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Backward Doubly Stochastic Differential Equations with Weak Regularity Coefficients: Existence in L^p

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Abstract:
This work is devoted to the study of multidimensional backward doubly stochastic differential equations under weak regularity assumptions on the coefficients. In particular, we introduce a new class of non-Lipschitz conditions and show that the associated BDSDEs admit a unique solution in Lp for any p > 1. Several classical results are recovered as particular cases of our analysis.

Keywords. Backward doubly stochastic differential equation, Non-Lipschitz coefficients, Mao's condition, Lp solution

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1 Introduction

We study the following backward doubly stochastic differential equation (hereafter abbreviated as BDSDE):

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s\,,\ \ t \in T, \qquad (1.1)$$

where $T \geq 0$ denotes a fixed time horizon and ξ is an R^d-valued random variable representing the terminal condition. The coefficient functions $f(\omega; t; y; z) : \Omega \times [0; T] \times \mathrm{R}^d \times \mathrm{R}^{d\times d} \Rightarrow \mathrm{R}^d;$ $g(\omega; t; y; z) : \Omega \times [0; T] \times \mathrm{R}^d \times \mathrm{R}^{d\times d} \Rightarrow \mathrm{R}^d;$ are assumed to be progressively measurable for each $(y; z)$ and are referred to as the generators of the BDSDE. Moreover, $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ are two independent standard Brownian motions defined on a complete probability space $(\Omega; \mathrm{F}; \mathrm{P})$. A solution of the BDSDE is a pair of adapted processes $(y; z)$.

Backward doubly stochastic differential equations were first introduced in the nonlinear framework by Pardoux and Peng [7], who proved the existence and uniqueness of L^2-solutions under Lipschitz continuity assumptions on the generators f and g. Since then, BDSDEs have attracted considerable attention and have become an important mathematical tool in various areas, including semilinear and quasilinear stochastic partial differential equations (see [2], [4], [5], [6], [7]), stochastic control (see [3]), and related fields.

Subsequent research has focused on weakening the classical Lipschitz conditions imposed on the coefficients. For example, Shi et al. [9] established the existence of L^2-

solutions in the one-dimensional case when the generator f is merely continuous and satisfies a linear growth condition with respect to y and z. Later, Bahlali et al. [2] obtained existence, uniqueness, and L^p-stability results for $p < 2$ when the generator f exhibits superlinear growth in both variables. More recently, Aman [1] investigated the well-posedness of BDSDEs in L^p, for $p > 1$, under monotonicity assumptions on the generator f.

Motivated by these developments, the present paper is devoted to the study of multidimensional BDSDEs with a newly introduced class of non-Lipschitz coefficients. We prove existence and uniqueness of solutions in L^p for any $p > 1$.

The paper is organized as follows. Section 2 collects several preliminary results and auxiliary lemmas. In Section 3, we state and prove our main theorem.

2 Assumptions and Definition

Let $(\Omega; F; P)$ be a complete probability space, and $T > 0$. Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independents standard Brownian motion defined on $(\Omega; F; P)$ with values in R^d and R, respectively. For $t \in [0, T]$, we put,

$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, and $\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B$ where $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B = \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. It should

be noted that (F_t) is not an increasing family of sub σ-fields, and hence it is not a filtration. However $(\mathcal{G}_t)$ is a filtration.

Let $M_T^p(0, T, \mathbb{R}^d)$ denote the set of d-dimensional, jointly measurable stochastic processes $\{\varphi_t; t \in [0, T]\}$, which satisfy :

(a) $\|\varphi\|_{M^p} := \left\{E\left[\left(\int_0^T |\varphi_t|^2 \, dt\right)^{p/2}\right]\right\}^{1/p} < +\infty$.

(b) φ_t is $\mathcal{F}_t - measurable, for$ any $t \in [0, T]$:

We denote by $S_T^p([0, T], \mathbb{R})$,; the set of continuous stochastic processes 't, which satisfy :

(a') $\|\varphi\|_{S^p} := (E[\sup_{t\in[0,T]} |\varphi_t|^p])^{1/p} < +\infty$.

(b') For every $t \in [0, T]$, $\varphi_t is \mathcal{F}_t$ −measurable.

And by L^p the set of all R^d-valued and F^T -measurable random vectors ξ such that $E|\xi|^p < +\infty$ We consider the following conditions,

H1) $dP \times dt$ $a.e.$, $\forall (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$

$$|f(t, y, z)| \leq \psi^{\frac{1}{p}}(|y|^p) + C|z| + \varphi_t + \theta_t,$$

where $C \geq 0$, both φ_t and θ_t are two nonnegative, (F_t)-progressively measurable processes with

$E\left[\int_0^T \varphi_t^p dt\right] < +\infty$ *and* $E[(\int_0^T \theta_t dt)^p] < +\infty$, *and* $\psi(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and concave function with $\psi(0) = 0$.

