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Fixed Point Theorems for Integral type Contraction in Metric Space

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Abstract:

In the present manuscript, a common fixed point theorem is proved for a pair of weakly compatible "self-maps $\wp$ and $\mathbb{Q}$ on a metric space (H, d^*) satisfying the following contractive inequality of integral type:

$$\int_0^{d^*(\mathbb{Q}\mu,\mathbb{Q}v)} \varrho(t)dt \le \beta\big(d^*(\wp\mu,\wp v)\big) \int_0^{J(\wp\mu,\wp v)} \varrho(t)dt,$$

where $(\varrho, \beta) \in \ \varrho_1 \times \varrho_3$ and for all μ, v in H,

where

$$J(\wp\mu,\wp v) = max\left\{d^*(\wp\mu,\wp v), d^*(\mathbb{Q}v,\wp v), \frac{d^*(\mathbb{Q}\mu,\wp\mu).d^*(\mathbb{Q}v,\wp v)}{1+d^*(\mathbb{Q}\mu,\mathbb{Q}v)}, \frac{d^*(\mathbb{Q}\mu,\wp\mu).d^*(\mathbb{Q}v,\ \wp v)}{1+d^*(\wp\mu,\wp v)}\right\}.$$

In addition to this, common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. and (CLR) properties are also proved.

Keywords: fixed point, weakly compatible maps, E.A. property, (CLR) property.

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1. Introduction

All around this paper we presume that $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ stands for the set of positive integers and

- $\varrho_1 = \{\ \varrho \mid \varrho :\ \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that ϱ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\delta \varrho(t)dt > 0$ for each $\delta > 0$ $\}$,

- $\varrho_2 = \{\, \varrho \mid \varrho : \mathbb{R}^+ \rightarrow [0,1)$ satisfies that $\lim_{s\rightarrow t} \sup \varrho(s) < 1$ for each $t \in \mathbb{R}^+\}$,
- $\varrho_3 = \{\, \varrho \mid \varrho \in \varrho_2$ and $\lim_{s\rightarrow +\infty} \sup \varrho(s) < 1\}$.

In 2002, Branciari [2] introduced the new concept of integral type contraction and proved the following fixed point result:

"Let (H, d^*) be a complete metric space and $\hat{T}$ be a self map on H satisfying

$$\int_0^{d^*(\hat{T}\mu,\hat{T}v)} \varrho(t)dt \le \beta \int_0^{d^*(\mu,v)} \varrho(t)dt \qquad \text{for all } \mu, v \text{ in } H,$$

where $\beta \in (0,1)$ is a constant and $\varrho \in \varrho_1$. Then $\hat{T}$ has a unique fixed point $b \in H$ such that $\lim_{n\rightarrow\infty} \hat{T}^n\mu = b$ for each $\mu \in H$."

Definition 1.1. A coincidence point of a pair of self – maps $\hat{P}, \hat{Q} : H \rightarrow H$ is a point $\mu \in H$ for which $\hat{P}\mu = \hat{Q}\mu$.

Definition 1.2. A common fixed point of a pair of self - mappings $\hat{P}, \hat{Q} : H \rightarrow H$ is a point $\mu \in H$ for which $\hat{P}\mu = \hat{Q}\mu = \mu$.

The concept of weakly compatible mappings was introduced by Jungck [5] to study common fixed point theorems:

Definition 1.3. Let (H, d^*) be a metric space. A pair of self – maps $\hat{P}, \hat{Q} : H \rightarrow H$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in H$ such that $\hat{P}\hat{Q}\mu = \hat{Q}\hat{P}\mu$, where μ is coincidence point of $\hat{P}$ and $\hat{Q}$.

