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# Inverse Domination Numbers to the Kronecker Product of Some Standard Graphs with Path

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**Abstract:**

Let G be a non-trivial connected graph with vertex set V (G). A subset D of V(G) is said to be a dominating set if all the elements of V-D are adjacent with at least one element in D. A minimum dominating set is a dominating set with minimum cardinality and its value is known as its domination number, denoted by γ (G).

If D′ is a dominating set in V-D is called the inverse dominating set of G with respect to D. The minimum cardinality taken over all inverse dominating sets of G is called the inverse domination number denoted by γ′. In this paper we find the inverse domination number of the resultant graphs obtained by the Kronecker product of some standard graphs with path.

Keywords: Domination, inverse domination, Kronecker product.

## Introduction:

The theory of domination began around 1960. In 1958, Berge formalized this concept. In 1962 O. Ore has used the term dominating set and domination number for the same concept in Graph Theory. This theory has been used in document summarization and designing secure for electrical grids. In 1991, the research paper of Kulli and Sigarkanti help to study the concept of inverse domination.

The Kronecker product $G_1(K)G_2$ of two graphs $G_1$ and $G_2$ is defined as $V(G_1\ (K)\ G_2)=V(G_1)\times V(G_2)$ and $E\ (G_1\ (K)\ G_2) = \{(u_1,v_1)\ (u_2,v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$
It is also named as the direct product and Tensor product.

**Definition: 1.1**

A walk in G is a finite non-null sequence $W=v_0e_1v_1e_2v_2,\ldots,e_kv_k$ whose terms are alternately vertics and edges. A walk is called a path if the vertices are distinct. A closed path is called a cycle. Number of edges in a cycle is called its length. Path and cycle of n vertices respectively denoted by $P_n$ and $C_n$

**Definition: 1: 2**

The open neighborhood of $v \in V(G)$ is the set of all vertices which are adjacent to $v$. That is $N(v) = \{u/uv \in E(G) \text{ and } u \in V(G)\}$, closed neighborhood of $v \in V(G)$ is defined as $N[v] = N(v) u \{v\}$

**Definition: 1:3**

The upper and lower sealing of $x$ is defined as

$$\lceil x \rceil = \begin{cases} x \text{ if } x \in N \\ [x+1] \text{ otherwise} \end{cases} \quad ; \quad \lfloor x \rfloor = \begin{cases} x \text{ if } x \in N \\ [x] \text{ otherwise} \end{cases}$$

**Preliminaries**

**Result 1:**

If $G = P_n$ then $\gamma'(G) = \left\lceil \frac{n}{3} \right\rceil$ for all $n > 3$

**Result 2:**

If $G = C_n$ then $\gamma'(G) = \gamma(G) = k$ if $n \equiv 0 \pmod 3$ & $n = 3k$

**Theorem 1.1**

Let $G_1$ and $G_2$ be any two paths of length m and n respectively. $G = G_1 (K) G_2$ then the domination and inverse domination number of G is

$$\gamma(G) = \begin{cases} n\left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 0 \pmod 3 \\ n\left[ \frac{m}{3} + 1 \right] & \text{if } m \equiv 1 \pmod 3 \\ n\left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 2 \pmod 3 \end{cases}$$

$$\gamma'(G) = \begin{cases} n\left[ \frac{m}{3} + 1 \right] & \text{if } m \equiv 0 \pmod 3 \\ n\left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 1, 2 \pmod 3 \text{ for all } m, n \text{ and } m \geq n. \end{cases}$$

Proof:

Let $G_1$ and $G_2$ be the path of length m and n respectively such that $m \geq n$.

Let $V(G_1) = \{ u_i / 1 \leq i \leq m\}$ ; $V(G_2) = \{ v_j / 1 \leq j \leq n\}$ and $d(u_1) = d(u_m) = d(v_1) = d(v_n) = 1$

G is the Kronecker product of $G_1$ and $G_2$.