H2) There exists a non decreasing and concave function $\rho(\cdot): \mathbb{R}^+ \to \mathbb{R}^+ with\, \rho(0) = 0, \rho(x) > 0\ for\ x > 0\ and \int_{0^+} \frac{dx}{\rho(x)} = +\infty\ such\ that\ dP \times dt,\quad \forall y_1, y_2 \in \mathbb{R}^d, z \in \mathbb{R}^{d\times d}$,

$$|f(t, y_1, z) - f(t, y_2, z)|^p \le \rho(|y_1 - y_2|^p).$$

H3) There exists a constant $C > 0$ such that $dP \times dt$-a.e., $\forall y \in \mathbb{R}^d, z_1, z_2 \in \mathbb{R}^{d\times d}$,

$$|f(t, y, z_1) - f(t, y, z_2)| \le C|z_1 - z_2|.$$

H4) $E[(\int_0^T |f(t, 0,0)|dt)^p] < +\infty, and\, E[(\int_0^T |g(t, 0,0)|dt)^p] < +\infty.$

H5) There exist constants $L > 0$ et $0 < \alpha < 1$; such that for every $(t, \omega) \in \Omega \times [0, T]\ and\ (y, z) \in \mathbb{R} \times \mathbb{R}^d$

$$|g\ (t;\ y;\ z)\ -\ g\ (t;\ y';\ z')\ |^2 \le L|y - y'| + \alpha|z - z'|^2$$

H6) Let ξ be a square integrable random variable which is F_T -mesurable.

Definition 2.1. A L^p solution of equation (1.1) is a $(\mathbb{R}^d \times \mathbb{R}^d)$-valued F_t-progressively measurable process $(Y_t, Z_t)_{0\le t\le T}$ *which satisfies equation* (1.1) *and* $(Y, Z) \in S^p(0, T, \mathbb{R}^d) \times M^p(0, T, \mathbb{R}^d)$.

We now introduce Propositions 2.1 and 2.2, both of which are essential to the proof of the main theorem.

Proposition 2.1. Let assumptions **(H1)-(H5)-(H6)** hold and let $(y_t, z_t)_{t\in T}$be a solution in L^p to BDSDE (1.1). Then there exists a constant $K_{\lambda,L,p,T}$ depending on λ, L, p and T such that for each $t \in [0, T]$,

$$\frac{1+\alpha}{2}(\int_t^T |z_s|^2 ds)^{\frac{p}{2}} \le \frac{K^2_{\lambda,L,p,T}}{2}[\sup_{s\in[t,T]} |y_s|^p + \psi(\sup_{s\in[t,T]} |y_s|^p) + \left[\int_t^T (\varphi_s + \theta_s)ds\right]^p]$$

Proof. Applying Itô's formula to $|y_t|^2$ leads to that

$$\begin{aligned} &|y_t|^2 + \int_t^T |z_s|^2 ds \\ &\qquad = |\xi|^2 + 2\int_t^T \langle y_s, f(s, y_s, z_s)\rangle ds + 2\int_t^T \langle y_s, g(s, y_s, z_s)dB_s\rangle \\ &\qquad - 2\int_t^T \langle y_s, z_s dW_s\rangle + \int_t^T |g(s, y_s, z_s)|^2 ds \end{aligned}$$

By assumption **(H1)** we have, for each $s \in [t, T]$,

$$2\langle y_s, f(s, y_s, z_s)\rangle \leq 2|y_s|\left(\psi^{\frac{1}{p}}(y_s|^p) + \lambda|z_s| + \varphi_s + \theta_s\right)$$
$$\leq 2\left(\sup_{s\in[t,T]}|y_s|\right)\left(\psi^{\frac{1}{p}}(y_s|^p) + \varphi_s + \theta_s\right) + 2\lambda^2 \sup_{s\in[t,T]}|y_s|^2 + \frac{|z_s|^2}{2}$$

and by assumption **(H5)**

$$\int_t^T |g(s, y_s, z_s)|^2 ds \leq \int_t^T (L|y_s|^2 + \alpha|z_s|^2 + |g(s,0,0)|^2) ds$$

Thus, in view of the inequality that $2ab \leq a^2 + b^2$ we can get that

$$(\frac{1}{2} - \alpha)\int_t^T |z_s|^2 ds$$
$$\leq (3 + 2\lambda^2 T + LT)\sup_{s\in[t,T]}|y_s|^2 + \left[\int_t^T \psi^{\frac{1}{p}}(y_s|^p) ds\right]^2 + \left[\int_t^T (\varphi_s + \theta_s) ds\right]^2$$
$$+ \left[\int_t^T |g(s,0,0)| ds\right]^2 + 2\int_t^T \langle y_s, g(s, y_s, z_s) dB_s\rangle + 2\left|\int_t^T \langle y_s, z_s dW_s\rangle\right|$$

