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The exponential behavior of a stochastic globally modified Cahn-Hilliard-Magnetohydrodynamic model with multiplicative noise

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Abstract

In this work, we explore the stability of weak solutions to a stochastic version of a globally modified coupled Cahn-Hilliard-Magnetohydrodynamic model with multiplicative noise. The model describes the flow of the mixture of two incompressible, immiscible fluids under the influence of an electomagnetic field with stochastic pertubations. This system consists of the globally modified Magnetohydrodynamic model for the velocity and magnetic field, coupled with a Cahn-Hilliard equation for the order (phase) parameter. We prove that the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions. We also show a result related to the stabilization of these equations.

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Keywords

Globally modified, Stochastic Magnetohydrodynamic, Cahn-Hilliard, Weak solutions, Stabilization, Stability.

1 Introduction

The stochastic Navier-Stokes equations describe the motion of fluids in the presence of random fluctuations, unlike deterministic equations which only take into account known forces and initial conditions (see [5, 9, 10, 11, 12, 18, 19, 30, 48, 54, 76]). These stochastic equations are used to model physical phenomena where uncertainties or random disturbances play an important role, such as turbulence, flows in porous media, or even fluid dynamics in complex media. By coupling the Navier-Stokes equations and the Maxwell equations of electromagnetism through the Lorentz force and Ohm's law, we obtain the magnetohydrodynamic (MHD) model which studies the behavior of a current-carrying fluid in the presence of an electromagnetic field, and which applies in particular to plasmas, even to sea water, and to several fields of applied science.

In previous years, the deterministic MHD system has been explored by famous scientists (e.g. Cowling [23], Chandrasekhar [20], Gerbeau [35], Planas [52], Schötzau [56], Temam [77], Sundar [62], Sango [55], Motyl [49], Sritharan [59]). Subsequently, the authors in [4, 22, 55] studied the existence and the uniqueness of the solutions for the stochastic MHD equations. The general framework is applied to the coupled stochastic Navier-Stokes, magnetohydrodynamics, and the Boussinesq equations. The authors in [75] established the existence and uniqueness of strong solutions for the stochastic three dimensional MHD equations with a multiplicative noise.

The study of the interaction of the electromagnetic field with the mixture of two incompressible and immiscible fluids is of great importance in engineering, and some analytical studies on the flow of the mixture of two immiscible fluids in a channel under an external magnetic field are carried out in MHD generators and pump accelerators (see [20, 22, 23, 28, 35, 36, 44, 45, 46, 47, 52, 55, 56, 58, 59, 62]). The Cahn-Hilliard-Magnetohydrodynamics (CH-MHD) system was introduced in [41, 74, 78] and describes the flow of the mixture of two incompressible, immiscible and electrically conducting fluids. The model consists of the Cahn-Hilliard, Navier-Stokes equations and Maxwell equations (magnetic field) which are nonlinearly coupled through convection, stresses and Lorentz forces. In [78], the authors proposed a fully discrete energy stable finite elements method with a semi-implicit scheme. The scheme preserves the mass conservation and the discrete energy law. They proved that there exist subsequences of discrete solutions which converge to a weak solution of the model for vanishing discretization parameters. The proof uses the stability of the scheme and the compactness method. In [74, 41], the authors prove the existence, uniqueness of the solutions for the stochastic CH-MHD and Allen-Cahn-Magnetohydrodynamics (AC-MHD) system driven by Jump noise respectively. When the magnetic field is withdrawn from the CH-MHD system, the remaining system is the Cahn-Hilliard-Navier-Stokes (CH-NS) system which is a diffuse interface model for incompressible isothermal mixture of two immiscible fluids and consists of the Navier-Stokes equations coupled with a convective Cahn-Hilliard equation for the relative concentration difference of both fluids (see [17, 24, 25, 31, 32, 40, 61, 65, 67, 68, 71, 73]). Moreover, stochastic partial differential equations serve as a mathematical model for systems involving two types of forces, one deterministic, the other random and these equations make it possible to take into account a random noise in the evolution of a phenomenon (see [24, 40, 62]). This may reflect, for instance, some environmental effects on the phenomena or some external random forces. In the theory of fluid dynamic, the long-time behavior of flows is a very interesting problem and the literature shows that the problem has been receiving very much attention over the three decades (see [5, 15, 16, 27, 39, 54, 64, 77]). The exponential behavior of a stochastic Cahn-Hilliard-Navier-Stokes (CH-NS) model with multiplicative noise was studied in [24, 71].

The above references motivate our work. We prove the stability of weak solutions to the stochastic 3D globally modified Cahn-Hilliard-Magnetohydrodynamic (GMCHMHD) model with multiplicative noise. Among other things, we investigate the exponential convergence in mean square and almost surely of the weak probability solutions to stationary solutions. It should be noted that the coupling between the stochastic Magnetohydrodynamic and Cahn-Hilliard system makes the analysis of the control problem more involved.

The article is structured as follows. In the next section, we present the stochastic 3D GMCHMHD model and the functional framework. In Section 3, we study the stability of weak solutions. We also study the stability of stationary solutions to the stochastic 3D GMCHMHD model, using the It$\hat{o}$ formula. In section fourth, we show a result related to the stabilization of these equations.

2 A stochastic GMCHMHD model and its mathematical setting

2.1 Governing equations

Assume that a finite time horizon $T > 0$ is given and the domain $\mathcal{M}$ of the fluid mixture is a bounded domain of $\mathbb{R}^3$ with smooth boundary. Then, we consider the system

$$\begin{cases} dv + [-\nu_1 \Delta v + (v\cdot\nabla)v - \mathcal{K}\mu\nabla\Phi + \nabla p - (b\cdot\nabla)b]dt = h_0^1(t)dt + h_1^1(v,\Phi)dt + h_2^1(t,v,\Phi)dW_t^1, \\ \dfrac{\partial b}{\partial t} + \nu\,\mathsf{curl}(\mathsf{curl} b) + (v\cdot\nabla)b = (b\cdot\nabla)v, \\ \dfrac{\partial \Phi}{\partial t} - \nu_3\Delta\mu + v\cdot\nabla\Phi = h_0^2(t) + h_1^2(v,\Phi) + h_2^2(t,v,\Phi)\dot{W}_t^2, \\ \mu = -\epsilon\Delta\Phi + \alpha f(\Phi), \\ \mathsf{div}(v) = \mathsf{div}(b) = 0. \end{cases} \tag{1}$$

In (1), the unknown functions are the random velocity v of the fluid mixture, its pressure p, the magnetic field b and the order (phase) parameter Φ. The external force $h_1(v,\Phi) = (h_1^1(v,\Phi), h_1^2(v,\Phi)), h_0(t) = (h_0^1(t), h_0^2(t))$ are given and the term $h_2(t,v,\Phi)\dot{W}_t = (h_2^1(t,v,\Phi)\dot{W}_t^1, h_2^2(t,v,\Phi)\dot{W}_t^2)$ denotes random external forces depending eventually on (v,Φ), where $\dot{W}_t = (\dot{W}_t^1, \dot{W}_t^2)$ represents the derivative with respect to time of a cylindrical Wiener process. The quantity μ is the chemical potential of the binary mixture which is given by the variational derivative of the following free energy functional.

$$\mathcal{F}(\Phi) = \int_{\mathcal{M}} \left(\frac{\epsilon}{2}|\nabla\Phi|^2 + \alpha F(\Phi)\right) ds,$$

where, e.g., $F(x) = \displaystyle\int_0^x f(\zeta)d\zeta$. The constants $\nu_1 > 0$, $\nu > 0$, $\nu_3 > 0$ and $\mathcal{K} > 0$ represent respectively the kinematic viscosity of the fluid, the magnetic resistivity, the mobility constant and the coefficient of capillarity. ϵ and $\alpha > 0$ are two physical quantities which characterize the interaction between the two phases. More precisely, ϵ has a relation with the zone of separation of the two fluids. Hereafter, suppose that $\epsilon \leq \alpha$.

The potential F is usually assumed to be of logarithmic type, but it is sometimes replaced by an approximate polynomial of the form $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, where γ_1 and γ_2 are two positive real constants. It should be noted, as in [26], that the first equation of the model can take the form:

$$dv + [-\nu_1\Delta v + (v\cdot\nabla)v - (b\cdot\nabla)b + \nabla\tilde{p}]dt = [-\mathcal{K}\mathsf{div}(\nabla\Phi\otimes\nabla\Phi) + h_0^1(t) + h_1^1(v,\Phi)]dt + h_2^1(t,v,\Phi)\dot{W}_t^1,$$

where $\tilde{p} = p - \mathcal{K}\left(\frac{\epsilon}{2}|\nabla\Phi|^2 + \alpha F(\Phi)\right)$, since $\mathcal{K}\mu\nabla\Phi = \nabla\left(\mathcal{K}\left(\frac{\epsilon}{2}|\nabla\Phi|^2 + \alpha F(\Phi)\right)\right) - \mathcal{K}\mathrm{div}(\nabla\Phi\otimes\nabla\Phi)$.
The tensor product $\nabla\Phi\otimes\nabla\Phi$ is the contribution which highlights the capillary forces coming from the tension on the boundary zone of the fluids.
For a vector-valued function $b = (b_1, b_2)$ on $\mathcal{M}$, define curl(b) $=\dfrac{\partial b_2}{\partial x} - \dfrac{\partial b_1}{\partial y}$ and for a scalar-valued function Φ on $\mathcal{M}$, define curl$(\Phi) = \left(\dfrac{\partial\Phi}{\partial y}, -\dfrac{\partial\Phi}{\partial x}\right)$.
Also we know that curl(curl(b))=grad(div(b)) - Δb is valid in the two dimensions case.

2.1 Governing equations

System (1) is supplemented with the following initial and boundary conditions:

$$\begin{cases} \partial_\eta \Phi = \partial_\eta \Delta\Phi = 0 \quad \text{on} \quad \partial\mathcal{M} \times (0,T), \\ v = 0 \quad \text{on} \quad \partial\mathcal{M}, \\ \mathsf{curl}(b) = 0 \quad \text{on} \quad \partial\mathcal{M}, \\ b \cdot \eta = 0 \quad \text{on} \quad \partial\mathcal{M}, \\ u(0,x) = u_0(x); \quad b(0,x) = b_0(x); \quad \Phi(0,x) = \Phi_0(x) \quad \text{in} \quad \mathcal{M}, \end{cases} \tag{2}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the unit outward normal to $\partial\mathcal{M}$.
From the first equation in (2), we deduce the conservation of the following quantity

$$\langle \Phi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \Phi(x,t)dx, \tag{3}$$

where $|\mathcal{M}|$ is the Lebesgue measure of $\mathcal{M}$. More exactly, we have

$$\langle \Phi(t) \rangle = \langle \Phi(0) \rangle = M_0, \quad \forall t \geq 0. \tag{4}$$

We deduce from (4) that one can sometimes suppose that the average of Φ is zero at the initial moment, starting from a displacement of the order parameter field. Hence, it will be zero for all positive instants. So, we suppose that

$$\langle \Phi(t) \rangle = \langle \Phi(0) \rangle = 0, \quad \forall t \geq 0. \tag{5}$$

We suppose that h_0^2, h_1^2 and h_2^2 are chosen such that (4) is satisfied, which is the case if we assume that

$$\langle h_0^2(t) \rangle = 0, \quad \langle h_1^2(u,\psi) \rangle = 0, \quad \langle h_2^2(u,\psi)\dot{W}_t^2 \rangle = 0, \quad \forall t \geq 0, \quad (u,\psi) \in \mathcal{H},$$

where $\mathcal{H}$ is defined by (3) below.

Remark 2.1. *If a stochastic perturbation of the Cahn-Hilliard system is considered, then the conservation of mass given by (5) does not hold.*

Now, we consider the function $F_N : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$F_N(r) = min\{1, \frac{N}{r}\}, \quad r \in \mathbb{R}^+,$$

for some (fixed) $N \in \mathbb{R}^+$. From [18], we recall the following properties of F_N.

Lemma 2.1. *The function F_N satisfies:*

$$|F_N(p) - F_N(r)| \leq \frac{|p-r|}{r}, \quad p, r \in \mathbb{R}^+, \quad r \neq 0,$$

$$|F_N(||v_1||) - F_N(||v_2||)| \leq \frac{||v_1 - v_2||}{||v_2||}, \quad v_1, v_2 \in V_1, \quad v_2 \neq 0,$$

$$|F_N(p) - F_N(r)| \leq \frac{|M-N|}{r} + \frac{|p-r|}{r}, \quad p, r, M, N \in \mathbb{R}^+, \quad r \neq 0.$$

Then, we consider the following 3D GMCHMHD model

$$\begin{cases} dv+[-\nu_1\Delta v+F_N(||v||)(v\cdot\nabla)v-\mathcal{K}\mu\nabla\Phi+\nabla p-(b\cdot\nabla)b]dt=h_0^1(t)dt+h_1^1(v,\Phi)dt+h_2^1(t,v,\Phi)dW_t^1, \\ \dfrac{\partial b}{\partial t}+\nu \mathit{curl}(\mathit{curl} b)+(v\cdot\nabla)b=(b\cdot\nabla)v, \\ \dfrac{\partial\Phi}{\partial t}-\nu_3\Delta\mu+v\cdot\nabla\Phi=h_0^2(t)+h_1^2(v,\Phi)+h_2^2(t,v,\Phi)\dot{W}_t^2, \\ \mu=-\epsilon\Delta\Phi+\alpha f(\Phi), \\ \mathit{div}(v)=\mathit{div}(b)=0, \end{cases} \tag{6}$$

in $\mathcal{M}\times(0+\infty)$, *where* $||v||$ *is a norm defined below.*

Recall that, as in [18], the GMNSE are indeed globally modified and the factors $F_N(||v||)$ *and* $F_N(||(v,\Phi)||)$ *depend respectively on the norms* $||v||$ *and* $||(v,\Phi)||$. *So, the GMCHNSE was studied in [24, 25, 65].*

2.2 Mathematical setting

Assume that the domain $\mathcal{M}$ is a bounded domain with a smooth boundary $\partial\mathcal{M}$ and that $f\in C^2(\mathbb{R})$ satisfies

$$\begin{cases} \lim\limits_{|r|\to+\infty} f'(r)>0, \\ |f^i(r)|\le c_f(1+|r|^{2-i}), \quad \forall r\in\mathbb{R}, \quad i=0,1,2, \end{cases} \tag{7}$$

where c_f is a positive real constant.

