * x) * (z * y)] * [(z * x) * (z * y)] = 0$$

by (ku_1), hence $(x * y) * [(z * x) * (z * y)] = 0$ that mean the condition (ku_1) and $(x * y) * [(z * y) * (z * y)] = 0$ are equivalent.

For any elements x and y of a KU-algebra, $y * x^n$ denoted by $(\overbrace{y * x) * x\ldots * x}^{ntimes}$.

Theorem 2.1 *[5] In a* KU*-algebra* X*, the following axioms are satisfied: For all* $x, y, z \in X$*, [(i)]*

1. $x \leq y$ imply $y * z \leq x * z$,
2. $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$,
3. $((y * x) * x) \leq y$.
4. $(y * x^3) = (y * x)$. We will refer to X is a KU-algebra unless otherwise indicated.

Definition 2.2 *[11], [12] Let* I *be a non empty subset of a* KU*-algebra* X*. Then* I *is said to be an ideal of* X*, if*

(I_1) $0 \in I$,

(I_2) $\forall y, z \in X$, if $(y * z) \in I$ and $y \in I$, imply $z \in I$.

Definition 2.3 *[5] Let* I *be a non empty subset of a* KU*-algebra* X*. Then* I *is said to be an* KU*-ideal of* X*, if*

(I_1) $0 \in I$

(I_2) $\forall x, y, z \in X$, if $x * (y * z) \in I$ and $y \in I$, imply $x * z \in I$.

Definition 2.4 *[6]* KU*-algebra* X *is said to be implicative if it satisfies* $(x * y^2) = (x * y) * (y * x^2)$.

Definition 2.5 *[6] KU-algebra X is said to be commutative if it satisfies $x \le y$ implies* $(y * y^2) = x$.

Lemma 2.1 *[6] Let X be a KU-algebra. X is KU-implicative iff X is KU-positive implicative and KU-commutative.*

Definition 2.6 *[6] A non empty subset A of a KU-algebra X is called a $KUsub$ implicative ideal of X, if $\forall x, y, z \in X$, [(i)]*

1. $0 \in A$
2. $z * ((x * y) * (y * x^2)) \in A$ and $z \in A$, imply $(x * y^2) \in A$.

Definition 2.7 *[6] Let $(X, *, 0)$ be a KU-algebra, a nonempty subset A of X is said to be a KU-postive implicative ideal if it satisfies, for all x, y, z in X. [(i)]*

1. $0 \in A$,
2. $z * (x * y) \in A$ and $z * x \in A$ imply $z * y \in A$.

Definition 2.8 *[6] A non empty subset A of a KU-algebra X is called a KU-sub commutative ideal of X, if [(i)]*

1. $0 \in A$
2. $z * \{((y * x^2) * y^2)\} \in A$ and $z \in A$, imply $(y * x^2) \in A$.

Definition 2.9 *[6] A nonempty subset A of a KU-algebra X is called a kp-ideal of X if it satisfies [(i)]*

1. $0 \in A$,
2. $(z * y) * (z * x) \in A,\ y \in A \Rightarrow x \in A$.

Definition 2.10 *[13] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set ($SVNS$) A in X is characterized by truth-membership function μ_A, indeterminacy-membership function ν_A and falsity-membership function λ_A. For each point x in X, $\mu_A(x)$, $\nu_A(x)$, $\lambda_A(x) \in [0,1]$. When X is continuous, a $SVNS$ A can be written as $A = \int_x \langle \mu(x), \nu x, \lambda(x) \rangle / x,\ x \in X$. When X is discrete, a $SVNS$ A can be written as $A = \sum_{i=1}^{n} \langle \mu(x_i), \nu(x_i), \lambda(x_i) \rangle | x_i \in X$. Consider parameters such as capability, trustworthiness and price of semantic Web services. These parameters are commonly used to define quality of service of semantic Web services.*

Definition 2.11 *[7] Let X be a non-empty set (Universe) A Pythagorean neutrosophic set (briefly, PNS) T and F as dependent neutrosophic components A on X is an object of the form* $\mathcal{P} = \{\langle x, \mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x), \lambda_{\mathcal{P}}(x) \rangle | x \in X\}$,

where $\mu_{\mathcal{P}}(x)$, $\nu_{\mathcal{P}}(x)$, $\lambda_{\mathcal{P}}(x)$ are the truth, indeterminacy and false respectively such that

$\mu, \nu, \lambda \in [0,1]$. Here when μ and λ are dependent components, then for all x in X; (i) $\mu + \lambda \leq 1$, (ii) $0 \leq \mu^2 + \lambda^2 \leq 1$, (iii) $0 \leq \mu^2 + \nu^2 + \lambda^2 \leq 2$.