Taking into account that $\psi(.)$ is nondecreasing and invoking the inequality $(a+b)^{p/2} \leq 2^p(a^{p/2} + b^{p/2})$, we deduce

$$(\int_t^T |z_s|^2 ds\ \ )^{\frac{p}{2}} \leq C_{\lambda,L,p,T}[\sup_{s\in[t,T]}|y_s|^p + \psi(\sup_{s\in[t,T]}|y_s|^p) + \left[\int_t^T (\varphi_s + \theta_s) ds\right]^p +$$
$$\left[\int_t^T |g(s,0,0)| ds\right]^p + |\int_t^T \langle y_s, g(s, y_s, z_s) dB_s\rangle|^{\frac{p}{2}} + \left|\int_t^T \langle y_s, z_s dW_s\rangle\right|^{\frac{p}{2}}] \qquad (2.1)$$

where $C_{\lambda,L,p,T} = 2^{p+6}(3 + 2\lambda^2 T + LT + T^p)$ and we have employed the fact that

$$\left[\int_t^T \psi^{\frac{1}{p}}(|y_s|^p) ds\right]^p \leq T^p \psi(\sup_{s\in[t,T]}|y_s|^p)$$

An application of the Burkholder-Davis-Gundy (BDG) inequality implies that, for any $t \in [0,T]$,

$$C_{\lambda,L,p,T}E[|\int_t^T \langle y_s, g(s, y_s, z_s) dB_s\rangle|^{\frac{p}{2}} + |\int_t^T \langle y_s, z_s dW_s\rangle|^{\frac{p}{2}}]$$
$$\leq L_{\lambda,L,p,T}E[(\int_t^T |y_s|^2 |g(s, y_s, z_s)|^2 ds)^{\frac{p}{4}} + (\int_t^T |y_s|^2 |z_s|^2 ds)^{\frac{p}{4}}]$$
$$\leq L_{\lambda,L,p,T}E[\sup_{s\in[t,T]}|y_s|^{\frac{p}{2}} \cdot (\int_t^T |g(s, y_s, z_s)|^2 ds + \int_t^T |z_s|^2)^{\frac{p}{4}}]$$
$$\leq \frac{L^2_{\lambda,L,p,T}}{2}E[(\int_t^T |y_s|^2 |g(s, y_s, z_s)|^2 ds)^{p/4} + (\int_t^T |y_s|^2 |z_s|^2 ds)^{p/4}]$$

$$\leq \frac{L^2_{\lambda,L,p,T}}{2} E[\sup_{s\in[t,T]} |y_s|^p] + \frac{1}{2} E[(\int_t^T |g(s,y_s,z_s)|^2 ds + \int_t^T |z_s|^2)^{p/2}]$$

$$\leq \frac{K^2_{\lambda,L,p,T}}{2} E[\sup_{s\in[t,T]} |y_s|^p] + \frac{1+\alpha}{2} E[(\int_t^T |z_s|^2)^{p/2}]$$

Coming back to estimate (2.1) we get, for each $t \in [0,T]$

$$\frac{1+\alpha}{2} (\int_t^T |z_s|^2 ds)^{\frac{p}{2}} \leq \frac{K^2_{\lambda,L,p,T}}{2} [\sup_{s\in[t,T]} |y_s|^p + \psi(\sup_{s\in[t,T]} |y_s|^p) + \left[\int_t^T (\varphi_s + \theta_s) ds\right]^p]$$

Observing that $\psi(\cdot)$ is a concave function, the desired result follows directly from Jensen's inequality together with Hölder's inequality. This completes the proof.

Proposition 2.2. Assume that **(H1)-(H5)-(H6)** holds and let $(y_t, z_t)_{t\in[0,T]}$ be an L^p solution of BDSDE (1.1). Then there exist positive constants C_p and $K_{\lambda,L,\alpha,p}$such that, for every $t \in [0,T]$

$$E[\sup_{s\in[t,T]} |y_s|^p] \leq e^{K_{\lambda,L,\alpha,p}(T-t)} \{C_p E[|\xi|^p] + C_p E[(\int_t^T \theta_s ds)^p] + \frac{1}{2} E[\int_t^T \varphi_s^p ds]$$
$$+ \frac{1}{2} \int_t^T \psi(E[|y_s|^p]) ds\}$$

Proof. From Itô formula applied to $|y_t|^p$, we get

$$|y_t|^p + p[(p-1)\wedge 1]/2 \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds$$
$$\leq |\xi|^p + p \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, f(s,y_s,z_s)\rangle ds$$
$$+ p \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, g(s,y_s,z_s)\rangle dB_s + p[(p-1)$$
$$\wedge 1]/2 \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |g(s,y_s,z_s)|^2 ds - p \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, z_s dW_s\rangle$$

From **(H1)** and **(H5)** we deduce that, for each $t \in [0,T]$,

$$|y_t|^p + p[(p-1)\wedge 1]/2 \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds$$
$$\leq |\xi|^p + p \int_t^T |y_s|^{p-1} [\psi^{\frac{1}{p}}(|y_s|^p) + \lambda|z_s| + \varphi_s + \theta_s] ds + p[(p-1)$$
$$\wedge 1]/2 \int_t^T |y_s|^{p-1} [L|y_s|^2 + \alpha|z_s|^2 + |g(s,0,0)|^2] ds$$
$$+ p \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, g(s,y_s,z_s)\rangle dB_s - p \int_t^T y_s|^{p-2} |1_{|y_s|\neq 0} \langle y_s, z_s dW_s\rangle$$