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.4. Let (H, d^*) be a metric space. Two self-maps $\hat{P}$ and $\hat{Q}$ on H are said to satisfy the E.A. property, if there exists a sequence $\{\mu_n\}$ in H such that,

$$\lim_{n\rightarrow\infty} \hat{P}\mu_n = \lim_{n\rightarrow\infty} \hat{Q}\mu_n = t, \text{ for some } t \in H.$$

In 2011, Sintunavarat *et al.* [9] introduced the notion of (CLR) property as follows:

Definition 1.5. Let (H, d^*) be a metric space. Two self-maps $\hat{P}$ and $\hat{Q}$ on H are said to satisfy the $(\mathrm{CLR}_{\hat{P}})$ property, if there exists a sequence $\{\mu_\mathrm{n}\}$ in H such that,

$$\lim_{n\to\infty} \hat{P}\mu_n = \lim_{n\to\infty} \hat{Q}\mu_n = \hat{P}\,\mathrm{t} \text{ for some } t \in H.$$

Lemma 1.6.[9] let $\varrho \in \varrho_1$ and $\{\mu_n\}_{n\in\mathbb{N}}$ be a non negative sequence with $\lim_{n\to\infty} \mu_n = b$. Then,

$$\lim_{n\to\infty} \int_0^{\mu_n} \varrho(t)dt = \int_0^b \varrho(t)dt.$$

2. Common fixed point theorems for integral type contraction

In this section, we shall prove some common fixed point theorems for integral type contractions for a pair of weakly compatible maps along with E.A. property and (CLR) property.

Theorem 2.1. Let (H, d^*) be a metric space and let $\wp$ and $\mathbb{Q}$ be two self– maps on H satisfying the followings:

(2.1) $\mathbb{Q}H \subseteq \wp H$.

(2.2) There exists a continuous mapping $\varrho: [0, \infty) \to [0, \infty)$ with $\varrho(0) = 0$ and $\varrho(\alpha) > \alpha$ for all $\alpha > 0$ such that:

$$\int_0^{d^*(\mathbb{Q}\mu, \mathbb{Q}v)} \varrho(t)dt \leq \beta\big(d^*(\wp\mu, \wp v)\big) \int_0^{J(\wp\mu, \wp v)} \varrho(t)dt,$$

where $(\varrho, \beta) \in \varrho_1 \times \varrho_3$ and for all μ, v in H,

where

$$J(\wp\mu, \wp v) = max\{d^*(\wp\mu, \wp v), \\ d^*(\mathbb{Q}v, \wp v), \frac{d^*(\mathbb{Q}\mu, \wp\mu).d^*(\mathbb{Q}v, \wp v)}{1 + d^*(\mathbb{Q}\mu, \mathbb{Q}v)}, \frac{d^*(\mathbb{Q}\mu, \wp\mu).d^*(\mathbb{Q}v, \wp v)}{1 + d^*(\wp\mu, \wp v)}\}.$$

If $\wp$ and $\mathbb{Q}$ are weakly compatible and $\wp H$ or $\mathbb{Q}H$ is complete, then $\wp$ and $\mathbb{Q}$ have a unique common fixed point.

Proof: Let $\mu_0$be arbitrary point in H. From (2.1), Since $\mathbb{Q}H \subseteq \wp H$, we can define a sequence $\{\mu_n\}$such that:

$$\mathbb{Q}\mu_\mathrm{n} = \wp\mu_{\mathrm{n}+1}.$$

Define a sequence $\{v_n\}$ in H by,

$$v_{\mathrm{n}} = \mathbb{Q}\mu_{\mathrm{n}} = \wp\mu_{\mathrm{n+1.}} \tag{2.3}$$

If $v_n = v_{n+1}$ for some n in ℕ, then there is nothing to prove.

Now, we assume that $v_n \neq v_{n+1}$ for all n in ℕ.