We define these basic operations on PNS which can be described as follows: Let X be a nonempty set (universe). A Pythagorean Neutrosophic set μ and λ as dependent neutrosophic components $\mathcal{P}$ and $\mathcal{Q}$ of the form $\mathcal{P} = \{\langle x, \mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x), \lambda_{\mathcal{P}}(x)\rangle | x \in X\}$ and $\mathcal{Q} = \{\langle x, \mu_{\mathcal{Q}}(x), \nu_{\mathcal{Q}}(x), \lambda_{\mathcal{Q}}(x)\rangle | x \in X\}$. The complement of $\mathcal{P}$ is $\mathcal{P}^c = \{\langle x, \lambda_{\mathcal{P}}(x), 1 - \nu_{\mathcal{P}}(x), \mu_{\mathcal{P}}(x)\rangle | x \in X\}$. The union and intersection of $\mathcal{P}$ and $\mathcal{Q}$ are [(i)]

1. $\mathcal{P} \cup \mathcal{Q} = \{max(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), min(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), min(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$;
2. $\mathcal{P} \cap \mathcal{Q} = \{min(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), max(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), max(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$.

3 Single valued Pythagorean neutrosophic sub implicative ideals of $\boldsymbol{KU}$-Algebras

Definition 3.1 *Let X be a KU-algebra, a Pythagorean neutrosophic set*

$A := \{\langle x, \mu_A(x), \nu_A(X), \lambda_A(x)\rangle | x \in X\}$

in X is called a single valued Pythagorean neutrosophic ideal (briefly $SVPNI$) of X if it satisfies the following conditions: [(i)]

1. $\mu_A(0) \geq \mu_A(x)$, $\nu_A(0) \geq \nu_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in X$,
2. $\forall x, y \in X$, $\mu_A(y) \geq \min\{\mu_A(x * y), \mu_A(x)\}$,
3. $\forall x, y \in X$, $\nu_A(y) \leq \max\{\nu_A(x * y), \nu_A(x)\}$,
4. $\forall x, y \in X$, $\lambda_A(y) \leq \max\{\lambda_A(x * y), \lambda_A(x)\}$.

Definition 3.2 *A non empty subset $A := \{\langle x, \mu_A(x), \nu_A(x), \lambda_A(x)\rangle | x \in X\}$ of a KU-algebra X is called a single valued Pythagorean neutrosophic sub implicative ideal (briefly $PNsubimpI$) of X, if* $\forall\, x, y, z \in X$,

1. $\mu_A(0) \geq \mu_A(x)$, $\nu_A(0) \geq \nu_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$,
2. $\mu_A(x * y^2) \geq \min\{\mu_A(z * ((x * y) * (y * x^2))), \mu_A(z)\}$,
3. $\nu_A(x * y^2) \leq \max\{\nu_A(z * ((x * y) * (y * x^2))), \nu_A(z)\}$,
4. $\lambda_A(x * y^2) \leq \max\{\lambda_A(z * ((x * y) * (y * x^2))), \lambda_A(z)\}$.

Example 3.1 *Let $X = \{a, b, c, d\}$ be a set with a binary operation $*$ defined by the following table:*

$*$	a	b	c	d
a	a	b	c	d
b	a	a	a	c

c	a	c	a	b
d	a	a	a	a

Let $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be an Pythagorean neutrosophic set in X defined by $\mu_A(a)=\mu_A(c)=0.9$, $\mu_A(d)=\mu_A(b)=0.4$, $\nu_A(a)=\nu_A(c)=0.8$, $\nu_A(d)=\nu_A(b)=0.4$ and $\lambda_A(a)=\lambda_A(c)=0.2$, $\lambda_A(d)=\lambda_A(b)=0.5$, by routine calculations we know that $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ is $PNSI$-ideal of algebra of X.