Let us introduce the functional framework of the problem (1)-(2).

Hereafter, if X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, then we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual.

Set

$$\mathcal{V}_1=\{u\in(C_c^\infty(\mathcal{M}))^2:\mathsf{div}(u)=0 \quad in \quad \mathcal{M}\}.$$

We denote by H_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^3$, and by V_1 the closure of $\mathcal{V}_1$ in $(H_0^1(\mathcal{M}))^3$.

H_1 is endowed with the scalar product $(.,.)_{L^2}$ and the corresponding norm $|\cdot|_{L^2}$. V_1 is endowed with the scalar product

$$((u,v))=\sum_{i=1}^{2}(\partial_{x_i}u,\partial_{x_i}v)_{L^2}$$

and the associated norm

$$||u||_{V_1}=((u,u))^{1/2}=||u||.$$

We now consider the operator A_0 defined by

$$A_0u=-\mathcal{P}\Delta u, \quad \forall u\in D(A_0)=H^2(\mathcal{M})\cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $L^2(\mathcal{M})$ onto H_1 (for more details see [24], page 37-38).

We also set

$$\mathcal{V}_2=\{b\in(C^\infty(\bar{\mathcal{M}}))^2:\mathsf{div}(b)=0 \quad and \quad b.\eta=0 \quad in \quad \partial\mathcal{M}\},$$

and denote by H_2 and V_2 the closure of $\mathcal{V}_2$ in $(L^2(\mathcal{M}))^3$ and $(H^1(\mathcal{M}))^3$, respectively.

H_2 is endowed with the scalar product $(.,.)_{L^2}$ and the corresponding norm $|\cdot|_{L^2}$. We have $H_1 = H_2$ (see [47], page 6).

V_2 is endowed with the scalar product $((\Phi,\psi))_{V_2} = (\mathsf{curl}\Phi, \mathsf{curl}\psi)_{L^2}$ and its corresponding norm $||\Phi||_{V_2} = \sqrt{((\Phi,\Phi))_{V_2}}$, which is equivalent to the norm in $H^1(\mathcal{M})$.

We introduce the operators $A_1 \in \mathcal{L}(V_2, V_2^*)$ defined by $\langle A_1 u, v\rangle = ((B,C))_2$ for all $B, C \in V_2$.

The operator A_1 can also be defined as an unbounded operator generated by the boundary value problem

$$\begin{cases} \mathsf{curl}(\mathsf{curl} b) = g \quad in \quad \mathcal{M}, \\ \mathsf{div}(b) = 0 \quad in \quad \mathcal{M}, \\ b.\eta = 0, \quad \mathsf{curl}(b) = 0 \quad on \quad \partial\mathcal{M}, \end{cases} \tag{8}$$

with domain

$$D(A_1) = \{C \in V_2 : A_1 C \in H_2\}.$$

For more details about the operator A_1, we refer the reader to [21].

We also consider the following linear nonnegative unbounded operator A_2 on $L^2(\mathcal{M})$:

$$A_2\Phi = -\Delta\Phi, \quad \forall \Phi \in D(A_2) = \{\Phi \in H^2(\mathcal{M}), \quad \partial_\eta \Phi = 0 \quad on \quad \partial\mathcal{M}\} \cap L_0^2(\mathcal{M}). \tag{9}$$

$D(A_2)$ is endowed with the norm $|A_2 \cdot|_{L^2} + |\langle\cdot\rangle|_{L^2}$, which is equivalent to the H^2-norm. We also consider the linear positive unbounded operator G defined on $L_0^2(\mathcal{M})$, the space of L^2-functions with null mean

$$G\Phi = -\Delta\Phi, \quad \forall \Phi \in D(G) = D(A_2) \cap L_0^2(\mathcal{M}). \tag{10}$$

G^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we consider G^s, for all $s \in \mathbb{R}$, knowing that $|G^{s/2} \cdot|_{L^2}$, $s > 0$, is an equivalent norm to the canonical H^s-norm on $D(G^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also, note that $A_2 = G$ on $D(G)$. If $\Phi - \langle\Phi\rangle \in D(G^{s/2})$, then we can say that $|G^{s/2}(\Phi - \langle\Phi\rangle)|_{L^2} + |\langle\Phi\rangle|_{L^2}$ is equivalent to the H^s-norm. It is worth recalling that $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, for all $s > 0$.

Set

$$H_3 = D(G^0) = L_0^2(\mathcal{M}), \quad V_3 = D(G^{1/2}). \tag{11}$$

H_3 is equipped with the norm $|\cdot|_{L^2}$, and V_3 with the norm $||\cdot||$, where $||\psi|| = |G^{1/2}\psi|_{L^2}$.

We now consider the bilinear operator B, and its trilinear form b_0 defined on $L^1(\mathcal{M}) \times W^{1,1}(\mathcal{M}) \times L^1(\mathcal{M})$ by setting

$$\mathsf{b}_0(u,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{M}} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

whenever the integrals make sense.

Let m_1, m_2, and m_3 be three real numbers. We recall that if the m_i are greater than or equal to 0 and satisfy

$$m_1 + m_2 + m_3 > 1 \quad or \quad m_1 + m_2 + m_3 = 1,$$

then, at least, two m_i (i=1,2,3) are different from zero and b_0 is a trilinear continuous form on $H^{m_1}(\mathcal{M}) \times H^{m_2+1}(\mathcal{M}) \times H^{m_3}(\mathcal{M})$. Moreover

$$|\mathsf{b}_0(u,v,w)| \leq c_1 |u|_{H^{m_1}} |v|_{H^{m_2+1}} |w|_{H^{m_3}}, \quad \forall (u,v,w) \in H^{m_1}(\mathcal{M}) \times H^{m_2+1}(\mathcal{M}) \times H^{m_3}(\mathcal{M}).$$

In particular, for $(m_1 = m_3 = 1, m_2 = 0)$, the trilinear form $\mathtt{b}_0$ is continuous on $(H^1(\mathcal{M}))^3$ (see [57]). Thus, we can define a bilinear map B on $H^1 \times H^1$ with values in $(H_1)^*$ such that

$$\langle B(u,v), w\rangle = \mathtt{b}_0(u,v,w), \quad \forall u,v,w \in H^1(\mathcal{M}).$$

We have

$$\mathtt{b}_0(u,v,v) = 0, \quad \forall u \in V_2, \quad v \in H^1(\mathcal{M}),$$

$$\mathtt{b}_0(u,v,w) = -\mathtt{b}_0(u,w,v), \quad \forall u \in V_2, \quad v,w \in H^1(\mathcal{M}).$$

We consider the bilinear operator B_1 (and its associated trilinear form b_1), and R_0, which are respectively defined from $D(A_0) \times D(A_2)$ in $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_2) \cap H^3(\mathcal{M}))$ in H_1.

More exactly,

$$(B_1(u,\Phi),\rho) = \int_{\mathcal{M}} [(u\cdot\nabla)\Phi]\rho dx = b_1(u,\Phi,\rho), \quad \forall u \in D(A_0), \quad \Phi,\rho \in D(A_2),$$

$$(R_0(\mu,\Phi),w) \quad = \quad \int_{\mathcal{M}} \mu[\nabla\Phi\cdot w]dx = b_1(w,\Phi,\mu), \quad \forall w \in D(A_0), \quad \Phi \in D(A_2)\cap H^3(\mathcal{M}), \quad \mu \in L^2(\mathcal{M}).$$

Note that

$$R_0(\mu,\Phi) = \mathcal{P}\mu\nabla\Phi.$$

As in [24], the operators B, B_1 and R_0 satisfy the following estimates:

$$|b_0(u,v,w)| \le c|u|_{L^2}^{1/2}||u||^{1/2}|A_0v|_{L^2}|w|_{L^2}, \quad \forall u \in V_1, \quad v \in D(A_0), \quad w \in H_1, \tag{12}$$

$$\begin{aligned} &|B(u,v)|_{V_1^*} \le c|u|_{L^2}^{1/4}||u||^{3/4}|v|_{L^2}^{1/4}||v||^{3/4}, \quad \forall u,v \in V_1,\\ &|B(u,v)|_{L^2} \le c||u||||v||^{1/2}|A_0v|_{L^2}^{1/2}, \quad \forall u \in V_1, \quad v \in D(A_0), \end{aligned} \tag{13}$$

$$\begin{aligned} &|B(v,b)|_{V_2^*} \le c|v|_{L^2}^{1/4}||v||^{3/4}|b|_{L^2}^{1/4}||b||^{3/4}, \quad \forall v \in V_1, \quad b \in D(A_1),,\\ &|B(v,b)|_{L^2} \le c||v||||b||^{1/2}|A_1b|_{L^2}^{1/2}, \quad \forall v \in V_1, \quad b \in D(A_1), \end{aligned} \tag{14}$$

$$\begin{aligned} &|B(b,v)|_{V_2^*} \le c|b|_{L^2}^{1/4}||b||^{3/4}|v|_{L^2}^{1/4}||v||^{3/4}, \quad \forall v \in V_1, \quad b \in D(A_1),\\ &|B(b,v)|_{L^2} \le c||b||||v||^{1/2}|A_1v|_{L^2}^{1/2}, \quad \forall v \in V_1, \quad b \in D(A_1), \end{aligned} \tag{15}$$

$$\begin{aligned} &|B_1(v,\Phi)|_{V_3^*} \le c|v|_{L^2}^{1/4}||v||^{3/4}|\Phi|_{L^2}^{1/4}||\Phi||^{3/4}, \quad \forall v \in V_1, \quad \forall\Phi \in V_3,\\ &|B_1(v,\Phi)|_{L^2} \le c||v||||\Phi||^{1/2}|A_2\Phi|_{L^2}^{1/2}, \quad \forall v \in V_1, \quad \Phi \in D(A_2), \end{aligned} \tag{16}$$

$$\begin{aligned} &|R_0(A_2\Phi,\rho)|_{V_1^*} \le c||\rho||^{1/2}|A_2\rho|_{L^2}^{1/2}|A_2\Phi|_{L^2}, \quad \Phi,\rho \in D(A_2),\\ &|R_0(A_2\Phi,\rho)|_{L^2} \le c|A_2\Phi|_{L^2}||A_2\Phi|_{L^2}^{1/2}|A_2^{3/2}\Phi|_{L^2}^{1/2}, \quad \forall\Phi \in D(A_2), \quad \rho \in D(A_2^{3/2}). \end{aligned} \tag{17}$$

Hereafter, we set

$$b_0^N(u,v,w) = F_N(||v||)b_0^N(u,v,w), \quad \langle B^N(u,v), w\rangle = b_0^N(u,v,w), \quad \forall u,v \in V_1.$$

It follows that

$$b_0^N(u,v,v) = 0, \quad \forall u,v \in V_1,$$

$$|b_0^N(u,v,w)| \le cN||u||||w||, \quad \forall u,v,w \in V_1,$$

$$||B^N(u,v)||_{V_1^*} \le cN||u||, \quad \forall u,v \in V_1.$$

2.2 Mathematical setting

Set

$$\mathcal{H} = H_1 \times H_2 \times V_3. \tag{18}$$

Then $\mathcal{H}$ is a complete metric space with respect to the norm

$$|(v, b, \Phi)|^2_{\mathcal{H}} = |v|^2_{L^2} + |b|^2_{L^2} + \epsilon|\nabla\Phi|^2_{L^2}. \tag{19}$$

We now consider the Hilbert space $\mathcal{U}$ defined by

$$\mathcal{U} = V_1 \times V_2 \times D(A_2^{3/2}), \tag{20}$$

with the associated norm

$$||(v, b, \Phi)||^2_{\mathcal{U}} = ||v||^2 + |\mathsf{curl}(b)|^2_{L^2} + \epsilon|A_2^{3/2}\Phi|^2_{L^2}. \tag{21}$$

Throughout the work, c will denote a generic positive real constant which depends on the domain $\mathcal{M}$, even though its values may vary from line to line. We consider $\nu_1 = \epsilon = \nu_3 = \nu = \alpha = \mathcal{K} = 1$.