First, observe that the nondecreasing and concave nature of $\psi(\cdot)$, together with the condition $\psi(0) = 0$, implies that has at most linear growth. Hence, the desired result follows from the preceding inequality,

dP – a.s.,

$\int_0^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds < +\infty.0$

In addition, an application of Young's inequality, namely $a^r b^{1-r} \leq ra + (1-r)b$ for all $a \geq 0$, $b \geq 0$ and *r in* (0*;* 1), combined with the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ yields,

$$p\int_t^T |y_s|^{p-1}(\psi^{\frac{1}{p}}(|y_s|^p) + \varphi_s)ds + p[(p-1)\wedge 1]/2\int_t^T |y_s|^{p-1}[L|y_s|^2 + |g(s,0,0)|^2]ds$$

$$= p\int_t^T \left\{ \left(\beta^{\frac{1}{p-1}}|y_s|^p\right)^{\frac{p-1}{p}} \left[\frac{1}{\beta}(\psi^{1p}(|y_s|^p) + \varphi_s)^p + L|y_s|^2 \right.\right.$$

$$\left.\left. + |g(s,0,0)|^2\right]^{\frac{1}{p}} \right\} ds$$

$$\leq (p-1)\beta^{\frac{1}{p-1}}\int_t^T |y_s|^p ds + \frac{2^p}{\beta}\int_t^T (\psi(|y_s|^p) + \varphi_s^p + |g(s,0,0)|^p)ds,$$

where $\beta > 0$ will be chosen later. From the inequality $ab \leq (a^2 + b^2)/2$ we get,

$p\lambda|y_s|^{p-1}|z_s| + \frac{p[(p-1)\wedge 1]}{2}\alpha\int_t^T |y_s|^{p-1}|z_s|^2 ds =$
$p\left(\frac{\sqrt{2}(\lambda+\alpha)\sqrt{1\wedge(p-1)}}{|}y_s|^{\frac{p}{2}}\right)\left(\sqrt{\frac{1\wedge(p-1)}{2}}|y_s|^{\frac{p-2}{2}}|z_s|\right) \leq \frac{p(\lambda+\alpha)^2}{1\wedge(p-1)}|y_s|^p + \frac{c(p)}{2}|y_s|^{p-2}1_{|y_s|\neq 0}|z_s|^2.$

We obtain,

$$|y_t|^p + \frac{c(p)}{2}\int_t^T |y_s|^{p-2}1_{|y_s|\neq 0}|z_s|^2 ds$$

$$\leq A_t + p\int_t^T |y_s|^{p-2}1_{|y_s|\neq 0}\langle y_s, g(s,y_s,z_s)\rangle dB_s$$

$$- p\int_t^T |y_s|^{p-2}1_{|y_s|\neq 0}\langle y_s, z_s dW_s\rangle, \qquad (2.2)$$

where $A_t = |\xi|^p + L_{\lambda,L,p,\beta}\int_t^T |y_s|^p ds + \frac{2^p}{\beta}\int_t^T(\psi(|y_s|^p) + \varphi_s^p + |g(s,0,0)|^p)ds +$
$p\int_t^T |y_s|^{p-1}\theta_s ds$

with

$$L_{\lambda,L,p,\beta} = (p-1)\beta^{\frac{1}{p-1}} + \frac{p(\lambda+L)^2}{[1 \wedge (p-1)]} > 0.$$

By the Burkholder-Davis-Gundy inequality, the process

$$M_t = \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, g(s,y_s,z_s)\rangle dB_s - \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} \langle y_s, z_s dW_s\rangle,$$

is a uniformly integrable martingale. Moreover, an application of Young's inequality yields

$$\begin{aligned}
E\left[\langle M,M\rangle_T^{1/2}\right] &\leq E\left[\sup_{s\in[0,T]} |y_s|^{p-1} \left(\int_0^T |g(s,y_s,z_s)|^2 ds + \int_0^T |z_s|^2 ds\right)^{1/2}\right] \\
&= E\left\{\left(\sup_{s\in[0,T]} |y_s|^p\right)^{\frac{p-1}{p}} \left[\left(\int_0^T |g(s,y_s,z_s)|^2 ds + \int_0^T |z_s|^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}\right\} \\
&\leq \frac{p-1}{p} E\left[\sup_{s\in[0,T]} |y_s|^p\right] + \frac{1}{p} E\left[\left(\int_0^T |g(s,y_s,z_s)|^2 ds + \int_0^T |z_s|^2 ds\right)^{p/2}\right] \\
&\leq (\frac{p-1}{p})(1+L) E\left[\sup_{s\in[0,T]} |y_s|^p\right] + \frac{1+\alpha}{p} E\left[\left(\int_0^T |z_s|^2 ds\right)^{p/2}\right] \\
&+ \frac{1}{p}(\int_0^T |g(s,0,0)|^2 ds)^{p/2} < +\infty.
\end{aligned}$$