Proposition 3.1 *Every $PNsubimpI$ of a KU-algebra X is order reversing.*

Proof. Let $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be $PNsubimpI$ of X and let $x,y,z\in X$ be such that $x\le z$, then $z*x=0$. Let $y=x$ in $(PNI2)$, $(PNSI3)$ and $(PNSI4)$, we get

$\mu_A(x)\ge \min\{\mu_A(z*x),\mu_A(z)\}=\min\{\mu_A(0),\mu_A(z)\}=\mu_A(z)$,

$\nu_A(x)\le \max\{\nu_A(z*x),\nu_A(z)\}=\max\{\nu_A(0),\nu_A(z)\}=\nu_A(z)$ and

$\lambda_A(x)\le \max\{\lambda_A(z*x),\lambda_A(z)\}=\max\{\lambda_A(0),\lambda_A(z)\}=\lambda_A(z)$.

This completes the proof

Lemma 3.1 *Let $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be a $PNsubimpI$ of KU-algebra X, if the inequality $y\star x\le z$ hold in X , then $\mu_A(y)\ge min\{\mu_A(x),\mu_A(z)\}$, $\nu_A(y)\le max\{\nu_A(x),\nu_A(z)\}$ and $\lambda_A(y)\le max\{\lambda_A(x),\lambda_A(z)\}$.*

Proof. Let $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be $PNsubimpI$ of X and let $x,y,z\in X$ be such that $y*x\le z$, then $z*(y*x)=0$ or $y*(z*x)=0$ i.e., $z*x\le y$, we get (by Proposition 3.1),

$$\mu_A(z*x)\ge\mu_A(y),\nu_A(z*x)\le\nu_A(y) \text{ and } \lambda_A(z*x)\le\lambda_A(y). \qquad (1)$$

Put in (PNI), $(PNSI2)$ and $(PNSI3)$, $x=y$, we get,

$\mu_A(x*x^2)\ge \min\left\{\mu_A(z*((\overbrace{x*x}^{0})*(\overbrace{x*x^2}^{x}))),\mu_A(z)\right\}=\min\{\mu_a(z*x),\mu_A(z)\}$,

i.e. $\mu_A(x)\ge\min\{\mu_A(z*x),\mu_A(z)\}\ge\min\{\mu_A(y),\mu_A(z)\}$ by (1)

$\nu_A(x*x^2)\le \max\left\{\nu_A(z*((\overbrace{x*x}^{0})*(\overbrace{x*x^2}^{x}))),\nu_A(z)\right\}=\max\{\nu_a(z*x),\nu_A(z)\}$,

i.e. $\nu_A(x)\le\max\{\nu_A(z*x),\nu_A(z)\}\le\max\{\nu_A(y),\nu_A(z)\}$ by (1), and

$\lambda_A(x*x^2)\le \max\{\lambda_A(z*((\overbrace{x*x}^{0})*(\overbrace{x*x^2}^{x}))),\lambda_A(z)\}=\max\{\lambda_a(z*x),\lambda_A(z)\}$,

i.e. $\lambda_A(x)\le\max\{\lambda_A(z*x),\lambda_A(z)\}\le\max\{\lambda_A(y),\lambda_A(z)\}$ by (1). This completes the proof.

Lemma 3.2 *If X is implicative KU-algebra, then every PN ideal of X is an $PNsubimpI$ of X.*

Proof. Let $A:=\{\langle x,\mu_A(x),\nu_A(X),\lambda_A(x)\rangle|x\in X\}$ be PN ideal of X. Substituting $x*y^2$ for y in $(PNSI2),(PNSI3)$ and $(PNSI4)$, we get

$\mu_A(x*y^2)\geq\min\{\mu_A(z*(x*y^2)),\mu_A(z)\}$, but KU-algebra is implicative i.e.

$\nu_A(x*y^2)\leq\max\{\nu_A(z*(x*y^2)),\nu_A(z)\}$ and

$\lambda_A(x*y^2)\leq\max\{\lambda_A(z*(x*y^2)),\lambda_A(z)\}$, but KU-algebra is implicative i.e.