Now $V(G) = \{u_i v_j / 1 \leq I \leq m; 1 \leq j \leq n\}$ with

$d(u_1 v_1) = d(u_1 v_n) = d(u_m v_l) = d(u_m v_n) = 1$

$d(u_i v_j) = d(u_i v_l) = d(u_i v_n) = d(u_n v_j) = 2$ for all $1 < i < m$ ; $1 < j < n$

$d(u_i v_j) = 4$ for all $2 \leq i < m$ ; $2 \leq j < n$

Case (i):

If m ≡ 0 (mod 3)

Choose D and D' are the subsets of V(G) such that

D = $\left\{u_{3i-1}\, v_j \;/\; 1 \le j \le ;\; 1 \le i \le \left\lceil\frac{m}{3}\right\rceil\right\}$ and

D' = $\left\{u_{3i-2}\;\; v_j \;;\; u_n v_j \;/ 1 \le \mathrm{j} \le \mathrm{n}; 1 \le \mathrm{i} \le \left\lceil\frac{m}{3}\right\rceil\right\}$

Clearly N[D] = N[D'] = V(G) is the required minimum dominating and inverse dominating sets of V(G).

=> |D| = n $\left\lceil\frac{m}{3}\right\rceil$& |D'| = (n + 1) $\left\lceil\frac{m}{3}\right\rceil$ …… (i)

If m ≡1 (mod 3)

Choose D and D' such that

D = $\left\{u_{3i-1}\, v_j \;\; ; u_n v_j \;/\; 1 \le \mathrm{j} \le \mathrm{n}\,;\; 1 \le \mathrm{i} \le \left\lceil\frac{m}{3}\right\rceil\right\}$

D' = $\left\{u_{3i-2}\;\; v_j \;/\; 1 \le \mathrm{j} \le \mathrm{n}\,;\; 1 \le \mathrm{i} \le \left\lceil\frac{m}{3}\right\rceil\right\}$

where D and D' are the required dominating and inverse dominating sets with

|D| = n $\left\lceil\frac{m}{3}\right\rceil$ & |D'| = n $\left\lceil\frac{m}{3}\right\rceil$ …… (ii)

If m ≡ 2 (mod 3)

Choose the subsets D and D' of V(G) such that

D = $\left\{u_{3i-1}\, v_j \;/ 1 \le \mathrm{j} \le \mathrm{n}\,;\; 1 \le \mathrm{i} \le \left\lceil\frac{m}{3}\right\rceil\right\}$

D' = $\left\{u_{3i-2}\;\; v_j \;/\; 1 \le \mathrm{j} \le \mathrm{n}\,;\; 1 \le \mathrm{i} \le \left\lceil\frac{m}{3}\right\rceil\right\}$

Now D and D′ are the minimum sets of V(G) such that

N[D] =N[D'] = V(G) and

|D|= n $\left\lceil\frac{m}{3}\right\rceil$ and |D'|= n $\left\lceil\frac{m}{3}\right\rceil$ ...... (iii)

From the above equations we get

$$\gamma\,(G) = \begin{cases} n\left\lceil\frac{m}{3}\right\rceil & \text{if m} \equiv 0 \text{ (mod 3)} \\ n\left[\frac{m}{3}+1\right] & \text{if m} \equiv 1 \text{ (mod 3)} \\ n\left\lceil\frac{m}{3}\right\rceil & \text{if m} \equiv 2 \text{ (mod 3)} \end{cases}$$

$$\gamma'\,(\mathrm{G}) = \begin{cases} n\left[\frac{m}{3}+1\right] & \text{if m} \equiv 0 \text{ (mod 3)} \\ n\left\lceil\frac{m}{3}\right\rceil & \text{if m} \equiv 1,2 \text{ (mod 3)} \end{cases}$$

Hence the proof

**Result 1.2**

If $G_1 = P_m$ ,$G_2 = P_n$ , $G = G_1(K)\, G_2$ then the domination and inverse domination numbers of G is $\gamma(G) \leq n\left\lceil \frac{m}{3} \right\rceil$ and $\gamma'(G) = n\left\lceil \frac{m}{3} + 1 \right\rceil$ for all m>n