Coming back to inequality (2.2), and taking the expectation, we get both

$$\frac{c(p)}{2} E\left[\int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds\right] \leq E[A_t] \quad (2.3)$$

and

$$E\left[\sup_{s\in[t,T]} |y_s|^p\right] \leq E[A_t] + L_p E\left[(\langle M,M\rangle_T - \langle M,M\rangle_t)^{1/2}\right], \quad (2.4)$$

where the BDG inequality is used in the final inequality. On the other hand, Young's inequality implies

$$\begin{aligned}
&L_p E\left[(\langle M,M\rangle_T - \langle M,M\rangle_t)^{1/2}\right] \\
&\leq L_p E\left[\sup_{s\in[t,T]} |y_s|^{p/2} \left(\int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |g(s,y_s,z_s)|^2 ds \right.\right. \\
&\left.\left. + \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds\right)^{1/2}\right] \\
&\leq \frac{1}{2} E\left[\sup_{s\in[t,T]} |y_s|^p\right] \\
&+ \frac{L_p^2}{2} E\left[\int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |g(s,y_s,z_s)|^2 ds + \int_t^T |y_s|^{p-2} 1_{|y_s|\neq 0} |z_s|^2 ds\right].
\end{aligned}$$

Coming back to inequalities (2.3) and (2.4), and applying Young's inequality, we obtain

$pL'_pE\left[\int_t^T |y_s|^{p-1}\theta_s ds\right] \leq \frac{1}{2}E\left[\sup_{s\in[t,T]}|y_s|^p\right] + \frac{L''_p}{2}E\left[\left(\int_t^T \theta_s ds\right)^p\right]$,;

from which we infer, upon reverting to the definition of A_t, that

$$E\left[\sup_{s\in[t,T]}|y_s|^p\right] \leq 2L'_pE\left[|\xi|^p + L_{\lambda,L,p,\beta}\int_t^T |y_s|^p ds + \frac{2^p}{\beta}\int_t^T (\psi(|y_s|^p) + \varphi_s^p)ds + \int_t^T |g(s,0,0)|^2 d\,s\right] + L''_pE\left[\left(\left(\int_t^T \theta_s ds\right)^p\right)\right].$$

With $\beta = 2^{p+2}L'_p$, an application of Fubini's theorem and Jensen's inequality, together with the concavity of , yields, for each t *in* $[0; T]$,

$$E\left[\sup_{s\in[t,T]}|y_s|^p\right] \leq 2L'_pE[|\xi|^p] + L''_pE\left[\left(\int_t^T \theta_s ds\right)^p\right] + \frac{1}{2}E\left[\int_t^T \varphi_s^p ds\right] + \frac{1}{2}\int_t^T \psi(E[|y_s|^p])ds + \frac{1}{2}E[\int_t^T |g(s,0,0)|^p ds] + 2L'_pL_{\lambda,L,p,\beta}\int_t^T [\sup_{s\in[t,T]}|y_s|^p]ds.$$

In conclusion, invoking Gronwall's inequality, we obtain, for all t *in* $[0; T]$,

$$E\left[\sup_{s\in[t,T]}|y_s|^p\right] \leq 2L'_pE[|\xi|^p] + L''_pE\left[\left(\int_t^T \theta_s ds\right)^p\right] + \frac{1}{2}E\left[\int_t^T \varphi_s^p ds\right] + \frac{1}{2}\int_t^T \psi(E[|y_s|^p])ds + \frac{1}{2}E[\int_t^T |g(s,0,0)|^p ds] + 2L'_pL_{\lambda,L,p,\beta}\int_t^T [\sup_{s\in[t,T]}|y_s|^p]ds.$$

Hence, Proposition 2.2 is proved.

3 Main result

Theorem 3.1. below is the principal result of this paper.

Theorem 3.1. Let *f, g* satisfying assumptions **(H2)-(H5).** Then, for any ξ *in* L^p , the BDSDE (1.1) has a unique solution in L^p.

We can construct the Picard approximate sequence of the BDSDE as follows:

$y_t^0 = 0;\quad y_t^n = \xi + \int_t^T f(s, y_s^{n-1}, z_s^n)ds + \int_t^T g(s, y_s^{n-1}, z_s^n)dB_s - \int_t^T z_s^n dW_s,\ t \in [0,T,$ (3.1)

Indeed, for each integer $n \geq 1$, it follows from **(H2),** and with $A > 0$ that

$$|f(s, y_s^{n-1}, 0)| \leq |f(s,0,0)| + \rho^{\frac{1}{p}}(|y_s^{n-1}|^p) \leq |f(s,0,0)| + A^{\frac{1}{p}}(|y_s^{n-1}| + 1),$$

and then

$$E\left[\left(\int_0^T |f(s, y_s^{n-1}, 0)|ds\right)^p\right]$$
$$\leq 2^p E\left[\left(\int_0^T |f(s, 0,0)|ds\right)^p\right] + A(2T)^p \left(E\left[\sup_{s\in[0,T]} |y_s^{n-1}|^p\right] + 1\right).$$

Hence, the generator $f(t, y_t^{n-1}, z_t)$ of BDSDE (3.1) satises **(H3)-(H4).** It then follows from **Theorem 4.1** in [1] that, for all $n \geq 1$, equation (3.1) admits a unique L^p-solution $(y_t^n, z_t^n)_{t\in[0,T]}$.