Let $(\Omega, \mathcal{P}, \mathcal{J})$ be a probability space on which an increasing and right continuous family $\{\mathcal{J}\}_{t\in[0,\infty)}$ of complete sub σ-algebra of $\mathcal{J}$ is defined and let $\beta_n(t)(n = 1, 2, 3, ...)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \mathcal{P}, \mathcal{J})$. We set

$$W_t(t) = \sum_{n=1}^{\infty} \sqrt{\lambda'}\beta_n(t)e_n, \quad t \geq 0,$$

where $\lambda'_n(n = 1, 2, 3, ...)$ are nonnegative real numbers such that $\sum\limits_{n=1}^{\infty} \lambda'_n < \infty$, and $\{e_n\}(n = 1, 2, 3, ...)$ is a complete orthogonal basis in the real and separable Hilbert space K. Let $\mathcal{Q} \in L(K, K)$ be the operator defined by $\mathcal{Q}e_n = \lambda'_n e_n$. The above K-valued stochastic process $W(t)$ is called a $\mathcal{Q}$-Wiener process.

Taking into consideration the above transformations, we rewrite (1), (2) as

$$\begin{cases} dv + [\nu_1 A_0 v + B^N(v, v) - R_0(\epsilon A_2\Phi, \Phi) - B^N(b, b)]dt = h_0^1(t)dt + h_1^1(v, \Phi)dt + h_2^1(t, v, \Phi)dW_t^1 \quad in \quad V_1^*, \\ \dfrac{db}{dt} + [B^N(v, b) - B^N(b, v) + \nu A_1 b] = 0 \quad in \quad V_2^*, \\ \dfrac{d\Phi}{dt} + A_2\mu + B_1(v, \Phi) = h_0^2(t) + h_1^2(v, \Phi) + h_2^2(t, v, \Phi)\dot{W}_t^2 \quad in \quad V_3^*, \\ \mu = \epsilon A_2\Phi + \alpha f(\Phi) \quad in \quad H^{-1}(\mathcal{M}), \\ (v, b, \Phi)(0) = (v_0, b_0, \Phi_0), \end{cases} \tag{22}$$

which is equivalent to

$$\begin{cases} v(t) + \displaystyle\int_0^t (\nu_1 A_0 v(s) + B^N(v(s), v(s)) - B^N(b(s), b(s)))ds = v_0 + \int_0^t R_0(\epsilon A_2\Phi(s), \Phi(s))ds \\ + \displaystyle\int_0^t (h_0^1(s) + h_1^1(v(s), \Phi(s)))ds + \int_0^t h_2^1(t, v, \Phi)dW_s^1, \\ b(t) + \displaystyle\int_0^t [B^N(v(s), b(s)) - B^N(b(s), v(s)) + \nu A_1 b(s)]ds = b_0, \\ \Phi(t) + \displaystyle\int_0^t (A_2\mu(s) + B_1(v(s), \Phi(s)))ds = \Phi_0 + \int_0^t (h_0^2(s) + h_1^2(v(s), \Phi(s)))ds + \int_0^t h_2^2(s, v, \Phi)dW_s^2, \\ \mu = \epsilon A_2\Phi + \alpha f(\Phi), \end{cases} \tag{23}$$

$\mathbb{P}$-a.s., and for all $t \in [0,T]$, where

$$h_0 = (h_0^1, h_0^2) \in L^2(0,\infty,\mathcal{H}), \quad h_1 = (h_1^1, h_1^2) : \mathcal{U} \to \mathcal{H}, \quad h_2 = (h_2^1, h_2^2) : [0,+\infty) \times \mathcal{U} \to \mathcal{L}(K,\mathcal{H}).$$

Remark 2.2. *In the weak formulation (22),* $A_2\Phi\nabla\Phi$ *replaces the term* $\mu\nabla\Phi$*. This is justified by the fact that* $f'(\Phi)\nabla\Phi$ *is the gradient of* $F(\Phi)$ *which can therefore be incorporated into the pressure gradient, see [26] for details.*

Definition 2.1. *A stochastic process* $(v,b,\Phi)(t), t \geq 0$ *is said to be a weak solution to (22) or (23) if*
i) $(v,b,\Phi)(t)$ *is* $\mathcal{J}_t-$ *adapted,*
ii) $(v,b,\Phi)(t) \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{U})$ *almost surely for all* $T > 0$,
iii) (v,b,Φ)*satisfied (23) as an identity in* $\mathcal{U}^*$*, almost surely for* $t \in [0,\infty)$.

Note that (23) implies that almost surely $(v,b,\Phi) \in \mathcal{C}(0,T;\mathcal{U}^*)$ and since $(v,b,\Phi)(\cdot)$ is also bounded in $\mathcal{H}$. As in [77], we can check that (v,b,Φ) is almost surely in $\mathcal{C}(0,T;\mathcal{H}_{weak})$ and the space of $\mathcal{H}$-valued weakly continuous functions on $[0,T]$.
So, we suppose that f satisfied the additional condition

$$\begin{aligned}&\langle \alpha A_2 f(\psi), \epsilon A_2\psi\rangle = \langle \alpha A_2^{1/2} f(\psi), \epsilon A_2^{3/2}\psi\rangle \geq -\kappa_0\epsilon|A_2^{3/2}\psi|_{L^2}^2, \quad \forall\psi \in D(A_2^{3/2}),\\ &\langle \alpha A_2 f(\Phi_1) - A_2 f(\Phi_2), \epsilon A_2(\Phi_1 - \Phi_2)\rangle \geq -\kappa_0\epsilon|A_2^{3/2}(\Phi_1-\Phi_2)|_{L^2}^2, \quad \forall\Phi_1,\Phi_2 \in D(A_2^{3/2}),\end{aligned} \tag{24}$$

where $\kappa_0 > 0$ is fixed constant.

We also set

$$\alpha_1 = \min(\nu_1, \nu, \epsilon - \kappa_0) > 0. \tag{25}$$

3 The exponential stability of solutions

In this part, we examine the moment exponential stability and almost we sure exponential stability of weak solutions to (22) assuming that they exist. Under some conditions, we discuss the long-time behavior of the weak solutions $(v,b,\Phi)(t)$. As in [24], we study the stability of stationary solutions to 3D GMCHMHD, using the It$\hat{o}$ formula.

We will use the notation

$$\begin{aligned}&||h_2(t,v,\Phi)||_{L^2(\mathcal{H})}^2 = tr(h_2(t,v,\Phi)Qh_2(t,v,\Phi)^*),\\ &\langle (x_1,x_2),(y_1,y_2)\rangle = \langle x_1,y_1\rangle_{L^2} + \langle x_2,y_2\rangle, \quad \forall (x_1,x_2),(y_1,y_2) \in \mathcal{H}.\end{aligned} \tag{26}$$

We assume that

$$h_0 = (h_0^1, h_0^2) \in \mathcal{H} \quad \textit{and} \quad h_1 = (h_1^1, h_1^2) : \mathcal{U} \to \mathcal{H}$$

satisfies

$$h_1(0,0) = 0, \quad ||h_1(v_1,\Phi_1) - h_1(v_2,\Phi_2)||_{\mathcal{U}^*}^2 \leq L_1||(v_1,\Phi_1) - (v_2,\Phi_2)||_{\mathcal{H}}, \quad \forall (v_1,\Phi_1),(v_2,\Phi_2) \in \mathcal{U}, \tag{27}$$

where $L_1 > 0$ is fixed.

We now consider the following stationary equation

$$\begin{cases} \nu_1 A_0 v^* + B^N(v^*,v^*) - B^N(b^*,b^*) - R_0(\epsilon A_2\Phi^*,\Phi^*) = h_0^1 + h_1^1(v^*,\Phi^*),\\ \nu A_1 b^* + B^N(v^*,b^*) - B^N(b^*,v^*) = 0,\\ \epsilon A_2^2\Phi^* + \alpha A_2 f(\Phi^*) + B_1(v^*,\Phi^*) = h_0^2 + h_1^2(v^*,\Phi^*).\end{cases} \tag{28}$$

To be more exact, by stationary solution, we mean an element $u^* = (v^*, b^*, \Phi^*) \in \mathcal{U}$ that satisfies $(28)_1, (28)_2$ and $(28)_3$ in V_1^*, V_2^* and V_3^* respectively.

3.1 Existence and uniqueness of stationary solution

Theorem 3.1. *Under the above assumptions and notations, if*

$$\alpha_1 - L_1 > 0, \tag{29}$$

then (28) has at least one solution u^*, which is in fact in $D(A_0) \times D(A_1) \times D(A_2^2)$. Moreover, any stationary solution $u^* = (v^*, b^*, \Phi^*)$ to (28) satisfied the following estimates:

$$||(v^*, b^*, \Phi^*)||_{\mathcal{U}} \leq (\alpha_1 - L_1)^{-1} ||h_0||_{\mathcal{U}^*} = K_1. \tag{30}$$

Moreover, if

$$\alpha_1 - (L_1 + 3cK_1) > 0, \tag{31}$$

then the stationary solution to (28) is unique.

Proof. To prove (30), by multiplying $(28)_1$ by v^*, $(28)_2$ by b^* and $(28)_2$ by $\epsilon A_2 \Phi^*$, to deduce that

$$\begin{aligned} &\nu_1 ||v^*||^2 + \nu ||b^*||^2 + \epsilon^2 |A_2^{3/2} \Phi^*|^2_{L^2} + \langle \alpha A_2^{1/2} f(\Phi^*), \epsilon A_2^{3/2} \Phi^* \rangle \\ &= \langle h_1^0 + h_1^1(v^*, \Phi^*), v^* \rangle + \langle h_0^2 + h_1^2(v^*, \Phi^*), \epsilon A_2 \Phi^* \rangle \leq ||h_0||_{\mathcal{U}^*} ||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + L_1 ||(v^*, b^*, \Phi^*)||^2_{\mathcal{U}}. \end{aligned} \tag{32}$$

This gives

$$(\alpha_1 - L_1) ||(v^*, b^*, \Phi^*)||^2_{\mathcal{U}} \leq ||h_0||_{\mathcal{U}^*} ||(v^*, b^*, \Phi^*)||_{\mathcal{U}}, \tag{33}$$

where $\alpha_1 = \min(\nu_1, \nu, \epsilon - \kappa_0)$.

We derive that

$$||(v^*, b^*, \Phi^*)||_{\mathcal{U}} \leq (\alpha_1 - L_1)^{-1} ||h_0||_{\mathcal{U}^*} = K_1, \tag{34}$$

and (30) is proved.

For the existence, let $\{(w_i, \beta_i, \psi_i), i = 1, 2, 3, ...\} \subset \mathcal{U}$ be an orthonomal basis of $\mathcal{H}$, where $\{w_i, i = 1, 2...\}$, $\{\beta_i, i = 1, 2...\}$, $\{\psi_i, i = 1, 2...\}$ are eigenvectors of A_0, A_1 and A_2 respectively.

We set $\mathcal{U}_m = \text{span}\{(w_1, \beta_1, \psi_1), ...(w_m, \beta_m, \psi_m)\}$.

We define the operator $\mathcal{Z}_m : \mathcal{U}_m \to \mathcal{U}_m$ by:

$$\begin{aligned} &\langle \mathcal{Z}_m u_1, u_2 \rangle = \langle \nu_1 A_0 v_1, v_2 \rangle + \langle \nu A_1 b_1, b_2 \rangle + \epsilon \langle A_2^2 \Phi_1, \epsilon A_2 \Phi_2 \rangle + \langle B^N(v_1, v_1), v_2 \rangle - \langle B^N(b_1, b_1), v_2 \rangle \\ &+ \langle B^N(v_1, b_1), b_2 \rangle - \langle B^N(b_1, v_1), b_2 \rangle + \langle B_1(v_1, \Phi_1), \epsilon A_2 \Phi_2 \rangle - \langle R_0(\epsilon A_2 \Phi_1, \Phi_1), v_2 \rangle + \langle \alpha A_2 f(\Phi_1), \epsilon A_2 \Phi_2 \rangle \\ &- \langle h_0^1 + h_1^1(v_1, \Phi_1), v_2 \rangle - \langle h_0^2 + h_1^2(v_1, \Phi_1), \epsilon A_2 \Phi_2 \rangle, \end{aligned} \tag{35}$$

for $u_1 = (v_1, b_1, \Phi_1), u_2 = (v_2, b_2, \Phi_2) \in \mathcal{U}_m$.

Since the right hand side is a continuous linear map from $\mathcal{U}_m$ to $\mathbb{R}$, by the Riesz theorem, each $\mathcal{Z}_m u_1 \in \mathcal{U}_m$ is well defined. we will check that $\mathcal{Z}_m$ is continuous.

Let $u_1 = (v_1, b_1, \Phi_1), u_2 = (v_2, b_2, \Phi_2) \in \mathcal{U}_m$. We set $u = (w, \beta, \psi) = (v_1, b_1, \Phi_1) - (v_2, b_2, \Phi_2)$.