**Theorem 1.3**

Let G be the Kronecker product of cycle of order m and a path of order n then the domination and inverse domination numbers is given as

$$\gamma(G) = \gamma'(G) = \begin{cases} n\left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 0 \pmod 3 \\ n\left\lceil \frac{m}{3} \right\rceil & \text{otherwise} \end{cases}$$

Proof :

Let G be the Kronecker product of $C_m$ and $P_n$. The vertex sets of $C_m$ and $P_n$ are taken as

$V(C_m) = \{ u_i / 1 \leq i \leq m \}$ and $V(P_n) = \{ v_j / 1 \leq j \leq n \}$ with

$d(u_i) = 2$ for all $i$, $u_i$ adjacent with $u_{i-1}$ and $u_{i+1}$ ;$1< i \leq n-1$

$d(v_1) = d(v_n)=1$ and $N(v_i)= \{v_{i-1}, v_{i+1}\}/ 1 \leq I \leq n$; i modulus n

Let $V(G) = \{ u_i v_j / 1 \leq i \leq m , 1 \leq j \leq n \}$ with

$d(u_i\, v_1) = d(u_i\, v_n) = 2$ for all $1 \leq i \leq m$ and

$d(u_i\, v_j) = 4$ for all $1 < i < m$ and $1 < j < n$

Case (i) If $n \equiv 0 \pmod 3$

Choose $D = \left\{ u_{3i-2}\, v_j \,/\, 1 \leq j \leq n\,;\, 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$ …… (i)

$D' = \left\{ u_{3i-1}\, v_j \,/\, 1 \leq j \leq n\,;\, 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$

Case (ii) If $n \equiv 1, 2 \pmod 3$

Choose $D = \left\{ u_{3i-2}\, v_j\,;\, u_{n-1}\, v_j \,/\, 1 \leq j \leq n;\, 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$ …… (ii)

$D' = \left\{ u_{3i-1}\, v_j\,;\, u_n\, v_j \,/\, 1 \leq j \leq n;\, 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$

In both cases the minimum disjoint sets D and D′ satisfy

$N[D] = N[D'] = V(G)$.

=> D and D′ are the required minimum dominating and inverse dominating set of V(G) with

$|D| = |D'| = n\left\lceil \frac{m}{3} \right\rceil$ if $m \equiv 0 \pmod 3$ and

$|D| = |D'| = n\left\lceil \frac{m}{3} \right\rceil$ if $m \equiv 1, 2 \pmod 3$.

$$\Rightarrow \gamma(G) = \gamma'(G) = \begin{cases} n\left\lceil \frac{m}{3} \right\rceil & m \equiv 0 \pmod 3 \\ n\left\lceil \frac{m}{3} \right\rceil & \text{otherwise} \end{cases}$$

**Theorem 1.4**

Let $G_1$ and $G_2$ are the cycles of order m and n respectively. G be the Kronecker product of $G_1$ and $G_2$ then $\gamma(G) = \gamma'(G) = \frac{mn}{4}$ where $m,n \equiv 0 \pmod 4$

Proof :

Let $V(G_1) = \{ u_i / 1 \le i \le m \}$ and $V(G_2) = \{ v_j / 1 \le j \le n \}$are the vertex sets of $G_1$ and $G_2$ respectively. $G = G_1(K)G_2$.

Now,$V(G) = \{ u_i v_j / 1 \le i \le m$ and $1 \le j \le n \}$.

Clearly, $d(u_iv_j) = 4$ and $N(u_iv_j) = \{u_{i-1}v_{j-1} ;u_{i-1}v_{j+1} ;u_{i+1}v_{j-1} ;u_{i+1}v_{j+1}\}$ for all i,j

Choose D and D′ are the subset of V(G) such that

$D = \left\{u_{4i-2}v_{4j-3};\ u_{4i-2}v_{4j-2}; u_{4i-}\ v_{4j-3}; u_{4i-1}v_{4j-2} / 1 \le i \le \frac{n}{4} \text{ for each i}, 1 \le i \le \frac{m}{4}\right\}$

$D' = \left\{u_{4i-2}\ v_{4j-1};\ u_{4i-2}\ v_{4j}; u_{4i-1}\ v_{4j-1};\ u_{4i-1}v_{4j} / 1 \le j \le \frac{n}{4} \text{ for each i}, 1 \le i \le \frac{m}{4}\right\}$

Clearly, D and D′ are disjoint minimum subsets of V(G) satisfies

$N[D] = N[D'] = V(G)$.