We next present Lemmas 3.1 and 3.2 regarding the $(y_t^n, z_t^n)_{t\in[0,T]}$

Lemma 3.1. Let the hypotheses of Theorem 3.1 hold. Then, there exists a constant $c_1 > 0$ depending solely on C, L and p, and a constant $K > 0$, depending solely on C, L, α, T and p such that for every $t \in [0; T]$ and all $n; m \geq 1$,

$$E\left[\sup_{s\in[t,T]} |y_s^{n+m} - y_s^n|^p\right] \leq \frac{e^{c_1(T-t)}}{2} \int_t^T \rho(E[|y_s^{n+m-1} - y_s^{n-1}|^p])ds. \quad (3.2)$$

and

$$E\left[\left(\int_t^T |z_s^{n+m} - z_s^n|^2 ds\right)^{p/2}\right] \leq K\left\{E\left[\sup_{s\in[t,T]} |y_s^{n+m} - y_s^n|^p\right] + \int_t^T \rho(E[|y_s^{n+m-1} - y_s^{n-1}|^p])ds\right\}. \quad (3.3)$$

Proof. It follows from (3.1) that $(y_t^{n+m} - y_t^n, z_t^{n+m} - z_t^n)_{t\in[0,T]}$is a L^p-solution of the following BDSDE

$$y_t = \int_t^T f_{n,m}(s, z_s)ds + \int_t^T g_{n,m}(s, z_s)dB_s - \int_t^T z_s dW_s, \ t \in [0, T] \quad (3.4)$$

where

$$f_{n,m}(s, z_s) := f(s, y_s^{n+m-1}, z + z_s^n) - f(s, y_s^{n-1}, z_s^n). f_{n,m}(s, z_s):$$
$$= f(s, y_s^{n+m-1}, z + z_s^n) - f(s, y_s^{n-1}, z_s^n).$$

and

$$g_{n,m}(s, z_s) := g(s, y_s^{n+m-1}, z + z_s^n) - g(s, y_s^{n-1}, z_s^n).$$

By **(H2)** and **(H3)** we have

$$|f_{n,m}(s, z)| \leq \rho^{\frac{1}{p}}(|y_s^{n+m-1} - y_s^{n-1}|^p) + C|z_s|,$$

and **(H5)**

$$|g_{n,m}(s, z)| \leq L^{\frac{1}{p}}(y_s^{n+m-1} - y_s^{n-1}|^p) + \alpha|z_s|,$$

This shows that assumption **(H1)** is satisfied for the generator $f_{n;m}(t;z)$ of BDSDE (3.4) with $\psi(\cdot)\equiv 0, \lambda = C, \theta_t \equiv 0$ *and* $\varphi_t = \rho^{\frac{1}{p}}(|y_t^{n+m-1} - y_t^{n-1}|^p)$ Consequently, conclusions (3.2) and (3.3) follow, since

$\rho(\cdot)$ is concave, by Propositions 2.1 and 2.2, and then by Fubini's theorem and Jensen's inequality.

Lemma 3.2. Let the hypotheses of Theorem 3.1 hold. Then, there exists $T_1 \in [0; T]$, independent of the terminal condition ξ, and a constant $M \geq 0$ such that, for every $n \geq 1$ and all $t \in [T_1; T]$,

$$E\left[\sup_{r\in[t,T]} |y_r^n|^p\right] \leq M.$$

Proof. Exploiting the hypotheses of Theorem 3.1, we deduce that

$$|f(s, y_s^{n-1}, z)| \leq \rho^{\frac{1}{p}}(|y_s^{n-1}|^p) + C|z| + |f(s,0,0)|$$

and

$$|g(s, y_s^{n-1}, z)| \leq L^{\frac{1}{p}}(|y_s^{n-1}|^p) + \alpha|z| + |g(s,0,0)|$$

Thus, assumption **(H1)** is satisfied for the generator $f(t, y_t^{n-1}, z)$ of BDSDE (3.1), with $\psi(\cdot)\equiv 0, \lambda = C, \theta_t = |f(t,0,0)|$ by (H4), and $\varphi_t = \rho^{\frac{1}{p}}(|y_t^{n-1}|^p)$. Consequently, since $\rho(\cdot)$ is concave, applying Proposition 2.2 together with Fubini's theorem and Jensen's inequality shows that there exist two positive constants c_2 and c_3 depending only on C, L, α and p, such that for every $n \geq 1$ and t *in* $[0; T]$, we have:

$$E\left[\sup_{r\in[t,T]} |y_r^n|^p\right] \leq \Lambda_t + \frac{e^{c_3(T-t)}}{2}\int_t^T \rho(E[|y_s^{n-1}|^p])ds \qquad (3.5)$$

where

$$\Lambda_t = c_2 e^{c_3(T-t)}\left\{E|\xi|^p + \backslash E\left[\left(\int_t^T |f(s,0,0)|ds + \int_t^T |g(s,0,0)|ds\right)^p\right]\right\} \geq 0.$$