For $u_3 = (v_3, b_3, \Phi_3) \in \mathcal{U}_m$, we have

$$\begin{aligned}
&\langle \mathcal{Z}_m u_1 - \mathcal{Z}_m u_2, u_3\rangle = \langle \nu_1 A_0 w, v_3\rangle + \langle \nu A_1 \beta, b_3\rangle + \langle B^N(b_2, b_2) - B^N(b_1, b_1), v_3\rangle + \langle B^N(v_1, b_1) - B^N(v_2, b_2), b_3\rangle \\
&+\langle B^N(b_2, v_2) - B^N(b_1, v_1), b_3\rangle + \epsilon\langle A_2^2\psi, \epsilon A_2\Phi_3\rangle + \langle B^N(v_1, v_1) - B^N(v_2, v_2), v_3\rangle \\
&+\langle B_1(v_1, \Phi_1) - B_1(v_2, \Phi_2), \epsilon A_2\Phi_3\rangle - \langle R_0(\epsilon A_2\Phi_1, \Phi_1) - R_0(\epsilon A_2\Phi_2, \Phi_2), v_3\rangle \\
&+\langle \alpha A_2 f(\Phi_1) - \alpha A_2 f(\Phi_2), \epsilon A_2\Phi_3\rangle - \langle h_1^1(v_1, \Phi_1) - h_1^1(v_2, \Phi_2), v_3\rangle - \langle h_1^2(v_1, \Phi_1) - h_1^2(v_2, \Phi_2), \epsilon A_2\Phi_3\rangle.
\end{aligned} \tag{36}$$

Note that

$$\langle B^N(v_1, v_1) - B^N(v_2, v_2), v_3\rangle = F_N(||v_1||)b_0(w, v_1, v_3)$$

$$+F_N(||v_2||)b_0(v_2, w, v_3) + (F_N(||v_1||) - F_N(||v_2||))b_0(v_2, v_1, v_3) = I_1 + I_2 + I_3,$$

$$\langle B^N(b_2, b_2) - B^N(b_1, b_1), v_3\rangle = (F_N(||b_2||) - F_N(||b_1||))b_0(b_2, b_1, v_3) - F_N(||b_2||)b_0(b_2, \beta, v_3)$$

$$-F_N(||b_1||)b_0(\beta, b_1, v_3) = I_4 + I_5 + I_6,$$

$$\langle B^N(v_1, b_1) - B^N(v_2, b_2), b_3\rangle = F_N(||b_1||)b_0(w, b_1, b_3)$$

$$+F_N(||b_2||)b_0(v_2, \beta, b_3) + (F_N(||b_1||) - F_N(||b_2||))b_0(v_2, b_1, b_3) = I_7 + I_8 + I_9,$$

$$\langle B^N(b_2, v_2) - B^N(b_1, v_1), b_3\rangle = (F_N(||v_2||) - F_N(||v_1||))b_0(b_2, v_1, b_3) - F_N(||v_2||)b_0(b_2, w, b_3)$$

$$-F_N(||v_1||)b_0(\beta, v_1, b_3) = I_{10} + I_{11} + I_{12}.$$

We have

$$\begin{aligned}
&|I_1| = F_N(||v_1||)|b_0(w, v_1, v_3)| \leq cN||w||||v_3||, \quad |I_2| = F_N(||v_2||)|b_0(v_2, w, v_3)| \leq cN||w||||v_3||, \\
&|I_3| = (F_N(||v_1||) - F_N(||v_2||))|b_0(v_2, v_1, v_3)| \leq cN||w||||v_3||||v_1||, \\
&|I_4| = (F_N(||b_2||) - F_N(||b_1||))|b_0(b_2, b_1, v_3)| \leq cN||\beta||||v_3||||b_1||, \\
&|I_5| = F_N(||b_2||)|b_0(b_2, \beta, v_3)| \leq cN||b_2||||v_3||, \quad |I_6| = F_N(||b_1||)|b_0(\beta, b_1, v_3)| \leq cN||\beta||||v_3||, \\
&|I_7| = F_N(||b_1||)|b_0(w, b_1, b_3)| \leq cN||w||||b_3||, \quad |I_8| = F_N(||b_2||)|b_0(v_2, \beta, b_3)| \leq cN||v_2||||b_3||, \\
&|I_9| = (F_N(||b_1||) - F_N(||b_2||))|b_0(v_2, b_1, b_3)| \leq cN||\beta||||b_3||||b_1||, \\
&|I_{10}| = (F_N(||v_2||) - F_N(||v_1||))|b_0(b_2, v_1, b_3)| \leq cN||w||||b_3||||v_1||, \\
&|I_{11}| = F_N(||v_2||)|b_0(b_2, w, b_3)| \leq cN||b_2||||b_3||, \quad |I_{12}| = F_N(||v_1||)|b_0(\beta, v_1, b_3)| \leq cN||\beta||||b_3||,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\langle \nu_1 A_0 w, v_3\rangle + \epsilon\langle A_2^2\psi, \epsilon A_1\Phi_3\rangle \leq c||u_1 - u_2||_{\mathcal{U}}||u_3||_{\mathcal{U}}, \\
&\langle \nu A_1\beta, b_3\rangle \leq c||u_1 - u_2||_{\mathcal{U}}||u_3||_{\mathcal{U}},
\end{aligned} \tag{38}$$

$$\begin{aligned}
&|\langle B_1(v_1, \Phi_1) - B_1(v_2, \Phi_2), \epsilon A_2\Phi_3\rangle| = |b_1(w, \Phi_1, \epsilon A_2\Phi_3) + b_1(v_2, \psi, \epsilon A_2\Phi_3)| \\
&\leq c\epsilon||w|||A_2\Phi_1|_{L^2}|A_2\Phi_3|_{L^2} + c\epsilon||v_2|||A_2\psi|_{L^2}|A_2\Phi_3|_{L^2},
\end{aligned} \tag{39}$$

$$\begin{aligned}
&|\langle R_0(\epsilon A_2\Phi_1, \Phi_1) - R_0(\epsilon A_2\Phi_2, \Phi_2), v_3\rangle| = |b_1(v_3, \Phi_1, \epsilon A_2\psi) + b_1(v_3, \psi, \epsilon A_2\Phi_2)| \\
&\leq c\epsilon||v_3|||A_2\Phi_1|_{L^2}|A_2\psi|_{L^2} + C\epsilon||v_3|||A_2\psi|_{L^2}|A_2\Phi_2|_{L^2},
\end{aligned} \tag{40}$$

$$\begin{aligned}
&|\langle \alpha A_2 f(\Phi_1) - \alpha A_2 f(\Phi_2), \epsilon A_2\Phi_3\rangle| = \alpha|\langle A_2^{1/2}(f(\Phi_1) - f(\Phi_2)), \epsilon A_2^{3/2}\Phi_3\rangle| \\
&\leq \epsilon M_2(|A_2\Phi_1|_{L^2}, |A_2\Phi_2|_{L^2}) + |A_2^{3/2}\Phi_3|_{l^2}|A_2^{3/2}\psi|_{L^2},
\end{aligned} \tag{41}$$

where hereafter M_2 denotes some monotone non-decreasing function depending only on the function f.

It follows from (37)-(41) that

$$|\langle \mathcal{Z}_m u_1 - \mathcal{Z}_m u_2, u_3\rangle| \leq c[1 + ||v_2|| + ||b_2|| + |A_1 b_1|_{L^2} + |A_2\Phi_1|_{L^2} + M_2(|A_2\Phi_2|_{L^2}, |A_2\Phi_2|_{L^2})]||u_1 - u_2||_{\mathcal{U}}||u_3||_{\mathcal{U}}, \tag{42}$$

which gives

$$||\mathcal{Z}_m u_1 - \mathcal{Z}_m u_2||_{\mathcal{U}^*} \leq c[1 + ||v_2|| + ||b_2|| + |A_1 b_1|_{L^2} + |A_2\Phi_1|_{L^2} + M_2(|A_2\Phi_2|_{L^2}, |A_2\Phi_2|_{L^2})]||u_1 - u_2||_{\mathcal{U}},$$

which proves that $\mathcal{Z}_m : \mathcal{U}_m \to \mathcal{U}^*$ is continuous.

For $u = (v, b, \Phi) \in \mathcal{U}_m$, we have

$$\begin{aligned}\langle \mathcal{Z}_m u, u\rangle &= \nu_1||v||^2 + \nu||b||^2 + \epsilon^2|A_2^{3/2}\Phi|_{L^2}^2 + \langle \alpha A_2 f(\Phi), \epsilon A_2\Phi\rangle - \langle h_0 + h_1(v, \Phi), (v, \epsilon A_2\Phi)\rangle \\ &\geq (\alpha_1 - L_1)||(v, b, \Phi)||_{\mathcal{U}}^2 - ||h_0||_{\mathcal{U}^*}||(v, b, \Phi)||_{\mathcal{U}}.\end{aligned}$$

Thus, if we take $K_1 = (\alpha_1 - L_1)^{-1}||h_0||_{\mathcal{U}^*}$, we obtain $\langle \mathcal{Z}_m u, u\rangle \geq 0, \forall u = (v, b, \Phi) \in \mathcal{U}_m$ with $||u||_{\mathcal{U}} = K_1$. Consequently by a Corollary of the Brouwer's fixed point theorem, for each $m \geq 1$, there exist $u_m = (v_m, b_m, \Phi_m) \in \mathcal{U}_m$ such that $\mathcal{Z}_m u_m = 0$ with $||u_m||_{\mathcal{U}} \leq K_1$.

From (28), we have

$$\begin{aligned}&\nu_1|A_0 v_m|_{L_2}^2 + \nu|A_1 b_m|_{L_2}^2 + \epsilon^2|A_2^2\Phi_m|_{L^2}^2 = -\langle B^N(v_m, v_m), A_0 v_m\rangle + \langle B^N(b_m, b_m), A_0 v_m\rangle \\ &+\langle R_0(\epsilon A_2\Phi_m, \Phi_m), A_0 v_m\rangle + \langle B^N(v_m, b_m), b_m\rangle - \langle B^N(b_m, v_m), b_m\rangle - \langle B_1(v_m, \Phi_m), \epsilon A_2^2\Phi_m\rangle \\ &-\alpha\langle A_2 f(\Phi_m), \epsilon A_2^2\Phi_m\rangle + \langle h_0 + h_1(v_m, \Phi_m), (A_0 v_m, \epsilon A_2^2\Phi_m\rangle).\end{aligned} \tag{43}$$

From [18, 32], we have

$$|-\langle B^N(v_m, v_m), A_0 v_m\rangle| \leq \frac{\nu_1}{8}|A_0 v_m|_{L^2}^2 + c||v_m||^2,$$

$$|\langle B^N(b_m, b_m), A_0 v_m\rangle| + |\langle B^N(b_m, v_m), b_m\rangle| \leq CN||b_m||^2(||A_0 v_m||^2 + ||v_m||^2).$$

Using (12)-(17) and interpolation, we can check that (see[24,25])

$$\begin{aligned}&|\langle R_0(\epsilon A_2\Phi_m, \Phi_m), A_0 v_m\rangle| = |b_1(A_0 v_m, \Phi_m, \epsilon A_2\Phi_m)| \leq c\epsilon|A_0 v_m|_{L^2}||\Phi_m||^{1/2}|A_2\Phi_m|_{L^2}^{1/2}|A_2^{3/2}\Phi_m|_{L^2} \\ &\leq c\epsilon|A_0 v_m|_{L^2}||\Phi_m||^{3/4}|A_2^{3/2}\Phi_m|_{L^2}^{5/4} \leq c\epsilon|A_0 v_m|_{L^2}||\Phi_m||^{3/4}|A_2\Phi_m|_{L^2}^{5/8}|A_2^2\Phi_m|_{L^2}^{5/8} \\ &\leq c\epsilon|A_0 v_m|_{L^2}||\Phi_m||^{3/4}||\Phi_m||^{5/16}|A_2^{3/2}\Phi_m|_{L^2}^{5/16}|A_2^2\Phi_m|_{L^2}^{5/8} \\ &\leq \tfrac{\nu_1}{8}|A_0 v|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_2^2\Phi|_{L^2}^2 + c||\Phi_m||^{34}.\end{aligned} \tag{44}$$

Similarly, we have (see [65, 67])

$$\begin{aligned}&|b_1(v_m, \Phi_m, \epsilon A_2^2\Phi)| \leq c\epsilon|v_m|_{L^2}^{1/2}||v_m||^{1/2}|A_2\Phi_m|_{L^2}|A_2^2\Phi_m|_{L^2} \\ &\leq c\epsilon|v|_{L^2}^{1/2}||v_m||^{1/2}||\Phi_m|^{21/32}|A_2^{3/2}\Phi_m|_{L^2}^{1/32}|A_2^2\Phi_m|_{L^2}^{21/16} \\ &\leq \tfrac{\nu_1}{8}|A_0 v_m|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_2^2\Phi_m|_{L^2}^2 + c||v_m||_{L^2}^{32/5}||\Phi_m||^{42/5}.\end{aligned} \tag{45}$$

As in [24], we also have

$$\alpha|\langle A_2 f(\Phi_m), \epsilon A_2^2\Phi_m\rangle| = \alpha|\langle f''(\Phi_m)(A_2^{1/2}\Phi_m)^2 + f'(\Phi_m)A_2\Phi_m, \epsilon A_2^2\Phi_m\rangle| \leq J_1 + J_2.$$

We have

$$\begin{aligned}&J_1 \equiv \alpha|\langle f''(\Phi_m)(A_2^{1/2}\Phi_m)^2, \epsilon A_2^2\Phi\rangle| \leq c\epsilon\int_{\mathcal{M}}|A_2^{1/2}\Phi_m|^2|A_2^2\Phi_m|dx \\ &\leq c\epsilon|A_2^{1/2}\Phi_m|_{L^4}^2|A_2^2\Phi_m|_{L^2} \\ &\leq c\epsilon|\Phi_m|^{1/2}|A_2\Phi_m|_{L^2}^{3/2}|A_2^2\Phi_m|_{L^2} \\ &\leq c\epsilon|\Phi_m|^{5/4}|A_2^{3/2}\Phi_m|_{L^2}^{3/4}|A_2^2\Phi_m|_{L^2} \\ &\leq \tfrac{\epsilon^2}{16}|A_2^2\Phi_m|_{L^2}^2 + c||\Phi_m||^{10}.\end{aligned} \tag{46}$$