$(x*y^2)=(x*y)*(y*x^2)$, hence

$\mu_A(x*y^2)\geq\min\{\mu_A(z*(x*y^2)),\mu(z)\}=\min\{\mu_A(z*(x*y)*(y*x^2)),\mu_A(z)\}$,

$\nu_A(x*y^2)\leq\max\{\nu_A(z*(x*y^2)),\nu(z)\}=\max\{\nu_A(z*(x*y)*(y*x^2)),\nu_A(z)\}$ and

$\lambda_A(x*y^2)\leq\max\{\lambda_A(z*(x*y^2)),\lambda(z)\}=\max\{\lambda_A(z*(x*y)*(y*x^2)),\lambda_A(z)\}$,

which shows that $A:=\{\langle x,\mu_A(x),\nu_A(x),\lambda_A(x)\rangle|x\in X\}$ is $PNsubimpI$ of X. This completes the proof.

Theorem 3.1 *Let $A:=\{\langle x,\mu_A(x),\nu_A(X),\lambda_A(x)\rangle|x\in X\}$ be PN set of KU-algebra X satisfying the conditions $(PNSI2)$, $(PNSI3)$ and $(PNSI4)$ then $A:=\{\langle x,\mu_A(x),\nu_A(X),\lambda_A(x)\rangle|x\in X\}$ satisfies the following inequalities;*

$(PNSI5)$ $\mu_A(x*y^2)\geq\mu_A((x*y)*((y*x^2))$

$(PNSI6)$ $\nu_A(x*y^2)\geq\nu_A((x*y)*((y*x^2))$

$(PNSI7)$ $\lambda_A(x*y^2)\leq\lambda_A((x*y)*((y*x^2))$.

Proof. Let $A:=\{\langle x,\mu_A(x),\nu_A(X),\lambda_A(x)\rangle|x\in X\}$ satisfying conditions $(PNSI2)$, $(PNSI3)$ and $(PNSI4)$ i.e.

$(PNSI2)$ $\mu_A(x*y^2)\geq\min\{\mu_A(z*((x*y)*(y*x^2))),\mu_A(z)\}$

$(PNSI3)$ $\nu_A(x*y^2)\leq\max\{\nu_A(z*((x*y)*(y*x^2))),\nu_A(z)\}$

$(PNSI4)$ $\lambda_A(x*y^2)\leq\max\{\lambda_A(z*((x*y)*(y*x^2))),\lambda_A(z)\}$

then by taking $z=0$ in $(PNSI2)$, $(PNSI3)$ and $(PNSI4)$ and using (PNI) $\mu_A(0)\geq\mu_A(x)$, $\nu_A(0)\leq\nu_A(x)$, $\lambda_A(0)\leq\lambda_A(x)$ and (ku_3) we get

$\mu_A(x*y^2)\geq\min\{\mu_A(0*((x*y)*(y*x^2))),\mu_A(0)\}=\mu_A((x*y)*(y*x^2))$.

$\nu_A(x*y^2)\leq\max\{\nu_A(0*((x*y)*(y*x^2))),\nu_A(z)\}=\nu_A((x*y)*((y*x^2))$.

$\lambda_A(x*y^2)\leq\max\{\lambda_A(0*((x*y)*(y*x^2))),\lambda_A(z)\}=\lambda_A((x*y)*(y*x^2))$. This completes the proof.

Theorem 3.2 *Every $PNsubimpI$ of a KU-algebra X is a PN-ideal, but the converse does not hold.*

Proof. Let $A := \{\langle x, \mu_A, \nu_A, \lambda_A \rangle | x \in X\}$ be $PNsubimpI$ of X; put $x = y$ in $(PNSI2), (PNSI3)$ and $(PNSI4)$, we get

$$\mu_A \overbrace{(x * x^2)}^{x} \geq \min\{\mu_A(z * ((x * x) * (x * x^2))), \mu_A(z)\}, \text{ then}$$

$$\mu_A(x) \geq \min\left\{\mu_A(z * (\overbrace{(x * x)}^{0} * \overbrace{(x * x)}^{1})), \mu_A(z)\right\} = \min\{\mu_A(z * x), \mu_A(z)\}$$

$$\nu_A \overbrace{(x * x^2)}^{x} \leq \max\{\nu_A(z * ((x * x) * (x * x^2))), \nu_A(z)\}, \text{ therefore}$$

$$\nu_A(x) \leq \max\left\{\nu_A(z * (\overbrace{(x * x)}^{0} * (\overbrace{(x * x)}^{1}))), \nu_A(z)\right\} = \max\{\nu_A(z * x), \nu_A(z)\}, \text{ and}$$

$$\lambda_A \overbrace{(x * x^2)}^{0} \leq \max\{\lambda_A(z * ((x * x) * (x * x^2))), \lambda_A(z)\}, \text{ we get}$$

$\lambda_A(x) \leq \max\{\lambda_A(z * (\overbrace{(x * x)}^{0} * \overbrace{(x * x^2)}^{1})), \lambda_A(z)\} = \max\{\lambda_A(z * x), \lambda_A(z)\}$. Hence $A := \{x, \mu_A, \nu_A, \lambda_A | x \in X\}$ is a PN-ideal of X. This completes the proof.