We prove that

$$\lim_{n\to\infty} d^*(v_n, v_{n+1}) = 0. \tag{2.4}$$

On substituting $\mu = \mu_n,\ v = \mu_{n+1}$ in (2.2) and using (2.3), we get

$$\begin{aligned} J(\wp\mu_n, \wp\mu_{n+1}) &= \max\left\{ d^*(\wp\mu_n, \wp\mu_{n+1}),\ d^*(\mathbb{Q}\mu_{n+1}, \wp\mu_{n+1}), \frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n).\, d^*(\mathbb{Q}\mu_{n+1}, \wp\mu_{n+1})}{1 + d^*(\mathbb{Q}\mu_n, \mathbb{Q}\mu_{n+1})}, \frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n).\, d^*(\mathbb{Q}\mu_{n+1},\ \wp\mu_{n+1})}{1 + d^*(\wp\mu_n, \wp\mu_{n+1})} \right\} \\ &= max\left\{ d^*(v_{n-1}, v_n), d^*(v_{n+1}, v_n), \frac{d^*(v_n, v_{n-1}).\, d^*(v_{n+1}, v_n)}{1 + d^*(v_n, v_{n+1})}, \frac{d^*(v_n, v_{n-1}).\, d^*(v_{n+1},\ v_n)}{1 + d^*(v_{n-1}, v_n)} \right\} \\ &= \max\{d^*(v_n, v_{n+1}),\ d^*(v_{n-1}, v_n)\}, \end{aligned}$$

since

$$\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},v_n)}{1+d^*(v_n,v_{n+1})} \leq d^*(v_n, v_{n-1})$$

and

$$\frac{d^*(v_n,v_{n-1}).d^*(v_{n+1},v_n)}{1+d^*(v_{n-1},v_n)} \leq d^*(v_{n+1}, v_n).$$

If $d^*(v_n, v_{n-1}) < d^*(v_{n+1}, v_n)$, we have

$$J(\wp\mu_n, \wp\mu_{n+1}) = d^*(v_{n+1}, v_n),$$

and

$$\begin{aligned} 0 &< \int_0^{d^*(v_n,v_{n+1,})} \varrho(t)dt \\ &= \int_0^{d^*(\mathbb{Q}\mu_n,\mathbb{Q}\mu_{n+1})} \varrho(t)dt \\ &\leq \beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big) \int_0^{J(\wp\mu_n,\wp\mu_{n+1})} \varrho(t)dt, \end{aligned}$$

$$= \beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big) \int_0^{d^*(v_{n+1}, v_n)} \varrho(t)dt$$

$$< \int_0^{d^*(v_{n+1}, v_n)} \varrho(t)dt,$$

which is a contradiction.

Hence

$$d^*(v_{n+1}, v_n) < d^*(v_n, v_{n-1}) \tag{2.5}$$

Hence the sequence $\{d^*(v_n, v_{n+1})\}$ is strictly decreasing and bounded below.

Thus, there exists $r \geq 0$, such that

$$\lim_{n\to\infty} d^*(v_n, v_{n+1}) = r, \tag{2.6}$$

Suppose that $r > 0$.

Then from (2.6) and Lemma 1.6, we get

$$0 < \int_0^r \varrho(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(v_n, v_{n+1})} \varrho(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}\mu_n, \mathbb{Q}\mu_{n+1})} \varrho(t)dt$$

$$\leq \limsup_{n\to\infty} \beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big) \int_0^{J(\wp\mu_n, \wp\mu_{n+1})} \varrho(t)dt$$

$$= \limsup_{n\to\infty}[\beta\big(d^*(\wp\mu_n, \wp\mu_{n+1})\big)\limsup_{n\to\infty} \int_0^{d^*(v_n, v_{n+1})} \varrho(t)dt]$$

$$< \limsup_{n\to\infty} \int_0^{d^*(v_n, v_{n+1})} \varrho(t)dt$$

$$< \int_0^r \varrho(t)dt,$$

which is a contradiction.

Thus, $r = 0$, which implies that

$$\lim_{n\to\infty} d^*\big(v_n, v_{n+1}\big) = 0. \tag{2.7}$$

Next, we prove that $\{v_n\}$ is a Cauchy sequence. Suppose that $\{v_n\}$ is not a cauchy sequence. Then there exists $\epsilon > 0$, such that for $k \in \mathbb{N}$, there are $m(k)$, $n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying:

(i) $m(k)$ and $n(k)$ are positive integers.