Similarly, we have

$$\begin{aligned}
J_2 &= \alpha|\langle f'(\Phi_m)A_2\Phi_m, \epsilon A_2^2\Phi\rangle| \le c\epsilon\int_{\mathcal{M}}(1+|\Phi_m|)|A_2\Phi||A_2^2\Phi_m|dx\\
&\le c\epsilon|A_2\Phi_m|_{L^2}^2|A_2^2\Phi_m|_{L^2} + c\epsilon||\Phi_m|||A_2\Phi_m|_{L^3}|A_2^2\Phi|_{L^3}\\
&\le c\epsilon|\Phi_m|^{1/2}|A_2^2\Phi_m|_{L^2}^{3/2} + c\epsilon||\Phi_m||||\Phi_m||^{1/2}|A_2^2\Phi_m|_{L^2}^{7/4}\\
&\le \tfrac{\epsilon^2}{16}|A_2^2\Phi_m|_{L^2}^2 + c||\Phi_m||^2 + c||\Phi_m||^{12}.
\end{aligned}\tag{47}$$

It follows that

$$\alpha|\langle A_2 f(\Phi_m), \epsilon A_2^2\Phi_m\rangle| \le \frac{\epsilon^2}{8}|A_2^2\Phi_m|_{L^2}^2 + c||\Phi_m||^{10} + c||\Phi_m||^2 + c||\Phi_m||^{12}.$$

We also have

$$\begin{aligned}
&|\langle h_1(v_m,\Phi_m),(A_0v_m,\epsilon A_2^2\Phi_m)\rangle| = |\langle h_1^1(v_m,\Phi_m),A_0v_m\rangle + \langle h_1^2(v_m,\Phi_m),\epsilon A_2^2\Phi_m\rangle|\\
&\le \tfrac{\nu_1}{8}|A_0v_m|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_2^2\Phi_m|_{L^2}^2 + c|h_1|_{L^2}^2,\\
&|\langle h_0(A_0v_m,\epsilon A_2^2\Phi_m)\rangle| = |\langle h_0^1, A_0v_m\rangle + \langle h_0^2,\epsilon A_2^2\Phi_m\rangle| \le \tfrac{\nu_1}{8}|A_0v_m|_{L^2}^2 + \tfrac{\epsilon^2}{8}|A_2^2\Phi_m|_{L^2}^2 + c|h_0|_{L^2}^2.
\end{aligned}\tag{48}$$

It follows from (43)-(48) that

$$|A_0v_m|_{L^2}^2 + \epsilon^2|A_2\Phi_m|_{L^2}^2 \le C,\tag{49}$$

where $C>0$ is independent of $m\ge 1$.

From (49), we deduce that the sequence $u_m = (v_m, b_m, \Phi_m)$ is bounded in $D(A_0)\times D(A_1)\times D(A_2^2)$ and consequently, we can extract a subsequence (still) denoted $u_m = (v_m, b_m, \Phi_m)$ that converges weakly in $D(A_0)\times D(A_1)\times D(A_2^2)$ and strongly in $\mathcal{U}$ to an element $u^* = (v^*, b^*, \Phi^*) \in D(A_0)\times D(A_1)\times D(A_2^2)$. As in [18], by passing to the limit in (42), we can check that $u^* = (v^*, b^*, \Phi^*)$ is a stationary solution to (28).

For the uniqueness, let $(v_1^*, b_1^*, \Phi_1^*), (v_2^*, b_2^*, \Phi_2^*)$ be two solutions and $(w,\beta,\psi) = (v_1^*, b_1^*, \Phi_1^*) - (v_2^*, b_2^*, \Phi_2^*)$. Then (w,β,ψ) satisfies

$$\begin{cases}
\nu_1A_0w + B^N(w,v_1^*) + B^N(v_2^*,w) - B^N(\beta,b_1^*) - B^N(b_2^*,\beta) - R_0(\epsilon A_2\Phi_2^*,\psi) - R_0(\epsilon A_2\psi,\Phi_1^*) = h_1^1(v_1^*,\Phi_1^*)\\
-h_1^1(v_2^*,\Phi_2^*),\\
\nu A_1\beta + B^N(w,b_1^*) + B^N(v_2^*,\beta) - B^N(\beta,v_1^*) - B^N(b_2^*,w) = 0,\\
\epsilon A_2^2\psi + \alpha A_2 f(\Phi_1^*) - \alpha A_2 f(\Phi_2^*) + B_1(v_2^*,\psi) + B_1(w,\Phi_1^*) = h_1^2(v_1^*,\Phi_1^*) - h_1^2(v_2^*,\Phi_2^*).
\end{cases}\tag{50}$$

Note that

$$\begin{aligned}
&\langle B^N(v_2^*,w),w\rangle = 0,\\
&\langle B^N(v_2^*,\beta),\beta\rangle = 0,\\
&-\langle B^N(b_2^*,\beta),w\rangle - \langle B^N(b_2^*,w),\beta\rangle = 0.
\end{aligned}\tag{51}$$

As in (37) -(41), we can check that

$$\begin{aligned}
&|F_N(||v_1^*||)b_0(w,v_1^*,w)| \le c||v_1^*||||w||^2, \quad |F_N(||b_1^*||)b_0(\beta,b_1^*,w)| \le c||\beta||||w||^2,\\
&|F_N(||b_1^*||)b_0(w,b_1^*,\beta)| \le c||w||||\beta||^2, \quad |F_N(||v_1^*||)b_0(\beta,v_1^*,\beta)| \le c||v_1^*||||\beta||^2,\\
&\langle R_0(\epsilon A_2\psi,\Phi_1^*,w\rangle = \langle B_1(w,\Phi_1^*),\epsilon A_2\psi\rangle, \quad |\langle R_0(\epsilon A_2\Phi_2^*,\psi),w\rangle \le c\epsilon|A_2\psi|_{L^2}|A_2\Phi_2^*|_{L^2}||w||,\\
&|\langle B_1(v_2^*,\psi),\epsilon A_2\psi\rangle| = c\epsilon|A_2\psi|_{L^2}^2||v_2^*||, \quad |\langle \alpha A_2f(\Phi_1^*) - \alpha A_2 f(\Phi_2^*),\epsilon A_2\psi\rangle| \ge -k_0|A_2^{3/2}\psi|_{L^2}^2,\\
&|\langle h_1^1(v_1^*,\Phi_1^*) - h_1^1(v_2^*,\Phi_2^*),w\rangle + \langle h_1^2(v_1^*,\Phi_1^*) - h_1^2(v_2^*,\Phi_2^*),\epsilon A_2\psi\rangle| \equiv |\langle h_1(v_1^*,\Phi_1^*) - h_1(v_2^*,\Phi_2^*),(w,\epsilon A_2\psi)\rangle|\\
&\le L_1||(w,\beta,\psi)||_{\mathcal{U}}^2.
\end{aligned}\tag{52}$$

Mulptiplying $(50)_1$, $(50)_2$ and $(50)_3$ by w, β and $\epsilon A_2\psi$ respectively and using $(3.38)-(3.39)$ yields

$$\nu_1||w||^2+\nu||\beta||^2+\epsilon^2|A_2^{3/2}\psi|_{L^2}^2-k_0\epsilon|A_2^{3/2}\psi|_{L^2}^2\leq c(||v_1^*||+||v_2^*||++||b_1^*||+||b_2^*||+\epsilon^{1/2}|A_2\Phi_2^*|_{L^2}+L_1)||(w,\beta,\psi)||_{\mathcal{U}}^2,$$

which gives

$$(\alpha_1-(L_1+3cK_1))||(w,\beta,\psi)||_{\mathcal{U}}^2\leq 0,$$

and $||(w,\beta,\psi)||_{\mathcal{U}}=0$, assuming (31) and the theorem is proved. ☐

Remark 3.1. *We note that condition* (29) *is satisfies if* $L_1>0,k_0$ *are enough and* $\alpha_1>0$ *is large enough. Condition* (31) *is satisfies if* $\alpha_1>0$ *is large enough,* L_1 *and* $|h_0|_{\mathcal{H}}$ *are small enough.*

3.2 Stability of the steady state solutions

In this section, we study the stability of the steady state solutions. We suppose that ν_1, ν and ϵ are large enough so that (28) has a unique solution (v^*,b^*,Φ^*). Firstly, we recall some preliminary definitions.

Definition 3.1. *We say that a weak solution* $(v,b,\Phi)(t)$ *to (22) converges to* $(v^*,b^*,\Phi^*)\in\mathcal{H}$ *exponentially in the mean square if there exists* $\eta>0$ *and* $M_0=M_0((v,b,\Phi)(0))>0$ *such that*

$$\mathbb{E}|(v,b,\Phi)(t)-(v^*,b^*,\Phi^*)|_{\mathcal{H}}^2\leq M_0e^{-\eta t},\quad t\geq 0. \tag{53}$$

If (v^*,b^*,Φ^*) *is a solution to (28), then we say that* (v^*,b^*,Φ^*) *is exponentially stable in the mean square provided that every weak solution to (22) converges to* (v^*,b^*,Φ^*) *exponentially in the mean square with the same exponential order* $\eta>0$.

Definition 3.2. *We say that a weak solution* $(v,b,\Phi)(t)$ *to (22) converges to* $(v^*,b^*,\Phi^*)\in\mathcal{H}$ *almost surely exponentially if there exists* $\eta>0$ *such that*

$$\lim_{t\to\infty}\frac{1}{t}\log|(v,b,\Phi)(t)-(v^*,b^*,\Phi^*)|_{\mathcal{H}}\leq-\eta. \tag{54}$$

If (v^*,b^*,Φ^*) *is a solution to (28), then we say that* (v^*,b^*,Φ^*) *is almost surely exponentially stable provided that every weak solution to (22) converges to* (v^*,b^*,Φ^*) *almost surely exponentially with the same constant* $\eta>0$.

Theorem 3.2. *Let* $(v^*,b^*,\Phi^*)\in\mathcal{U}$ *be the unique solution to (28). We assume that* h_1 *satisfies (26)-(27) and* h_2 *verifies*

$$||h_2(t,v,\Phi)||_{L^2(\mathcal{H})}^2\leq\varphi(t)+(\zeta+\delta(t))|(v,b,\Phi)(t)-(v^*,b^*,\Phi^*)|_{\mathcal{H}}^2, \tag{55}$$

where $\zeta>0$ *is a constant and* $\varphi(t),\delta(t)$ *are nonnegative integrable functions such that there exist real numbers* $\rho>0,M_\delta\geq 1,M_\varphi\geq 1$ *with*

$$\varphi(t)\leq M_\varphi e^{-\rho t},\quad \delta(t)\leq M_\delta e^{-\rho t},\quad t\geq 0. \tag{56}$$

We also assume that

$$\lambda^{-1}\zeta+c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}}+2L_1-2\alpha_1<0, \tag{57}$$

where c_1 *is defined by (66) below.*

Then any weak solution $(v,b,\Phi)(t)$ *to (22) converges to* (v^*,b^*,Φ^*) *exponentially in the mean square. More exactly, there exist real numbers* $\eta\in(0,\rho),M_0=M_0((v,b,\Phi)(0))>0$ *such that*

$$\mathbb{E}|(v,b,\Phi)(t)-(v^*,b^*,\Phi^*)|_{\mathcal{H}}^2\leq M_0e^{-\eta t},\quad \forall t>0. \tag{58}$$

Proof. Firstly, we choose $\eta \in (0, \rho)$ such that

$$\lambda^{-1}(\zeta+\eta) + c_1||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1 - 2\alpha_1 < 0. \tag{59}$$

Let us set

$$(w, \beta, \psi) = (v^*, b^*, \Phi^*) - (v, b, \Phi). \tag{60}$$

Applying It$\hat{o}$ formula to $e^{\eta t}|(w,\beta,\psi)(t)|^2_{\mathcal{H}}$ gives

$$\begin{aligned}
&e^{\eta t}|(w,\beta,\psi)(t)|^2_{\mathcal{H}} = |(w,\beta,\psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}|(w,\beta,\psi)(s)|^2_{\mathcal{H}} ds - 2\nu_1 \int_0^t e^{\eta s}\langle A_0 v(s), w(s)\rangle ds \\
&-2\nu \int_0^t e^{\eta s}\langle A_1 b(s), \beta(s)\rangle ds - 2\epsilon^2 \int_0^t e^{\eta s}\langle A_2^2\Phi(s), A_2\psi(s)\rangle ds \\
&-2\int_0^t e^{\eta s}\langle B^N(v(s),v(s)) - B^N(b(s),b(s)) - R_0(\epsilon A_2\Phi(s),\Phi(s)), w(s)\rangle ds \\
&-2\int_0^t e^{\eta s}\langle B^N(v(s),b(s)) - B^N(b(s),v(s)), \beta(s)\rangle ds - 2\int_0^t e^{\eta s}\langle B_1(v(s),\Phi(s)) + \alpha A_2 f(\Phi(s)), \epsilon A_2\psi(s)\rangle ds \\
&+2\int_0^t e^{\eta s}\langle h_0 + h_1(v(s),\Phi(s)), (w, \epsilon A_2\psi(s)\rangle ds + 2\int_0^t e^{\eta s}\langle h_2(v(s),\Phi(s)), (w,\epsilon A_2\psi(s)\rangle dW_s \\
&+\int_0^t e^{\eta s}||h_2(s,v(s),\Phi(s))||^2_{L^2(\mathcal{H})} ds.
\end{aligned} \tag{61}$$