Hence D and D′ are the required dominating and inverse dominating sets of V(G) with

$|D| = |D'| = \frac{mn}{4}$

⇨ $\gamma(G) = \gamma'(G) = \frac{mn}{4}$

**Theorem 1.5**

$G_1$ is a path graph of order m and $G_2$ be a complete graph of order n, then G is the Kronecker product of $G_1$ and $G_2$ then $\gamma(G) = \gamma'(G) = m$.

Proof :

Let $V(G_1) = \{ u_i / 1 \le i \le m \}$and $V(G_2) = \{ v_j / 1 \le j \le n \}$ be the vertex sets of $P_m$ and $K_n$ respectively.

$V(G) = \{ u_i v_j / 1 \le i \le m ; 1 \le j \le n \}$be the vertex set of G; where $G = P_m (K)K_n$ with

$d(u_i v_j) = 2 (n - i)$ for all i; $1 < i < m$ and $d(u_i v_j) = n - 1$ for each i = 1, m.

Clearly, $N(u_i v_j) = \{ u_{i-1} v_k , u_{i+1} v_k / j \ne k \}$.

Let $D_{ik} = \{ u_i v_k / 1 \le i \le m \} : 1 \le k \le n$ …… (i)

$D_{kj} = \{ u_k v_j / 1 \le j \le n \} , 1 \le k \le m$. …… (ii)

From eqn (i)-for any fixed k, no two elements of $D_{ik}$ are adjacent in G. In similar, from eqn (ii) for any fixed k, no two elements of $D_{kj}$ are adjacent in G.

Choose $D_{ik} \leq V(G)$ such that

$D_{i1} = \{ u_i v_1 / 1 \leq i \leq m \}$

$D_{i2} = \{ u_i v_2 / i \leq i \leq m \}$

⋮ ⋮ ⋮

$D_{in} = \{ u_i v_n / 1 \leq i \leq m \}$ satisfies

$N[D_{i1}] = N[D_{i2}] = \ldots = N[D_{in}] = V(G)$.

Hence, each $D_{ik}$, $1 \leq k \leq n$, is the minimum set which satisfies the condition

$N[D_{ik}] = V(G)$ for all i = 1, 2, …; m for fixed k = 1, 2, …, n.

Also $D_{ik} \cap Dit = \emptyset$ if $k \neq t$

Hence, every $D_{ik} \subseteq V(G)$ for any fixed k = 1, 2, …, n is the required dominating set and another $Dit \subseteq V(G)$ such that $1 \leq t \leq n$ and $t \neq k$ are the dominating and inverse dominating sets of V(G).

Therefore, each dominating set $D_{ik}$ for any fixed k of V(G) we have (n −1) inverse dominating sets with same cardinality.

That is, $D_{il}, 1 \leq i \leq m$, is a dominating set of V(G),

then $D_{ik}$, $1 \leq i \leq m$, $1 < k \leq n$, are the inverse dominating sets of G.

Hence, $\gamma(G) = \gamma'(G) = m$.

Hence, the result.

**Theorem 1.6**

Graph G is the Kronecker product of $C_m$ and $K_{1,n}$ then the domination and inverse domination numbers is given by

$$\gamma(G) = \begin{cases} m, & \text{if m is even and } m \equiv 0 \pmod 4 \\ 2\left\{\left\lceil\frac{m}{4}\right\rceil + \left\lceil\frac{m}{4}\right\rceil\right\}, & \text{if m is even and } \frac{m}{2} \text{ is ood} \\ \left\lceil\frac{m}{4}\right\rceil + 3\left\lceil\frac{m}{4}\right\rceil, & \text{if } m \equiv 1 \pmod 4 \\ 3\left\lceil\frac{m}{4}\right\rceil + \left\lceil\frac{m}{4}\right\rceil, & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

$$\gamma'(G) = \begin{cases} m\,, & \text{if m is even and } m \equiv 0 \pmod 4 \\ 4\left\lceil\frac{m}{4}\right\rceil + 2(n-1), & \text{if m is even and } \frac{m}{2} \text{ is ood} \\ \left\lceil\frac{m}{4}\right\rceil + n - 1 & \text{if } m \equiv 1 \pmod 4 \\ \left\lceil\frac{m}{4}\right\rceil + 3\left\lceil\frac{m}{4}\right\rceil + n & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