The following example shows that the converse of Theorem 3.2 may not be true.

Example 3.2 *Let $X = \{a, b, c, d, e\}$ in which the operation $*$ is given by the table*

*	a	b	c	d	e
a	a	b	c	d	e
b	a	a	b	d	e
c	a	a	a	d	e
d	a	a	a	a	e
e	a	a	a	a	a

Then $(X,*,0)$ is a KU-Algebra. Define a fuzzy set $\mu_(A): X \to [0,1]$, $\nu_(A): X \to [0,1]$ and $\lambda_(A): X \to [0,1]$ by $\mu_A(a) = 0.9, \mu_A(b) = \mu_A(c) = \mu_A(d) = \mu_A(e) = 0.4$, $\nu_A(a) = 0.8, \nu_A(b) = \nu_A(c) = \nu_A(d) = \nu_A(e) = 0.4$, $\lambda_A(a) = 0.2, \lambda_A(b) = \lambda_A(c) = \lambda_A(d) = \lambda_A(e) = 0.5$, we get for $z = a$, $x = a$ and $y = c$, L.H.S of $(PNSI1)$ $\mu_A((b*c)*c) = \mu_A(b) = 0.4$, $\nu_A((b*c)*c) = \nu_A(b) = 0.4$, $\lambda_A((b*c)*c) = \lambda_A(b) = 0.5$,

R.H.S of $(PNSI1)$ $\min\{\mu_A(a*(b*c)*((c*b)*b)), \mu_A(a)\} = \mu_A(a) = 0.9$,

$\min\{\nu_A(a*(b*c)*((c*b)*b)), \nu_A(a)\} = \nu_A(a) = 0.8$,

$\max\{\lambda_A(a*(b*c)*((c*b)*b)), \lambda_A(a)\} = \lambda_A(a) = 0.2$.

i.e. In this case

$\mu_A(x*y^2) \ngeq \min = min\{\mu_A(z*((x*y)*(y*x^2))), \mu_A(z)\}$,

$\nu_A(x*y^2) \ngeq \min = min\{\nu_A(z*((x*y)*(y*x^2))), \nu_A(z)\}$,

$\lambda_A(x*y^2) \ngeq \min = min\{\lambda_A(z*((x*y)*(y*x^2))), \lambda_A(z)\}$.

We now give a condition for a PN-ideal to be a $PNsubimpI$.

Theorem 3.3 *Every PN-ideal $A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ of X satisfying the conditions $(PNSI5), (PNSI6)$, and $(PNSI7)$ is a $PNsubimpI$ of X.*

Proof. Let $A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ be an PN-ideal of X satisfying conditions $(PNSI5)$, $(PNSI6)$, and $(PNSI7)$, we get $\mu_A(x*y^2) \geq \{\mu_A(x*y)*(y*x^2)\}$, $\nu_A(x*y^2) \leq \{\nu_A(x*y)*(y*x^2)\}$ and $\lambda_A(x*y^2) \leq \{\lambda_A((x*y)*(y*x^2))\}$. Therefore $\mu_A(x*y^2) \geq \min\{\mu_A(z*((x*y)*(y*x^2))), \mu_A(z)\}$,

$\nu_A(x*y^2) \leq \max\{\nu_A(z*((x*y)*(y*x^2))), \nu_A(z)\}$, and

$$\lambda_A(x*y^2) \leq \max\{\lambda_A(z*((x*y)*(y*x^2))), \lambda_A(z)\}$$

by (Definition of PN-ideal $(PN2), (PN3), (PN4)$), we get

$\mu_A(x*y^2) \geq \mu_A((x*y)*(y*x^2)) \geq \min\{\mu_A(z*((x*y)*(y*x^2))), \mu_A(z)\}$,

$\nu_A(x*y^2) \leq \nu_A((x*y)*(y*x^2)) \leq \max\{\nu_A(z*((x*y)*(y*x^2))), \nu_A(z)\}$ and

$\lambda_A(x*y^2) \leq \lambda_A((x*y)*(y*x^2)) \leq \max\{\lambda_A(z*((x*y)*(y*x^2))), \lambda_A(z)\}$,

which proves the condition $(PNSI2), (PNSI3), (PNSI4)$. This completes the proof.