We also know that (v^*, b^*, Φ^*) satisfies

$$\begin{aligned}
&\int_0^t e^{\eta s}\langle \nu_1 A_0 v^* + B^N(v^*,v^*) - B^N(b^*,b^*) - R_0(\epsilon A_2\psi^*, \Phi^*), w(s)\rangle ds \\
&+\int_0^t e^{\eta s}\langle \nu A_1 b^* + B^N(v^*,b^*) - B^N(b^*,v^*), \beta(s)\rangle ds + \int_0^t e^{\eta s}\langle \epsilon A_2^2\Phi^*) + B_1(v^*,\Phi^*) + \alpha A_2 f(\Phi^*), \epsilon A_2\psi(s)\rangle ds \\
&= \int_0^t e^{\eta s}\langle h_0^1 + h_1^1(v^*,\Phi^*), w(s)\rangle ds + \int_0^t e^{\eta s}\langle h_0^2 + h_1^2(v^*,\Phi^*), \epsilon A_2\psi(s)\rangle ds \\
&= \int_0^t e^{\eta s}\langle h_0 + h_1(v^*,\Phi^*), (w,\epsilon A_2\psi)(s)\rangle ds.
\end{aligned} \tag{62}$$

Using (61)-(62), we derive that

$$\begin{aligned}
&e^{\eta t}\mathbb{E}|(w,\beta,\psi)(t)|^2_{\mathcal{H}} = \mathbb{E}|(w,\beta,\psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|^2_{\mathcal{H}} ds - 2\nu_1 \int_0^t e^{\eta s}\mathbb{E}||w(s)||^2 ds \\
&-2\nu \int_0^t e^{\eta s}\mathbb{E}||\beta(s)||^2 ds - 2\epsilon^2 \int_0^t e^{\eta s}\mathbb{E}|A_2^{3/2}\psi(s)|^2_{L^2} ds - 2\int_0^t e^{\eta s}\mathbb{E}F_N(||v^*||)b_0(w,v^*,w) ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(||v^*||) - F_N(||v||))b_0(v,v^*,w) ds + 2\int_0^t e^{\eta s}\mathbb{E}F_N(||b||)b_0(b,\beta,w) ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}F_N(||b^*||)b_0(\beta,b^*,w) ds - 2\int_0^t e^{\eta s}\mathbb{E}(F_N(||b||) - F_N(||b^*||))b_0(b,b^*,w) ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}F_N(||b^*||)b_0(w,b^*,\beta) ds + 2\int_0^t e^{\eta s}\mathbb{E}(F_N(||b^*||) - F_N(||b||))b_0(v,b^*,\beta) ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(||v||) - F_N(||v^*||))b_0(b,v^*,\beta) ds + 2\int_0^t e^{\eta s}\mathbb{E}F_N(||v||)b_0(b,w,\beta) ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}b_1(w,\psi,\epsilon A_2\Phi^*) ds - 2\int_0^t e^{\eta s}\mathbb{E}b_1(v^*,\psi,\epsilon A_2\psi) ds - 2\alpha \int_0^t e^{\eta s}\mathbb{E}\langle A_2 f(\Phi)(s) - A_2 f(\Phi^*), \epsilon A_2\psi(s)\rangle ds \\
&+\int_0^t e^{\eta s}\mathbb{E}||h_2(s,v(s),\Phi(s))||^2_{L^2(\mathcal{H})} ds + 2\int_0^t e^{\eta s}\mathbb{E}\langle h_1(v(s),\Phi(s)) - h_1(v^*,\Phi^*), (w,\epsilon A_2\psi)(s)\rangle ds,
\end{aligned} \tag{63}$$

since

$$\begin{aligned}
&2\langle B^N(v^*,v^*),w\rangle - 2\langle B^N(v,v),w\rangle = 2F_N(||v^*||)b_0(w,v^*,w) + 2(F_N(||v^*||) - F_N(||v||))b_0(v,v^*,w),\\
&-2\langle B^N(b^*,b^*),w\rangle + 2\langle B^N(b,b),w\rangle = -2F_N(||b||)b_0(b,\beta,w) - 2F_N(||b^*||)b_0(\beta,b^*,w)\\
&+2(F_N(||b||) - F_N(||b^*||))b_0(b,b^*,w),\\
&2\langle B^N(v^*,b^*),\beta\rangle - 2\langle B^N(v,b),\beta\rangle = 2F_N(||b^*||)b_0(w,b^*,\beta) + 2(F_N(||b^*||) - F_N(||b||))b_0(v,b^*,\beta),\\
&-2\langle B^N(b^*,v^*),\beta\rangle + 2\langle B^N(b,v),\beta\rangle = -2F_N(||v||)b_0(b,w,\beta) - 2F_N(||v^*||)b_0(\beta,v^*,\beta)\\
&+2(F_N(||v||) - F_N(||v^*||))b_0(b,v^*,\beta).
\end{aligned}\tag{64}$$

Note that

$$\begin{aligned}
&2F_N(||v^*||)|b_0(w,v^*,w)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2(F_N(||v^*||) - F_N(||v||))|b_0(v,v^*,w)| \le c_1||v^*||||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2F_N(||b||)b_0|(b,\beta,w)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2F_N(||b^*||)|b_0(\beta,b^*,w)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2(F_N(||b||) - F_N(||b^*||))|b_0(b,b^*,w)| \le c_1||b^*||||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2F_N(||b^*||)|b_0(w,b^*,\beta)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2(F_N(||b^*||) - F_N(||b||))|b_0(v,b^*,\beta)| \le c_1||b^*||||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2F_N(||v||)|b_0(b,w,\beta)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2F_N(||v^*||)|b_0(\beta,v^*,\beta)| \le c_1N||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2(F_N(||v||) - F_N(||v^*||))|b_0(v,v^*,\beta)| \le c_1||v^*||||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2|b_1(v^*,\psi,\epsilon A_2\psi)| \le c_1\epsilon||v^*|||A_2\psi|^2_{L^2} \le c_1\epsilon||(v^*,b^*,\Phi^*)||_{\mathcal{U}}||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&2|b_1(w,\psi,\epsilon A_2\Phi^*)| \le c\epsilon||w|||A_2\psi|_{L^2}|A_2\Phi^*|^2_{L^2} \le c\epsilon||(v^*,b^*,\Phi^*)||_{\mathcal{U}}||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&-\alpha\langle A_2f(\Phi) - A_2f(\Phi^*),\epsilon A_2\psi(s)\rangle \le \epsilon\kappa_0|A_2^{3/2}\psi|^2_{L^2} \le \epsilon\kappa_0||(w,\beta,\psi)||^2_{\mathcal{U}},\\
&|\langle h_1(v(s),\Phi(s)) - h_1(v^*,\Phi^*),(w,\epsilon A_2\psi)\rangle| \le L_1||(w,\beta,\psi)||^2_{\mathcal{U}}.
\end{aligned}\tag{65}$$

It follows that

$$\begin{aligned}
&2F_N(||v^*||)|b_0(w,v^*,w)| + (F_N(||v^*||) - F_N(||v||))||b_0(v,v^*,w)| + 2F_N(||b||)b_0|(b,\beta,w)|\\
&+2F_N(||b^*||)|b_0(\beta,b^*,w)| + 2(F_N(||b||) - F_N(||b^*||))|b_0(b,b^*,w)| + 2F_N(||b^*||)|b_0(w,b^*,\beta)|\\
&+2(F_N(||b^*||) - F_N(||b||))|b_0(v,b^*,\beta)| + 2F_N(||v||)|b_0(b,w,\beta)| + 2F_N(||v^*||)|b_0(\beta,v^*,\beta)|\\
&+2(F_N(||v||) - F_N(||v^*||))|b_0(b,v^*,\beta)| + 2|b_1(v^*,\psi,\epsilon A_2\psi)| + 2|b_1(w,\psi,\epsilon A_2\Phi^*)|\\
&-2\alpha\langle A_2f(\Phi) - A_2f(\Phi^*),\epsilon A_2\psi(s)\rangle + 2\langle h_1(v(s),\Phi(s)) - h_1(v^*,\Phi^*),(w,\epsilon A_2\psi)\rangle|\\
&\le [c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1 + 2\epsilon\kappa_0]||(w,\beta,\Phi)||^2_{\mathcal{U}},
\end{aligned}\tag{66}$$

for some $c_1 > 0$.

We derive from (65)-(66) that

$$\begin{aligned}
&e^{\eta t}\mathbb{E}|(w,\beta,\psi)(t)|^2_{\mathcal{H}} = \mathbb{E}|(w,\beta,\psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|^2_{\mathcal{H}}ds\\
&-2\alpha_1\int_0^t e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|^2_{\mathcal{U}}ds + [c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1]\int_0^t e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|^2_{\mathcal{U}}ds\\
&+\int_0^t e^{\eta s}(\varphi(s) + (\zeta + \delta(s))\mathbb{E}|(w,\beta,\Phi)(s)|^2_{\mathcal{U}}ds.
\end{aligned}\tag{67}$$

In (67), we use the fact that

$$\nu_1||w||^2 + \nu||\beta||^2 + \epsilon^2|A_2^{3/2}\psi|^2_{L^2} - \epsilon\kappa_0|A_2^{3/2}\psi|^2_{L^2} \ge \alpha_1||(w,\beta,\psi)||^2_{\mathcal{U}}.$$

We recall that

$$-2\alpha_1 + [c_1||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1] + \lambda^{-1}(\eta + \zeta) < 0. \tag{68}$$

It follows from (66)-(68) that

$$e^{\eta t}\mathbb{E}(|(w, \beta, \psi)(t)|_{\mathcal{H}}^2) \leq \mathbb{E}(|(w, \beta, \psi)(0)|_{\mathcal{H}}^2) + \int_0^t e^{\eta s}(\varphi(s) + \delta(s))|(w, \beta, \psi)(s)|_{\mathcal{H}}^2 ds. \tag{69}$$

Using the Gronwall lemma, we derive that there exists $M_0 > 0$ such that

$$\mathbb{E}(|(w, \beta, \psi)(t)|_{\mathcal{H}}^2) \leq M_0 e^{-\eta t}, \quad \forall t > 0, \tag{70}$$

which proves (58). □

Theorem 3.3. *The hypotheses are the same as in theorem 3.2. Then any weak solution* $(v, b, \Phi)(t)$ *to (22) converges to the stationary solution* (v^*, b^*, Φ^*) *of (28) almost surely exponentially.*

Proof. Let N be a positif integer and by the Itô formula, for any $t \geq N$, we have

$$\begin{aligned}
|(w, \beta, \psi)(t)|_{\mathcal{H}}^2 &= |(w, \beta, \psi)(N)|_{\mathcal{H}}^2 - 2\nu_1\int_N^t ||w(s)||^2 ds - 2\nu\int_N^t ||\beta(s)||^2 ds - 2\epsilon^2\int_N^t |A_2^{3/2}\psi(s)|_{L^2}^2 ds \\
&-2\int_N^t \mathbb{E}F_N(||v^*||)b_0(w, v^*, w)ds - 2\int_N^t \mathbb{E}(F_N(||v^*||) - F_N(||v||))b_0(v, v^*, w)ds \\
&-2\int_N^t \mathbb{E}F_N(||v||)b_0(v, w, w)ds + 2\int_N^t \mathbb{E}F_N(||b||)b_0(b, \beta, w)ds + 2\int_N^t \mathbb{E}F_N(||b^*||)b_0(\beta, b^*, w)ds \\
&-2\int_N^t \mathbb{E}(F_N(||b||) - F_N(||b^*||))b_0(b, b^*, w)ds - 2\int_N^t \mathbb{E}(F_N(||b^*||) - F_N(||b||))b_0(v, b^*, \beta)ds \\
&-2\int_N^t \mathbb{E}F_N(||b^*||)b_0(w, b^*, \beta)ds - 2\int_N^t \mathbb{E}F_N(||b||)b_0(v, \beta, \beta)ds + 2\int_N^t \mathbb{E}F_N(||v^*||)b_0(\beta, v^*, \beta)ds \\
&+2\int_N^t \mathbb{E}F_N(||v||)b_0(b, w, \beta)ds - 2\int_N^t \mathbb{E}(F_N(||v||) - F_N(||v^*||))b_0(b, v^*, \beta)ds \\
&-2\int_N^t b_1(v^*, \psi, \epsilon A_2\psi))ds + 2\int_N^t b_1(w, \psi, \epsilon A_2\Phi^*))ds - 2\alpha\int_N^t \langle A_2 f(\Phi) - A_2 f(\Phi^*), \epsilon A_2\psi\rangle ds \\
&+2\int_N^t \langle h_1(v(s), \Phi(s)) - h_1(v^*, \Phi^*), (w, \epsilon A_2\psi)(s)\rangle ds + 2\int_N^t ||h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds \\
&+2\int_N^t \langle (w, \epsilon A_2\psi)(s), h_2(s, v(s), \Phi(s))dW_s(s)\rangle.
\end{aligned} \tag{71}$$

By the Burkholder-Davis-Gundy lemma, we have

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{N\leq t\leq N+1}\int_N^t \langle (w, \epsilon A_2\psi)(s), h_2(s, v(s), \Phi(s))dW_s(s)\rangle\right] \\
&\leq \eta_1\left[\mathbb{E}\int_N^{N+1} |(w, \beta, \psi)(s)|_{\mathcal{H}}^2 ||h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds\right]^{1/2} \\
&\leq \eta_1\left[\mathbb{E}\left(\sup_{N\leq t\leq N+1}|(w, \beta, \psi)(t)|_{\mathcal{H}}^2 \int_N^{N+1} ||h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds\right)\right]^{1/2} \\
&\leq \eta_2\int_N^{N+1} \mathbb{E}|h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds + \frac{1}{2}\mathbb{E}\sup_{N\leq t\leq N+1} |(w, \beta, \psi)(s)|_{\mathcal{H}}^2,
\end{aligned} \tag{72}$$

where $\eta_1 > 0, \eta_2 > 0$ are some constant.