Proof:

Let $C_m$ and $K_{,,n}$ are the cycle of order m and complete bipartite graph of order n+1 respectively. with

$V(C_m) = \{u_i / 1 \le i \le m\}$,$V(K_{1,n}) = \{v_j / 1 \le j \le n\}$ Such that $\deg(v_0) = n$.

Let G be the Kronecker product of $C_m$ and $K_{1,n}$ then

$V(G) = \{u_i v_j / 1 \le i \le m, 0 \le j \le n\}$ such that

$d(u_i v_0) = 2n$ and $d(u_i v_j) = 2$ for all $1 \le i \le m, 1 \le j \le n$.

Case (i): m is even and m ≡ 0 (mod 4)

then G is two disjoint components $G_1$ and $G_2$ with

$V(G_1) = \{u_{2i}v_0; u_{2i-1} v_j /1 \le i \le \frac{m}{2}; 1 \le j \le n\}$ and

$V(G_2) = \{u_{2i-1} v_0; u_{2i} v_j / 1 \le i \le \frac{m}{2}; 1 \le j \le n\}$ with

$d(u_{2i}v_0) = d(u_{2i-1} v_0) = 2n$ and $d(u_{2i-1} v_j) = d(u_{2i}v_j) = 2$ for all $1 \le i \le \frac{m}{2}; 1 \le j \le n$.

Since $G_1$ and $G_2$ are the disjoint components of G.

Hence, dominating and inverse dominating sets of $G_1$ and $G_2$ respectively are the dominating and inverse dominating sets of G.

Choose $D_1$ and $D_1'$ are the disjoint minimum subsets of $V(G_1)$ such that

$$\left.\begin{array}{l} D_1 = \left\{u_{4i-2}v_0 \,; u_{4i-3}v_1 \,/\, 1 \le i \le \frac{m}{4}\right\} \\ D_1' = \left\{u_{4i-2}v_0 \,;\, u_{4i-3}v_1 \,/\, 1 \le i \le \frac{m}{4}\right\} \end{array}\right\} \quad \ldots\ldots(i)$$

are the minimum subsets of $G_1$ which satisfy $N[D_1] = N[D_1'] = V(G_1)$

In similar choose $D_2$ and $D_2'$ such that

$$\left.\begin{array}{l} D_2 = \left\{u_{4i-3}\, v_0 \,;\, u_{4i-2}\, v_1 \,/\, 1 \le i \le \frac{m}{4}\right\} \ \& \\ D_2' = \left\{u_{4i-1}\, v_0 \,;\, u_{4i}\, v_1 \,/\, 1 \le i \le \frac{m}{4}\right\} \end{array}\right\} \quad \ldots\ldots (ii)$$

are the minimum disjoint subsets of $G_2$ satisfies

$N[D_2] = N[D_2'] = V(G_2)$.

Therefore, $D = D_1 \cup D_2$

$D = \left\{u_{4i-2}v_0 \,;\, u_{4i-3}v_0; u_{4i-3}v_1 \,;\, u_{4i-2}v_1 \,/\, 1 \le i \le \frac{m}{4}\right\}$ and

$D' = D_1' \cup D_2'$

$D' = \left\{u_{4i-}v_0;\, u_{4i-}v_1;\, u_{4i-1}\, v_0;\, u_{4i}v_1/\, 1 \le i \le \frac{m}{4}\right\}$

are the required minimum dominating and inverse dominating sets of G with

$|D| = |D'| = 4\left(\frac{m}{4}\right)$

$\Rightarrow |D| = |D'| = m$ …… ⓐ

Case (ii): If m is even and $\frac{m}{2}$ is odd.