Theorem 3.4 *Let $A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ be PN-ideal of X. Then the following are equivalent: [(i)]*

1. $A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ is an $PNsubimpI$ of X,
2. $\mu_A(x*y^2) \geq \mu_A(z*((x*y)*(y*x^2))), \nu_A(x*y^2) \leq \nu_A(z*((x*y)*(y*x^2)))$, and $\lambda_A(x*y^2) \leq \lambda_A(z*((x*y)*(y*x^2)))$,

3. $\mu_A(x*y^2) \geq \mu_A((x*y)*(y*x^2))$, $\nu_A(x*y^2) \geq \nu_A(x*y)*((y*x^2))$, and $\lambda_A(x*y^2) \leq \lambda_A((x*y)*(y*x^2))$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be $PNSI$-ideal of X. By $(PNSI2),(PNSI3),(PNSI4)$ and $(PN1)$ we have

$\mu_A(x*y^2) \geq \min\{\mu_A(0*((x*y)*(y*x^2))),\mu_A(0)\} = \mu_A(0*((x*y)*(y*x^2)))$ i.e.

$\mu_A(x*y^2) \geq \mu_A((x*y)*(y*x^2))$,

$\nu_A(x*y^2) \leq \max\{\nu_A(0*((x*y)*(y*x^2))),\nu_A(0)\} = \nu_A(0*((x*y)*(y*x^2)))$ i.e.

$$\nu_A(x*y^2) \leq \nu_A((x*y)*(y*x^2))$$

and $\lambda_A(x*y^2) \leq \max\{\lambda_A(0*((x*y)*(y*x^2))),\lambda_A(0)\} = \lambda_A(0*((x*y)*(y*x^2)$ i.e.

$\lambda_A(x*y^2) \leq \lambda_A((x*y)*(y*x^2))$.

(ii) $\Rightarrow$ (iii): Since $(x*y)*(y*x^2) \leq x*y^2$, by Lemma 3.1 we obtain,

$\mu_A(x*y^2) \geq \mu_A((x*y)*(y*x^2))$, $\nu_A(x*y^2) \leq \nu_A((x*y)*(y*x^2))$, and

$\lambda_A(x*y^2) \leq \lambda_A((x*y)*(y*x^2))$.

Combining (ii) we have

$\mu_A(x*y^2) \geq \mu_A((x*y)*(y*x^2))$, $\nu_A(x*y^2) \geq \nu_A((x*y)*(y*x^2))$, and

$\lambda_A(x*y^2) \leq \lambda_A(x*y)*(y*x^2)$.

(iii) $\Rightarrow$ (i): Since $[(z*((x*y)*(y*x^2)))]*[(x*y)*(y*x^2)] =$

$$[(x*y)*(z*((y*x^2)))]*[(x*y)*(y*x^2)] \leq$$

$[(z*(y*x^2))]*[(y*x^2)] = [(z*(y*x^2))]*[0*(y*x^2)] \leq 0*z = z$.

By (Lemma 3.1) we obtain

$\mu_A((x*y)*(y*x^2)) \geq \min\{\mu_A((x*y)*(y*x^2)),\mu_A(z)\}$,

$\nu_A(x*y)*((y*x^2)) \leq \max\{\nu_A((x*y)*(y*x^2)),\nu_A(z)\}$ and

$\lambda_A((x*y)*(y*x^2)) \leq \max\{\lambda_A((x*y)*(y*x^2)),\lambda_A(z)\}$.

From (iii), we have $\mu_A(x*y^2) \geq \min\{\mu_A(z*((x*y)*((y*x^2))),\mu_A(z)\}$,

$\nu_A(x*y^2) \leq \max\{\nu_A(z*((x*y)*(y*x^2))),\nu_A(z)\}$, and

$\lambda_A(x*y^2) \leq \max\{\lambda_A(z*((x*y)*(y*x^2))),\lambda_A(z)\}$.