Therefore as in (58)-(67), we obtain that

$$\begin{aligned}
&\mathbb{E}[\sup_{N\leq t\leq N+1}|(w, \beta, \psi)(t)|_{\mathcal{H}}^2] \leq \mathbb{E}|(w, \beta, \psi)(N)|_{\mathcal{H}}^2 - 2\alpha_1\int_N^{N+1} \mathbb{E}||(w, \beta, \psi)(s)||_{\mathcal{U}}^2 ds \\
&+[\zeta\lambda_1^{-1} + c_1||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1]\int_N^{N+1} \mathbb{E}||(w, \beta, \psi)(s)||_{\mathcal{U}}^2 ds \\
&+\eta_0\int_N^{N+1} \mathbb{E}||h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds + \frac{1}{2}\mathbb{E}\sup_{N\leq t\leq N+1} |(w, \beta, \psi)(t)|_{\mathcal{H}}^2,
\end{aligned} \tag{73}$$

for some $\eta_0 > 0$.

It follows from (55), (59) and (73) that

$$\tfrac{1}{2}\mathbb{E}\sup_{N\le t\le N+1}|(w,\beta,\psi)(t)|^2_{\mathcal{H}} \le \mathbb{E}(w,\beta,\psi)(N)|^2_{\mathcal{H}} + \eta_0\int_N^{N+1}(\varphi(s)+(\zeta+\delta(s))\mathbb{E}|(w,\beta,\psi,(s)|^2_{\mathcal{H}}ds. \tag{74}$$

Since

$$\varphi(t)\le M_\varphi e^{-\rho t},\quad \delta(t)\le M_\delta e^{-\rho t},\quad \eta\in(0,\rho),\quad M\varphi\ge 1,\quad M_\delta\ge 1, \tag{75}$$

it follows from Theorem 3.2 that there exist $M_1 = M_1((v,b,\Phi)(0)\ge 1$ such that

$$\mathbb{E}\left(\sup_{N\le t\le N+1}|(w,\beta,\psi)(t)|^2_{\mathcal{H}}\right)\le M_1 e^{-\eta N}, \tag{76}$$

and the proof of the Theorem follows from the Borel-Cantelli lemma. ☐

Theorem 3.4. *Let $(v^*,b^*,\Phi^*)\in\mathcal{U}$ be the unique solution to (28). Furthermore, we assume that*

$$\begin{aligned}&h_2(v^*,\Phi^*)=0,\quad \forall t\ge 0,\\ &||h_2(t,v_1,\Phi_1)-h_2(t,v_2,\Phi_2)||_{L^2(\mathcal{H})}\le c_h|(v_1,\Phi_1)-(v_2,\Phi_2)|_{\mathcal{H}},\quad \forall(v_1,\Phi_1),(v_2,\Phi_2)\in\mathcal{H}.\end{aligned} \tag{77}$$

If

$$-2\alpha_1 + c_h\lambda_1^{-1} + c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1 < 0, \tag{78}$$

then any weak solution to (22) converges to (v^*,b^*,Φ^*) exponentially in the mean square. That is, there exists $\eta>0$ such that

$$\mathbb{E}|(v,b,\Phi)(t)-(v^*,b^*,\Phi^*)|^2_{\mathcal{H}}\le\mathbb{E}|(v_0,b_0,\Phi_0)-(v^*,b^*,\Phi^*)|^2_{\mathcal{H}}e^{-\eta t},\quad\forall t\ge 0. \tag{79}$$

Moreover, the path-wise exponential stability with probability one of (v^*,b^*,Φ^*) also holds true.

Proof. We start with the equality

$$\begin{aligned}&v(t)-v^* = v(0)-v^*-\int_0^t[\nu_1A_0(v-v^*)+B^N(v,v)-B^N(b,b)-B^N(v^*,v^*)]ds\\ &+\int_0^t[R_0(\epsilon A_2\Phi,\Phi)-R_0(\epsilon A_2\Phi^*,\Phi^*)+h_1^1(v,\Phi)-h_1^1(v^*,\Phi^*)]ds+\int_0^t(h_2^1(s,v,\Phi)-h_2^1(s,v^*,\Phi^*))dW_t^1,\\ &b(t)-b^*=b(0)-b^*-\int_0^t[\nu A_1(b-b^*)+B^N(v,b)-B^N(b,v)]ds,\\ &\Phi(t)-\Phi^*=\Phi(0)-\Phi^*-\epsilon\int_0^t A_2^2(\Phi-\Phi^*)ds-\int_0^t[B_1(v,\Phi)-B_1(v^*,\Phi^*)]ds\\ &-\alpha\int_0^t[A_2f(\Phi)-A_2f(\Phi^*)]ds+\int_0^t(h_2^2(v,\Phi)-h_2^2(v^*,\Phi^*))dW_t^2.\end{aligned} \tag{80}$$

Let $\eta > 0$ small enough and fixed later. By the Itô formula, we have

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(w,\beta,\psi)(t)|_{\mathcal{H}}^2 = \mathbb{E}e^{\eta s}|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|_{L^2}^2 ds - 2\nu_1 \int_0^t e^{\eta s}\mathbb{E}||w(s)||^2 ds \\
&-2\nu \int_0^t e^{\eta s}\mathbb{E}||\beta(s)||^2 ds - 2\epsilon^2 \int_0^t e^{\eta s}\mathbb{E}|A_2^{3/2}\psi(s)|_{L^2}^2 ds - 2\int_0^t e^{\eta s}\mathbb{E}F_N(||v^*||)b_0(w,v^*,w)ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(||v^*||) - F_N(||v||))b_0(v,v^*,w)ds - 2\int_0^t e^{\eta s}\mathbb{E}F_N(||v||)b_0(v,w,w)ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}F_N(||b||)b_0(b,\beta,w)ds + 2\int_0^t e^{\eta s}\mathbb{E}F_N(||b^*||)b_0(\beta,b^*,w)ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}(F_N(||b||) - F_N(||b^*||))b_0(b,b^*,w)ds - 2\int_0^t e^{\eta s}\mathbb{E}(F_N(||b^*||) - F_N(||b||))b_0(v,b^*,\beta)ds \\
&-2\int_0^t e^{\eta s}\mathbb{E}F_N(||b^*||)b_0(w,b^*,\beta)ds - 2\int_0^t e^{\eta s}\mathbb{E}F_N(||b||)b_0(v,\beta,\beta)ds + 2\int_0^t e^{\eta s}\mathbb{E}F_N(||v^*||)b_0(\beta,v^*,\beta)ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}F_N(||v||)b_0(b,w,\beta)ds - 2\int_0^t e^{\eta s}\mathbb{E}(F_N(||v||) - F_N(||v^*||))b_0(b,v^*,\beta)ds \\
&+2\epsilon\int_0^t e^{\eta s}\mathbb{E}b_1(w,\psi,A_2\Phi^*)ds - 2\epsilon\int_0^t e^{\eta s}\mathbb{E}b_1(v^*,\psi,A_2\psi)ds - 2\int_0^t e^{\eta s}\mathbb{E}\langle\alpha A_2 f(\Phi) - \alpha A_2 f(\Phi^*), \epsilon A_2\psi(s)\rangle ds \\
&+2\int_0^t e^{\eta s}\mathbb{E}\langle h_1(v(s),\Phi(s)) - h_1(v^*,\Phi^*), (w,\epsilon A_2\psi)(s)\rangle ds + \int_0^t e^{\eta s}\mathbb{E}||h_2(v(s),\Phi(s)) - h_2(v^*,\Phi^*)||_{L^2(\mathcal{H})}^2 ds.
\end{aligned} \tag{81}$$

It follows from (81) and (15)-(17) that

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(w,\beta,\psi)(t)|_{\mathcal{H}}^2 \le \mathbb{E}|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|_{\mathcal{H}}^2 ds \\
&+[-2\alpha_1 + c_h\lambda_1^{-1} + c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1]\int_0^t e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|_{\mathcal{H}}^2 ds \\
&\le \mathbb{E}|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 + (\eta + \kappa_2\lambda_1)\int_0^t e^{\eta s}|(w,\beta,\psi)(s)|_{\mathcal{H}}^2 ds \le \mathbb{E}|(w,\beta,\psi)(s)|_{\mathcal{H}}^2,
\end{aligned} \tag{82}$$

where

$$\kappa_2 = -2\alpha_1 + c_h\lambda_1^{-1} + c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} < 0, \tag{83}$$

and η is chosen such that

$$\eta + \kappa_2\lambda_1 < 0. \tag{84}$$

It follows from (82) that

$$\mathbb{E}e^{\eta t}|(w,\beta,\psi)(t)|_{\mathcal{H}}^2 \le \mathbb{E}|(w,\beta,\psi)(0)|_{\mathcal{H}}^2, \tag{85}$$

and the proof of the first part of the theorem follows as that of Theorem 3.3. The rest of the theorem is proved using a similar method to the one in the proof of theorem 3.4. □

Theorem 3.5. *We assume that there exists a constant* $\zeta > 0$ *and* $h_0 = 0$ *such that*

$$||h_2(t,v,\Phi)||_{L^2(\mathcal{H})}^2 \le \varphi(t) + (\zeta + \delta(t))|(v,b,\Phi)|_{\mathcal{H}}^2, \tag{86}$$

where $\varphi(t), \delta(t)$ *satisfy (56). We also suppose that* $h_1 : [0,\infty] \times \mathcal{U} \to \mathcal{U}^*$ *satisfies*

$$\langle h_1(t,v,\Phi), (v,\epsilon A_2\Phi)\rangle \le \alpha(t) + (c_3 + \beta(t))|(v,b,\Phi)|_{\mathcal{H}}^2, \tag{87}$$

where $c_3 > 0, \alpha(t), \beta(t)$ *are integrable functions such that there exist real numbers* $\rho > 0, M_\alpha \ge 1, M_\beta \ge 1$, *with*

$$\alpha(t) \le M_\alpha e^{-\rho t}, \quad \beta(t) \le M_\beta e^{-\rho t}, \quad t \ge 0. \tag{88}$$

Furthermore, let

$$2\alpha_1 > \zeta\lambda_1^{-1} + 2c_3\lambda_1^{-1}. \tag{89}$$

Then any weak solution $(v,b,\Phi)(t)$ *to (22) converges to zero almost surely exponentially.*

Proof. Let $\eta \in (0, \rho)$ be such that ☐

$$2\alpha_1 > \lambda_1^{-1}(\zeta + \eta) + 2c_3\lambda_1^{-1}. \tag{90}$$

Then, we have

$$\begin{aligned}
&\mathbb{E}e^{\eta t}|(v,b,\Phi)(t)|_{\mathcal{H}}^2 = \mathbb{E}|(v,b,\Phi)(0)|_{\mathcal{U}}^2 + \int_0^t \eta e^{\eta s}\mathbb{E}|(w,\beta,\psi)(s)|_{\mathcal{H}}^2 ds - 2\nu_1 \int_0^t e^{\eta s}\mathbb{E}||(v(s)||^2 ds \\
&-2\nu \int_0^t e^{\eta s}\mathbb{E}||(b(s)||^2 ds - 2\epsilon^2 \int_0^t e^{\eta s}\mathbb{E}|A_2^{3/2}\Phi(s)|_{L^2}^2 ds - 2\epsilon \int_0^t e^{\eta s}\mathbb{E}\langle A_2 f(\Phi), \epsilon A_2 f(\Phi)\rangle ds \\
&+2 \int_0^t e^{\eta s}\mathbb{E}\langle h_1(v(s),\Phi(s)), (v, \epsilon A_2\Phi(s))ds + \int_0^t e^{\eta s}\mathbb{E}||h_2(s, v(s), \Phi(s))||_{L^2(\mathcal{H})}^2 ds \\
&\leq \mathbb{E}|(v,b,\Phi)(0)|_{L^2}^2 + (-2\alpha_1 + \lambda_1^{-1}(\zeta+\eta) + 2c_3\lambda_1^{-1}) \int_0^t e^{\eta s}\mathbb{E}|(v,b,\Phi)(s)|_{\mathcal{U}}^2 ds \\
&+ \int_0^t e^{\eta s}\mathbb{E}(2\alpha(s) + \varphi(s) + (\beta(s) + \delta(s))|(v,b,\Phi)(s)|_{\mathcal{H}}^2) ds \\
&+\mathbb{E}|(v,b,\Phi)(0)|_{\mathcal{H}}^2 + \int_0^t e^{\eta s}\mathbb{E}(2\alpha(s) + \varphi(s) + (\beta(s) + \delta(s))|(v,b,\Phi)(s)|_{\mathcal{H}}^2) ds,
\end{aligned} \tag{91}$$

which gives

$$\mathbb{E}e^{\eta t}|(v,b,\Phi)(t)|_{\mathcal{H}}^2 \leq \mathbb{E}|(v,b,\Phi)(0)|_{\mathcal{H}}^2 + \int_0^t e^{\eta s}(\varphi(s) + 2\alpha(s) + (2\beta(s) + \delta(s))\mathbb{E}|(v,b,\Phi)(s)|_{\mathcal{H}}^2) ds. \tag{92}$$

We obtain that any weak solution to (22) converges to zero exponentially in the mean square, applying the Gronwall lemma. Using the same method as in the proof of Theorem 3.3, we can finish the proof .