Now, let $M = 2\Lambda_0 + 2AT$ and $T_1 = \max\left\{T - \frac{\ln 2}{c_1}, T - \frac{\ln 2}{c_3}, T - \frac{1}{2A}, 0\right\}, c_1$ where c_1 is defined in Lemma 3.1. Then for each t *in* $[T_1; T]$, we have

$\frac{1}{2}e^{c_1(T-t)} \leq 1,\ \frac{1}{2}e^{c_3(T-t)} \leq 1$ and $A(T-t) \leq \frac{1}{2}$. (3.6)

Hence, it follows from (3.5) and (3.6) that

$E\left[\sup_{r\in[t,T]} |y_r^n|^p\right] \leq \Lambda_0 + \int_t^T (E[|y_s^{n-1}|^p])ds\,\rho,\ \ t \in [T_1, T]$ (3.7)

Noting that $\rho(\cdot)$ is nondecreasing, we deduce from (3.7), and (3.6) that for every t *in* $[T_1; T]$,

$$E\left[\sup_{r\in[t,T]}|y_r^1|^p\right]\le \Lambda_0\le M,$$

$$E\left[\sup_{r\in[t,T]}|y_r^2|^p\right]\le \Lambda_0+\int_t^T\rho(M)ds\le \Lambda_0+A(M+1)(T-t)\le \Lambda_0+\frac{M}{2}+AT=M,$$

$$E\left[\sup_{r\in[t,T]}|y_r^3|^p\right]\le \Lambda_0+\int_t^T\rho(M)ds\le \Lambda_0+A(M+1)(T-t)\le \Lambda_0+\frac{M}{2}+AT=M.$$

An induction argument yields that, for every $n\ge 1$ and every t *in* $[T_1; T]$,

$$E\left[\sup_{r\in[t,T]}|y_r^n|^p\right]\le M$$

The proof of Lemma 3.2 is therefore complete.

Lemmas 3.1 and 3.2 yield the proof of Theorem 3.1.

Proof. (Proof of Theorem 3.1.)

Existence: We define $\{\varphi_n(t)\}_{n\ge1}$as follows:

$\varphi_0(t)=\int_t^T\rho(M)ds\,;\quad \varphi_{n+1}(t)=\int_t^T\rho(\varphi_n(s))ds.$ (3.8)

Consequently, for every t *in* $[T_1; T]$, by the proof of Lemma 3.2 we have

$$\varphi_0(t)=\int_t^T\rho(M)ds\le M,$$

$$\varphi_1(t)=\int_t^T\rho\big(\varphi_0(s)\big)ds\le\int_t^T\rho(M)ds=\varphi_0(t)\le M,$$

$$\varphi_2(t)=\int_t^T\rho(\varphi_1(s))ds\le\int_t^T\rho(\varphi_0(s))ds=\varphi_1(t)\le M.$$

An induction argument shows that, for every $n\ge 1$, $\varphi_n(t)$satisfies

$$0\le\varphi_{n+1}(t)\le\varphi_n(t)\le\cdots\le\varphi_1(t)\le\varphi_0(t)\le M.$$

Then, for each t *in* $[T_1; T]$, the sequence $\{\varphi_n(t)\}_{n\ge1}$ converges; we denote its limit by $\varphi_n(t)$. Letting $n\to\infty$in (3.8), and since $\rho(\cdot)$ is continuous with $\rho(\varphi_n(s))\le\rho(M)$ for all $n\ge 1$, the Lebesgue dominated convergence theorem implies that, for every t *in* $[T_1; T]$,

$\varphi(t)=\int_t^T\rho(\varphi(s))ds:$

Then Bihari's inequality (see Lemma 3.1 in [8]) yields that for each t *in* $[T_1; T]$, $\varphi(t)=0$.

Now, for all t *in* $[T_1; T]$, $n; m\ge 1$, thanks to Lemma 3.2, (3.2) in Lemma 3.1 and inequality (3.6) we have,

$$E\left[\sup_{r\in[t,T]}|y_r^n|^p\right]\leq M,$$

$$E\left[\sup_{r\in[t,T]}|y_r^{1+m}-y_r^1|^p\right]\leq\int_t^T\rho(E[|y_s^m|^p])ds\leq\int_t^T\rho(M)ds=\varphi_0(t)\leq M,$$

$$E\left[\sup_{r\in[t,T]}|y_r^{2+m}-y_r^2|^p\right]\leq\int_t^T\rho(E[|y_s^{1+m}-y_s^1|^p])ds\leq\int_t^T\rho\big(\varphi_0(s)\big)ds=\varphi_1(t)\leq M,$$

$$E\left[\sup_{r\in[t,T]}|y_r^{3+m}-y_r^3|^p\right]\leq\int_t^T\rho(E[|y_s^{2+m}-y_s^2|^p])ds\leq\int_t^T\rho(\varphi_1(s))ds=\varphi_2(t)\leq M.$$

Using induction, we obtain that

$$E\left[\sup_{T_1\leq r\leq T}|y_r^{n+m}-y_r^n|^p\right]\leq\varphi_{n-1}(T_1)\to 0,\ \ n\to\infty.$$

This shows that $\{y_t^n\}_{n\geq1}$is a Cauchy sequence in $S^p([T_1,T];\mathbb{R}^d)$ Moreover, since $\rho(\cdot)$ is continuous, it follows from (3.3) in Lemma 3.1 that $\{z_t^n\}_{n\geq1}$ is a Cauchy sequence in $M^p([T_1,T];\mathbb{R}^{d\times d})$.