Hence $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$ be $PNSI$-ideal of X. The proof is complete.

Theorem 3.5 *A single valued Pythagorean neutrosophic set* $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x\in X\}$

of a KU-algebra X is a PNsubimpI of X if and only if $A_{t,s,m} = \{\langle x \in X | \mu_A \geq t, \nu_A \leq s, \lambda_A \leq m\rangle\} \neq \Phi$, *is a sub-implicative ideal of X.*

Proof. Suppose that $A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ is a single valued Pythagorean neutrosophic sub-implicative ideal of X $A_{t,s,m} \neq \Phi$ for any $t, s, m \in (0,1]$, there exists $x \in A_{t,s,m}$ so that $\mu_A \geq t$, $\nu_A \leq s$, $\lambda_A \leq m$. It follows from $(PF1)$ that $\mu_A(0) \geq \mu_A(x) \geq t$, $\nu_A(0) \leq \nu_A(x) \leq s$, $\lambda_A(0) \leq \mu_A(x) \leq m$ so that $0 \in A_{t,s,m}$. Let $x, y, z \in X$ be such that

$z * ((x * y) * (y * x^2)) \in A_{t,s,m}$ and $z \in A_{t,s,m}$. Using $(PNSI2), (PNSI3), (PNSI4)$,

we know that

$$\mu_A(x * y^2) \geq \min\{\mu_A(z * ((x * y) * (y * x^2))), \mu_A(z)\} = \min\{t, t\} = t,$$

$\nu_A(x * y^2) \leq \max\{\nu_A(z * ((x * y) * (y * x^2))), \nu_A(z)\} = \max\{s, s\} = s$, and

$\lambda_A(x * y^2) \leq \max\{\lambda_A(z * ((x * y) * (y * x^2))), \lambda_A(z)\} = \max\{m, m\} = m$,

thus $x * y^2 \in A_{t,s,m}$. Hence $A_{t,s,m}$ is a sub-implicative ideal of X. Conversely, suppose that $A_{t,s,m} \neq \Phi$ is a sub-implicative ideal of X, for every $t, s, m \in (0,1]$ and any $x \in X$, let $\mu_A(x) = t$, $\nu_A(x) = s$ and $\lambda_A(x) = m$. Then $x \in A_{t,s,m}$. Since $0 \in A_{t,s,m}$, it follows that $\mu_A(0) \geq t = \mu_A(x)$, $\nu_A(0) \geq s = \nu_A(x)$, $\lambda_A(0) \leq m = \lambda_A(x)$ so that $\mu_A(0) \geq \mu_A(x)$, $\nu_A(0) \leq \nu_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$, for all $x \in X$. Now, we need to show that

$A := \{\langle x, \mu_A, \nu_A, \lambda_A\rangle | x \in X\}$ satisfies $(PNSI2), (PNSI3), (PNSI4)$.

If not, then there exist $a, b, c \in X$ such that

$$\mu_A(a * b^2) \leq \min\{\mu_A(c * ((a * b) * (b * a^2))), \mu_A(c)\}$$

$\nu_A(a * b^2) \geq \max\{\nu_A(c * ((a * b) * (b * a^2))), \nu_A(c)\}$, and

$\lambda_A(a * b^2) \geq \max\{\lambda_A(c * ((a * b) * (b * a^2))), \lambda_A(c)\}$.

Taking

$$t_0 = \frac{1}{2}(\mu_A(a * b^2)) + \{\mu_A(c * ((a * b) * (b * a^2))), \mu_A(c)\}$$

$s_0 = \frac{1}{2}(\nu_A(a * b^2)) + \{\nu_A(c * ((a * b) * (b * a^2))), \nu_A(c)\}$ and

$m_0 = \frac{1}{2}(\lambda_A(a * b^2)) + \{\lambda_A(c * ((a * b) * (b * a^2))), \lambda_A(c)\}$,

then we have

$$\mu_A(a * b^2) < t_0 < \{\mu_A(c * ((a * b) * (b * a^2))), \mu_A(c)\}$$

$$\nu_A(a * b^2) > s_0 > \{\nu_A(c * ((a * b) * (b * a^2))), \nu_A(c)\}$$

$$\lambda_A(a * b^2) > m_0 > \{\lambda_A(c * ((a * b) * (b * a^2))), \lambda_A(c)\}$$

Hence $c*((a*b)*(b*a^2)) \in A_{t,s,m}$ and $c \in A_{t,s,m}$, but $a*b^2 \notin A_{t,s,m}$ which means that $A_{t,s,m}$ is not a sub-implicative ideal of X, this is contradiction. Therefore $A:=\{\langle x,\mu_A,\nu_A,\lambda_A\rangle | x \in X\}$ is a single valued Pythagorean neutrosophic sub-implicative ideal of X.