In this case, we get the graph G as the two disjoint components of $G_1$ and $G_2$, each components having odd number of vertices with degree 2n.

Let $V(G_1) = \{u_{2i}\, v_{0;}\ u_{2i-1}\, v_j \,/\, 1 \le i \le \frac{m}{2};\ 1 \le j \le n\}$ and

$V(G_2) = \{u_{2i-1}\, v_0\ u_{2i}\, v_j \,/\, 1 \le i \le \frac{m}{2};\ 1 \le j \le n\}$

Since G is the union of $G_1$ and $G_2$, clearly dominating and inverse dominating sets of G is the union of the dominating and inverse dominating sets of $G_1$ and $G_2$ respectively.

Choose $D_1$ and $D_1'$ are the disjoint subsets of $V(G_1)$ such that

$D_1 = \{\, u_{4i-2}\, v_{0\,;}\ u_{4k+1} v_1 \,/\, 1 \le i \le \left\lceil\frac{m}{4}\right\rceil;\ 1 \le k \le \left\lceil\frac{m}{4}\right\rceil\}$ &
$D_1' = \{\, u_{4i} v_0\,;\ u_{4k-1}\, v_1\,;\ u_1 v_j \,/\, 1 \le i \le \left\lceil\frac{m}{4}\right\rceil;\ 1 \le k \le \left\lceil\frac{m}{4}\right\rceil\text{-}1;\ 1 \le j \le n\}$ …… (iii)

Clearly, $D_1$ and $D_1'$ are minimum sets satisfies

$N[D_1] = N[D_1'] = V(G_1)$.

Hence, $D_1$ and $D_1'$ are the minimum dominating sets of $G_1$.

In similar, choose $D_2$ and $D_2'$ are the minimum disjoint subsets of $G_2$.

$D_2 = \{\, u_{4i-3}\, v_0;\ u_{4k}\, v_1 \,/ 1 \le i \left\lceil\frac{m}{4}\right\rceil;\ 1 \le k \le \left\lceil\frac{m}{4}\right\rceil \}$&
$D_2' = \{\, u_{4i-1}\, v_0\ u_{4k+2}\ v_1\,;\ u_n v_j \,/\, 1 \le i \le \left\lceil\frac{m}{4}\right\rceil;\ 1 \le k \left\lceil\frac{m}{4}\right\rceil - 1\,;\ 1 \le j \le n \,\}$ ……(iv)

Clearly the sets $D_2$ and $D_2'$ satisfies

$N[D_2] = N[D_2'] = V(G_2)$

Hence, $D_2$ and $D_2'$ are the minimum dominating and inverse dominating sets of $G_2$.

Hence, $D = D_1 \cup D_2$

$D = \{\, u_{4i-2} v_0;\ u_{4k+1} v_{1;}\ u_{4i-3} v_{0;}\ u_{4k} v_1 \,/\, 1 \le i \left\lceil\frac{m}{4}\right\rceil;\ 1\le k \le \left\lceil\frac{m}{4}\right\rceil\}$ and
$D_2' = \{\, u_{4i} v_0;\ u_{4k-1} v_{1;} u_1\, v_j\,; u_{4i-1} v_0; u_{4k+2} v_1;\ u_n v_j\,; 1 \le i \le \left\lceil\frac{m}{4}\right\rceil\ \ 1\le k \le \left\lceil\frac{m}{4}\right\rceil - 1; 1 \le j \le n\,\}$ …(v)

are the required dominating and inverse dominating sets of G.

$|D| = 2\{\left\lceil\frac{m}{4}\right\rceil + \left\lceil\frac{m}{4}\right\rceil\}$ and
$|D'| = 2\left\lceil\frac{m}{4}\right\rceil + 2\{\left\lceil\frac{m}{4}\right\rceil - 1\} + 2n$
$\quad = 4\left\lceil\frac{m}{4}\right\rceil + 2(n\text{-}1)$ …… ⓑ