4 Stabilization of the 3D GMCHMHD model(22)

In this section, we discuss the stabilization of the 3D GMCHMHD model (22). As in [62], it is enough to consider a one dimensional Wiener process for that purpose.

We suppose that $h_0 \in \mathcal{H}$ and h_2 is defined by

$$h_2(t, v, \Phi) = \sigma(v - v^*, \Phi - \Phi^*), \quad \forall (v, \Phi) \in \mathcal{H}, \tag{93}$$

for some $\sigma \in \mathbb{R}$ and We also assume that

$$|h_1(v_1,\Phi_1) - h_1(v_2,\Phi_2)|_{\mathcal{H}} \leq L_1|(v_1,\Phi_1) - (v_2,\Phi_2)|_{\mathcal{H}}, \quad \forall (v_1,\Phi_1),(v_2,\Phi_2) \in \mathcal{H}, \quad h_1(0,0) \neq 0. \tag{94}$$

Lemma 4.1. *Let $(v^*, b^*, \Phi^*) \in \mathcal{U}$ be the unique solution to (28). If h_1 satisfies (94) and*

$$2\alpha_1 - c_1[||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1] > 0, \tag{95}$$

where L_1 is the Lipschitz constant of h_1 given in (94), then the stationary solution (v^*, b^*, Φ^*) to (28) is exponentially stable.

Proof. Let (v, b, Φ) be a solution to the deterministic system. We will only sketch the proof as it is similar to the proof of Theorem 10.2 of [77]. ☐

$$\begin{cases} \dfrac{dv}{dt} + \nu_1 A_0 v + B^N(v,v) - B^N(b,b) - R_0(\epsilon A_2\Phi, \Phi) = h_0^1 + h_1^1(v,\Phi), \\ \dfrac{db}{dt} + \nu A_1 b + B^N(v,b) - B^N(v,b) = 0, \\ \dfrac{d\Phi}{dt} + A_2\mu + B_1(v,\Phi) = h_0^2 + h_1^2(v,\Phi), \\ \mu = \epsilon A_2\Phi + \alpha f(\Phi), \\ (v,b,\Phi)(0) = (v_0, b_0, \Phi_0). \end{cases} \tag{96}$$

Let

$$(w,\beta,\psi) = (v,b,\Phi) - (v^*, b^*, \Phi^*). \tag{97}$$

Then (w,β,ψ) satisfies

$$\begin{cases} \dfrac{dw}{dt} + \nu_1 A_0 w + B^N(v,w) - B^N(w,v^*) - B^N(\beta,b^*) - B^N(b,\beta) - R_0(\epsilon A_2\Phi^*, \psi) - R_0(\epsilon A_2\psi, \Phi^*) \\ = h_1^1(v,\Phi) - h_1^1(v^*,\Phi^*), \\ \dfrac{d\beta}{dt} + \nu A_1\beta + B^N(w,b^*) + B^N(v,\beta) - B^N(\beta,v^*) - B^N(b,w) = 0, \\ \dfrac{d\psi}{dt} + \epsilon A_2^2\psi + B_1(w,\Phi) + B_1(v^*,\psi) + \alpha A_2 f(\Phi) - \alpha A_2 f(\Phi^*) = h_1^2(v,\Phi) - h_1^2(v^*,\Phi^*), \\ (w,\beta,\psi)(0) = (v_0,b_0,\Phi_0) - (v^*,b^*,\Phi^*). \end{cases} \tag{98}$$

Let

$$y = |(w,\beta,\psi)|_{\mathcal{H}}^2. \tag{99}$$

Then, multiplying $(98)_1$ by w, $(98)_2$ by β $(98)_3$ by $\epsilon A_2\psi$ and adding the resulting equalities, we have

$$\frac{dy}{dt} + 2\alpha_1 ||(w,\beta,\psi)||_{\mathcal{U}}^2 \leq [c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1]||(w,\beta,\psi)||_{\mathcal{U}}^2. \tag{100}$$

Assuming that

$$\sigma_0 = 2\alpha_1 - [c_1||(v^*,b^*,\Phi^*)||_{\mathcal{U}} + 2L_1] > 0, \tag{101}$$

we derive that

$$\frac{dy}{dt} + \kappa_2 y \leq 0, \tag{102}$$

where

$$\kappa_2 = \lambda_1\sigma_0 > 0. \tag{103}$$

It is follows that

$$y(t) \leq y(0)e^{-\kappa_2 t}, \quad \forall t \geq 0, \tag{104}$$

and the lemma is proved.

If the lipschitz constant L_1 of h_1 is sufficiently large such that $\kappa_2 < 0$, then we do not know that (v^*, b^*, Φ^*) is exponentially stable or not. However, the following result related to the stabilization of the 3D GMCHMHD systems holds true.

Theorem 4.1. *Suppose that h_1 satisfies (94). Let $(v^*, b^*, \Phi^*) \in \mathcal{U}$ be the unique solution to (28) and $\kappa_2 < 0$, where κ_2 is given by (103). Assuming that σ is any real number such that*

$$\lambda_1\kappa_2 + \sigma^2 > 0. \tag{105}$$

Then there exists $\Omega_0 \subset \Omega, \mathcal{P}(\Omega_0) = 0$*, such that for* $\omega \notin \Omega_0$*, there exists* $T(\omega) > 0$ *such that*

$$|(v, b, \Phi)(t) - (v^*, b^*, \Phi^*)|^2_{\mathcal{H}} \leq |(v, b, \Phi)(0) - (v^*, b^*, \Phi^*)|^2_{\mathcal{H}} e^{-\eta t}, \quad \forall t \geq T(\omega), \tag{106}$$

where $\eta > 0$ *is given below and* $(v, b, \Phi)(t)$ *is any weak solution to (11) with the function* h_2 *given by*

$$h_2(t, x, y) = \sigma(x - v^*, y - \Phi^*), \quad \forall (x, y) \in \mathcal{H}. \tag{107}$$

Proof. Let

$$(w, \beta, \psi)(t) = (v, b, \Phi)(t) - (v^*, b^*, \Phi^*). \tag{108}$$

Applying the Itô formula $|(w, \beta, \psi)(t)|^2_{\mathcal{H}}$, we derive as in (61)-(63) that

$$\begin{aligned}
&|(w,\beta,\psi)(t)|^2_{\mathcal{H}} = |(w,\beta,\psi)(0)|^2_{\mathcal{H}} - 2\nu_1 \int_0^t ||w(s)||^2 ds - 2\nu \int_0^t ||\beta(s)||^2 ds - 2\epsilon^2 \int_0^t |A_2^{3/2}\psi(s)|^2_{L^2} ds \\
&-2\int_0^t \mathbb{E}F_N(||v^*||)b_0(w, v^*, w)ds - 2\int_0^t \mathbb{E}(F_N(||v^*||) - F_N(||v||))b_0(v, v^*, w)ds - 2\int_0^t \mathbb{E}F_N(||v||)b_0(v, w, w)ds \\
&+2\int_0^t \mathbb{E}F_N(||b||)b_0(b, \beta, w)ds + 2\int_0^t \mathbb{E}F_N(||b^*||)b_0(\beta, b^*, w)ds - 2\int_0^t \mathbb{E}(F_N(||b||) - F_N(||b^*||))b_0(b, b^*, w)ds \\
&-2\int_0^t \mathbb{E}(F_N(||b^*||) - F_N(||b||))b_0(v, b^*, \beta)ds - 2\int_0^t \mathbb{E}F_N(||b^*||)b_0(w, b^*, \beta)ds - 2\int_0^t \mathbb{E}F_N(||b||)b_0(v, \beta, \beta)ds \\
&+2\int_0^t \mathbb{E}F_N(||v^*||)b_0(\beta, v^*, \beta)ds + 2\int_0^t \mathbb{E}F_N(||v||)b_0(b, w, \beta)ds - 2\int_0^t \mathbb{E}(F_N(||v||) - F_N(||v^*||))b_0(b, v^*, \beta)ds \\
&-2\int_0^t b_1(v^*, \psi, \epsilon A_2\psi)ds + 2\int_0^t b_1(w, \psi, \epsilon A_2\Phi^*)ds - 2\alpha\int_0^t \langle A_2 f(\Phi) - A_2 f(\Phi^*), \epsilon A_2\psi\rangle ds \\
&+\int_0^t ||h_2(s, v(s), \Phi(s))||^2_{L^2(\mathcal{H})} ds + 2\int_0^t \langle (w, \epsilon A_2\psi), h_2(s, v, \Phi)dW_s(s)\rangle \\
&+2\int_0^t \langle h_1(v, \Phi) - h_1(v^*, \Phi^*), (w, \epsilon A_2\psi)\rangle ds.
\end{aligned} \tag{109}$$

Using (65), we also have

$$\begin{aligned}
&-2\nu_1||w(s)||^2 - 2\nu||\beta(s)||^2 - 2\epsilon^2|A_2\psi(s)|^2_{L^2} + 2F_N(||v^*||)|b_0(w, v^*, w)| + (F_N(||v^*||) - F_N(||v||))|b_0(v, v^*, w)| \\
&+2F_N(||b||)b_0|(b, \beta, w)| + 2F_N(||b^*||)|b_0(\beta, b^*, w)| + 2(F_N(||b||) - F_N(||b^*||))|b_0(b, b^*, w)| \\
&+2F_N(||b^*||)|b_0(w, b^*, \beta)| + 2(F_N(||b^*||) - F_N(||b||))|b_0(v, b^*, \beta)| + 2F_N(||v||)|b_0(b, w, \beta)| \\
&+2F_N(||v^*||)|b_0(\beta, v^*, \beta)| + 2(F_N(||v||) - F_N(||v^*||))|b_0(b, v^*, \beta)| + 2|b_1(v^*, \psi, \epsilon A_2\psi)| + 2|b_1(w, \psi, \epsilon A_2\Phi^*)| \\
&-2\alpha\int_0^t \langle A_2 f(\Phi) - A_2 f(\Phi^*), \epsilon A_2\psi\rangle \leq [-2\alpha_1 + c_1||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1]||(w, \beta, \psi)(s)||^2_{\mathcal{U}} \\
&\leq [-2\alpha_1 + c_1||(v^*, b^*, \Phi^*)||_{\mathcal{U}} + 2L_1]\lambda_1|(w, \beta, \psi)(s)|^2_{\mathcal{H}}.
\end{aligned} \tag{110}$$

Let

$$2\eta = \lambda_1\kappa_2 + \sigma^2 > 0, \tag{111}$$

where κ_2 is given by (103).

It follows from (109)-(111) that

$$
\begin{aligned}
&\log |(w,\beta,\psi)(t)|_{\mathcal{H}}^2 \\
&= \int_0^t \frac{1}{|(w,\beta,\psi)(t)|_{\mathcal{H}}^2}\left(-2\nu_1||w(s)||^2 - 2\nu||\beta(s)||^2 ds - 2\epsilon^2|A_2^{3/2}\psi(s)|_{L^2}^2 + \sigma^2|(w,\beta,\psi)(s)|_{\mathcal{H}}^2\right) ds \\
&+ \log|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 \\
&- \int_0^t \frac{2}{|(w,\beta,\psi)(s)|_{\mathcal{H}}^2}(b_0(w,v^*,w) + b_0(v,v^*,w) + b_0(b,\beta,w) + b_0(\beta,b^*,w) + b_0(b,b^*,w) + b_0(w,b^*,\beta) \\
&+2b_0(v,b^*,\beta) + b_0(b,w,\beta) + b_0(\beta,v^*,\beta) + b_0(b,v^*,\beta) + b_1(v^*,\psi,\epsilon A_2\psi) - b_1(w,\psi,\epsilon A_2\Phi^*))ds \\
&+ \int_0^t \frac{2}{|(w,\beta,\psi)(s)|_{\mathcal{H}}^2}(\langle h_1(v,\Phi) - h_1(v^*,\Phi^*),(w,\epsilon A_2\psi)\rangle - \alpha\langle A_2 f(\Phi) - A_2 f(\Phi^*),\epsilon A_2\psi\rangle)ds \\
&+2\int_0^t \frac{\sigma|(w,\beta,\psi)(s)|_{\mathcal{H}}^2}{|(w,\beta,\psi)(s)|_{\mathcal{H}}^2} dW_s(s) - \frac{1}{2}\int_0^t \frac{4\sigma^2|(w,\beta,\psi)(s)|_{\mathcal{H}}^4}{|(w,\beta,\psi)(s)|_{\mathcal{H}}^4} ds \\
&\leq \log|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 - 2\eta t + 2\sigma W_t(t).
\end{aligned}
\tag{112}
$$

Since almost surely, we have

$$
\lim_{t\to\infty} \frac{W_t(t)}{t} = 0, \tag{113}
$$

we can find $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that for each $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that for all $t \geq T(\omega)$, we have

$$
\frac{2\sigma W_t(t)}{t} \leq \eta. \tag{114}
$$

Therefore, for $T(\omega) > 0$, we derive from (112) that

$$
\log|(w,\beta,\psi)(t)|_{\mathcal{H}}^2 \leq \log|(w,\beta,\psi)(0)|_{\mathcal{H}}^2 - \eta t, \tag{115}
$$

which proves (106). □

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Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethical Conduct

All subjects gave their informed consent for inclusion before they participated in the study.

Data Availability Statements

The data that support the findings of this study are openly available in this manuscript.

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