4 Conclusion

In this paper, we have introduced the notion of single-valued Pythagorean neutrosophic sub-implicative ideals in KU-algebras and investigated several of their fundamental properties. We believe that these definitions and results can be naturally extended to other fuzzy algebraic structures such as hypergroups, hypersemigroups, and hyperrings. It is our expectation that this study will provide a foundation for further exploration in the theory of BCK/BCI-KU-algebras. Moreover, the results obtained here may find potential applications in diverse areas including engineering, soft computing, and medical diagnosis. In our future research on single-valued Pythagorean neutrosophic sub-commutative ideal structures of KU-algebras, the following directions may be considered: Establishing single-valued Pythagorean neutrosophic (s-weakâ€“strong) hyper KU-ideals in hyper KU-algebras. Deriving further results on single-valued Pythagorean neutrosophic ideals in hyper KU-algebras and exploring their applications. Investigating the structure of single-valued Pythagorean neutrosophic dot (s-weakâ€“strong) hyper KU-ideals in hyper KU-algebras.

References

[1] A. A. A. Agboola and B. Davvaz, *Introduction to Neutrosophic BCI/BCK-algebras*, Int. J. Math. Math. Sci., (2015), Article ID 370267.

[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy sets and systems, 20 (1) (1986), 87-96.

[3] B. Davvaz, S. M. Mostafa and F. Kareem, *Neutrosophic ideals of neutrosophic KU-algebras*, Gazi University Journal of Science, 30 (4) (2017), 463-472.

[4] S. M. Mostafa, M. A. Abdel Naby and O. R. Elgendy, *Intuitionistic fuzzy KU-ideals in KU-algebras*, Int. J. of Mathematical Sciences and Applications, 1 (3) (2011).

[5] S. M. Mostafa, M. A. Abd-Elnaby and M. M. M. Yousef, *Fuzzy ideals of KU-algebras*, Int. Math. Forum, 6 (63) (2011), 3139-3149.

[6] S. M. Mostafa, R. A. K. Omar and O. W. Abd El- Baseer, *Sub implicative ideals of KU-Algebras*, International Journal of Modern Science and Technology, 2 (5) (2017), 223-227.

[7] R. Radha, A. S. A. Mary, R. Prema and S. Broumi, *Neutrosophic Pythagorean sets with dependent neutrosophic Pythagorean components and its improved correlation coefficients*, Neutrosophic Sets Syst., **46** (1) (2021), 77-86.

[8] F. Smarandache, *Neutrosophy and neutrosophic logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico*, Gallup, NM 87301, USA (2002).

[9] F. Smarandache, *Neutrosophic set, a generalization of the intuitionistic fuzzy sets*, Inter. J. Pure Appl. Math., **24** (2005), 287-297.

[10] F. Smarandache, *A Unifying field in logics: Neutrosophic logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, American Research Press, Rehoboth, NM, USA, 2003.

[11] C. Prabpayak and U. Leerawat, *On ideals and congruence in KU-algebras*, Scientia Magna Journal, 5 (1) (2009), 54-57.

[12] C. Prabpayak and U. Leerawat, *On isomorphisms of KU-algebras*, Scientiamagna journal, **5** (3) (2009), 25-31.

[13] H. Wang, F. Smarandache, Y. Zhang and R. Sunderraman, *Single valued neutrosophic sets*, Technical sciences and applied Mathematics, (2012), 10-14.

[14] N. Yaqoob, S. M. Mostafa and M. A. Ansari, *On cubic KU-ideals of KU-algebras*, ISRN Algebra, (2013) Article ID935905.

[15] L. A. Zadeh, *Fuzzy sets*, Inform. Control, 8 (1965), 338â€“353.