Case (iii) If m is odd, and m ≡1(mod)4

for all odd m, we get G is a connected simple graph with

$V(G) = \{\, u_i v_j \,/\, 1 \le i \le m\,;\ 0 \le j \le n\,\}$ with

$d(u_iv_0) = 2n; 1 \leq i \leq m$ ; $d(u_iv_j) = 2$ for all $1 \leq i \leq m$ ; $1 \leq j \leq n$

also, $N(u_{2i}v_0) = \{ u_{2i-1}v_j; u_{2i+1}v_j \,/\, 1 \leq i \leq \left[\frac{m}{4}\right]; 1 \leq j \leq n \}$ and

$N(u_{2i-1}v_0) = \{u_{2i-2}v_j; u_{2i}v_j \,/\, 1 \leq i \leq \left[\frac{m}{4}\right]; 1 \leq j \leq n\}$

Also, $N(u_1v_0) = \{ u_mv_j; u_2v_j \,/\, 1 \leq j \leq n \}$

In this case we choose D and D′ as

$$\left.\begin{array}{l} D = \{ u_{4i-3}v_0; u_{4j}v_0; u_{4k-2}v_1; u_{4l-3}v_1 / 1 \leq i \leq \left[\frac{m}{4}\right]; 1 \leq j; k; l \leq \left[\frac{m}{4}\right] - \} \\ D' = \{ u_{4i-1}v_0; u_{4i-2}v_0; u_{4j}v_1; u_{4k-1}v_1; u_nv_i / 1 \leq i; j \leq \left[\frac{m}{4}\right]; 1 \leq k \leq \left[\frac{m}{4}\right] - 1; 1 \leq l \leq n\} \end{array}\right\} \quad \ldots\ldots (vi)$$

Clearly D and D′ are the disjoint subsets of V(G) satisfies N[D] = N[D′] = V(G) with

$$\left.\begin{array}{l} |D| = \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] - \& \\ |D'| = 3\left[\frac{m}{4}\right] + \left[\frac{m}{4}\right] - 1 + n; \\ |D'| = 4\left[\frac{m}{4}\right] + n - 1. \end{array}\right\} \quad \ldots\ldots (c)$$

(Case (iv) If m is odd and m ≡ 3 (mod 4))

In this case G is a connected graph similar to in Case (iii) but the dominating and inverse dominating sets are different from Case (iii).

Let $V(G) = \{ u_iv_j \,/\, 1 \leq i \leq m ; 0 \leq j \leq n \}$ with

$$\left.\begin{array}{l} \text{Choose } D = \{u_{4i-3}v_0; u_{4i-2}v_0; u_{4i-2}v_1; u_{4j-1}; v_t / 1 \leq i \leq \left[\frac{m}{4}\right]; 1 \leq j \leq \left[\frac{m}{4}\right]\} \\ \text{and } D' = \{u_{4i-1}v_0; u_{4j}v_0; u_{4i-3}v_1; u_{4i-3}v_1; u_mv_t \,/\, 1 \leq i \leq \left[\frac{m}{4}\right]; 1 \leq jk, l \leq \left[\frac{m}{4}\right]; 1 \leq t \leq n\} \end{array}\right\} \quad \ldots\ldots (vii)$$

Where D and D' are the minimum disjoint subsets of V(G) with such that
N[D] = N[D'] = V(G) with

$$\left.\begin{array}{l} |D| = 3\left[\frac{m}{4}\right] + \left[\frac{m}{4}\right] \\ |D'| = \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] + n \end{array}\right\} \quad \ldots\ldots (d)$$

From (a), (b), (c) and (d) we get

$$\gamma(G) = \begin{cases} m, & \text{if m is even and } m \equiv 0 \pmod 4 \\ 2\left\{\left[\frac{m}{4}\right] + \left[\frac{m}{4}\right]\right\} & \text{if m is even and } \frac{m}{2} \text{ ood} \\ \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] & \text{if } m \equiv 1 \pmod 4 \\ \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

$$\gamma'(G) = \begin{cases} m & \text{if m is even \& } m \equiv 0 \pmod 4 \\ 4\left[\frac{m}{4}\right] 2(n-1) & \text{if m is even and } \frac{m}{2} \text{ is odd} \\ \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] & \text{if } m \equiv 1 \pmod 4 \\ \left[\frac{m}{4}\right] + 3\left[\frac{m}{4}\right] & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

**Corollary 1.7**

By Theorem 1.6, GS = $C_m$ (K) $K_{1,n}$ then γ(G) = m for all m ≥ 4.