Let us denote their limits by $(y_t)_{t\in[T_1,T]}$, $(z_t)_{t\in[T_1,T]}$, respectively. By letting $n\to\infty$ in (3.1), we then obtain that $(y_t; z_t)$ is an L^p-solution to the BDSDE (1.1) on $[T_1; T]$.

Since, by Lemma 3.2, T_1 does not depend on the terminal condition ξ, we can iterate this argument to establish existence on $[T - l(T - T_1); T]$, for each l, and hence on the entire interval $[0; T]$. This completes the proof of existence.

Uniqueness: Let $\left(y_t^i,z_t^i\right)_{t\in[0,T]}$, $(i = 1; 2)$ be two solutions in L^p of the BDSDE (1.1). It follows that $(y_t^i,z_t^i)_{t\in[0,T]}$, $(i=1,2)$ is a solution in L^p to the following BDSDE:

$$y_t=\int_t^T\hat{\mathrm{f}}(s,y_s,z_s)ds+\int_t^T\hat{\mathrm{g}}(s,y_s,z_s)dB_s-\int_t^T z_s dW_s,\ \ t\in[0,T],\qquad(3.9)$$

where

$$\hat{\mathrm{f}}(s,y,z):=f(s,y+y_s^2,z+z_s^2)-f(s,y_s^2,z_s^2).$$

and

$$\hat{\mathrm{g}}(s,y,z):=g(s,y+y_s^2,z+z_s^2)-g(s,y_s^2,z_s^2).$$

By **(H5)**, we get

$$|\hat{\mathrm{g}}(s,y,z)|\leq L^{\frac{1}{p}}(|y|^p)+\alpha|z|$$

and by **(H2)** and **(H3)** we have

$$|\hat{\mathrm{f}}(s,y,z)|\leq\rho^{\frac{1}{p}}(|y|^p)+C|z|,$$

This shows that assumption **(H1)** is satisfied for the generator $\hat{f}(t, y, z)$ of BDSDE (3.9), with $\psi(\cdot) = \rho(\cdot), \lambda = C, \varphi_t \equiv 0, \theta_t \equiv 0$. Then, by Proposition 2.1 and Proposition 2.2, there exist a constant $c_4 > 0$, depending only on C, L and p, and a constant $c_5 > 0$, depending only on C, L p and T, such that for all t *in* $[0; T]$,

$$E[|y_t^1 - y_t^2|^p] \le \frac{e^{c_4(T-t)}}{2} \int_t^T \rho(E[|y_s^1 - y_s^2|^p])ds \qquad (3.10)$$

and

$$E\left[\left(\int_t^T |z_s^1 - z_s^2|^2 ds\right)^{p/2}\right] \le c_5 \left\{E\left[\sup_{s\in[t,T]} |y_s^1 - y_s^2|^p\right] + \rho\left(E\left[\sup_{s\in[t,T]} |y_s^1 - y_s^2|^p\right]\right)\right\}. \qquad (3.11)$$

Then, from (3.10), Bihari's inequality (see Lemma 3.1 in [8]) implies that, for each t *in* $[0; T]$, $E[|y_t^1 - y_t^2|^p] = 0$. which shows that $y_t^1 = y_t^2$ for all t *in* $[0; T]$ almost surely. It then immediately follows from (3.11) that

$z_t^1 = z_t^2$ for all t *in* $[0; T]$ almost surely. This completes the proof of Theorem 1.

4 Conclusion

In this paper, we have studied multidimensional backward doubly stochastic differential equations (BDSDEs) with weakly regular, non-Lipschitz coefficients. By introducing a new class of generator conditions that generalize the classical Lipschitz framework, we established the existence and uniqueness of solutions in L^p for any $p > 1$. Our approach relied on careful estimates, the construction of appropriate sequences of approximating solutions, and the application of tools such as Bihari's inequality and the Lebesgue dominated convergence theorem.

These results extend and unify several previous works in the literature, which either considered one-dimensional cases, restricted $p < 2$, or required monotonicity assumptions. Consequently, our findings provide a broader and more flexible framework for analyzing BDSDEs, with potential applications to semilinear and quasilinear stochastic partial differential equations, stochastic control, and related stochastic systems.

Future work may explore further relaxations of the coefficient regularity, the treatment of more general noise structures, or the development of numerical schemes for such BDSDEs.

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