Proof :

By theorem 1.6

$$\gamma(G) = \begin{cases} m, & \text{if m is even and } m \equiv 0 \pmod 4 \\ 2\left\{\left\lceil \frac{m}{4} \right\rceil + \left\lceil \frac{m}{4} \right\rceil\right\}, & \text{if m is even and } \frac{m}{2} \text{ } is \text{ ood} \\ \left\lceil \frac{m}{4} \right\rceil + 3\left\lceil \frac{m}{4} \right\rceil, & \text{if } m \equiv 1 \pmod 4 \\ 3\left\lceil \frac{m}{4} \right\rceil + \left\lceil \frac{m}{4} \right\rceil, & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

Case (i): If m ≡ 0 (mod 4), no need to prove.

Case (ii): If m is even and $\frac{m}{2}$ is odd

Let m = 4k + 2.

$\gamma(G) = 2\left\{\left\lceil \frac{m}{4} \right\rceil + \left\lceil \frac{m}{4} \right\rceil\right\}$

$= 2\left\{\left\lceil \frac{4k+2}{4} \right\rceil + \left\lceil \frac{4k+2}{4} \right\rceil\right.$ $\quad \left\{\because \left\lceil \frac{4k+2}{4} \right\rceil + \frac{4k+4}{4} \;\&\; \left\lceil \frac{4k+2}{4} \right\rceil = \frac{4k}{4}\right\}$

$= 2\left\{\left\lceil \frac{4k+2+2}{4} \right\rceil + \left\lceil \frac{4k}{4} \right\rceil\right\}$

$= 2\left\{\left\lceil \frac{8k+4}{4} \right\rceil\right\}$

$= 2\left\{\left\lceil \frac{2(4k+2)}{4} \right\rceil\right\}$

= 4k + 2

= m. …… (ii)

Case (iii):
If m ≡ 1 (mod 4); let m = 4k + 1.
$\gamma(G) = \left\lceil \frac{m}{4} \right\rceil + 3\left\lceil \frac{m}{4} \right\rceil$

$= \left\lceil \frac{4k+1}{4} \right\rceil + 3\left\lceil \frac{4k+4}{4} \right\rceil$ $\quad$ {∵ [m ≡ 1 (mod 4)

$= \frac{4k+1+3}{4} + 3\left\lceil \frac{4k}{4} \right\rceil$ $\quad 3\left\lceil \frac{4k+4}{4} \right\rceil = \left\lceil \frac{4k+1+3}{4} \right\rceil = \left\lceil \frac{4k+1}{4} \right\rceil = \frac{4k}{4}\}$

$= \frac{16k+4}{4}$

$\left\lceil \frac{4k+1}{4} \right\rceil$ = m …… (iii)

Case (iv):
If m ≡ 3 (mod 4); let m = 4k + 3.

$\gamma(G) = 3 \left\lceil \frac{m}{4} \right\rceil + \left\lfloor \frac{m}{4} \right\rfloor$ [∵m ≡ 3 (mod 4)

$= 3\left\lceil \frac{4k+3}{4} \right\rceil + \left\lfloor \frac{4k+3}{4} \right\rfloor$ $\left\lceil \frac{4k+3}{4} \right\rceil = \left\lceil \frac{4k+3}{4} \right\rceil$ & $\left\lfloor \frac{4k+3}{4} \right\rfloor = \left\lfloor \frac{4k}{4} \right\rfloor$]

$= 3\left\lceil \frac{4k+3+1}{4} \right\rceil + \left\lfloor \frac{4k}{4} \right\rfloor$

$= 3\left(\frac{4k+4}{4}\right) + \frac{4k}{4}$

$= \frac{16k+12}{4}$

$= \frac{4(4k+3)}{4} = m$ …… (iv)

From equations (i), (ii), (iii) and (iv), we get

γ(G) = m.

Hence the proof.

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