(ii) $d^*\big(v_{n(k)}, v_{m(k)}\big) > \epsilon$.

(iii) $m(k)$ is the smallest even number such that the condition (ii) holds, that is $d^*(v_{n(k)}, v_{m(k)-1}) \leq \epsilon$.

Therefore,

$$\begin{aligned}\epsilon &< d^*(v_{n(k)}, v_{m(k)}) \\ &\leq d^*(v_{n(k)}, v_{m(k)-1}) + d^*(v_{m(k)-1}, v_{m(k)}) \\ &\leq \epsilon + d^*(v_{m(k)-1}, v_{m(k)}). \qquad (2.8)\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^*(v_{n(k)}, v_{m(k)}) = \epsilon. \qquad (2.9)$$

$$\begin{aligned}\epsilon &\leq d^*(v_{n(k)-1}, v_{m(k)-1}), \\ &\leq d^*(v_{n(k)-1}, v_{m(k)-2}) + d^*(v_{m(k)-2}, v_{m(k)-1}), \\ &\leq \epsilon + d^*(v_{m(k)-2}, v_{m(k)-1}).\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^*(v_{n(k)-1},\ v_{m(k)-1}) = \epsilon. \qquad (2.10)$$

Substituting $\mu = \mu_{n(k)},\ v = \mu_{m(k)}$ in (2.2), we get

$$\begin{aligned}J(\wp\mu_{n(k)}, \wp\mu_{m(k)}) = \max\Big\{&d^*(\wp\mu_{n(k)}, \wp\mu_{m(k)}), d^*(\mathbb{Q}\mu_{m(k)}, \wp\mu_{m(k)}), \\ &\frac{d^*(\mathbb{Q}\mu_{n(k)}, \wp\mu_{n(k)}).\, d^*(\mathbb{Q}\mu_{m(k)}, \wp\mu_{m(k)})}{1 + d^*(\mathbb{Q}\mu_{n(k)}, \mathbb{Q}\mu_{m(k)})}, \\ &\frac{d^*(\mathbb{Q}\mu_{n(k)}, \wp\mu_{n(k)}).d^*(\mu c_{m(k)}, \wp\mu_{m(k)})}{1+d^*\ (\wp\mu_{n(k)}, \wp\mu_{m(k)})}\Big\}. \\ = \max\Big\{&d^*(v_{n(k)-1}, v_{m(k)-1}), \\ &d^*(v_{m(k)}, v_{m(k)-1}), \\ &\frac{d^*(v_{n(k)}, v_{n(k)-1}).\, d^*(v_{m(k)}, v_{m(k)-1})}{1 + d^*(v_{n(k)}, v_{m(k)})}, \\ &\frac{d^*(v_{n(k)}, v_{n(k)-1}).\, d^*(v_{m(k)},\ v_{m(k)-1})}{1 + d^*\ (v_{n(k)-1}, v_{m(k)-1})}\Big\}.\end{aligned}$$

Taking limit as $k \to \infty$ and using (2.7), (2.8), (2.9) and (2.10), we have

$$\lim_{k\to\infty} J(\wp\mu_{n(k)}, \wp\mu_{m(k)}) = \max\{\epsilon, \epsilon, \frac{\epsilon.\epsilon}{1+\epsilon}, \frac{\epsilon.\epsilon}{1+\epsilon}\}. = \epsilon.$$

And

$$0 < \int_0^\epsilon \varrho(t)dt$$

$$= \limsup_{\alpha\to\infty} \int_0^{d^*(v_{n(k)}, v_{m(k)})} \varrho(t)dt$$

$$= \limsup_{\alpha\to\infty} \int_0^{d^*(\mathbb{Q}c_{n(k)}, \mathbb{Q}c_{m(k)})} \varrho(t)dt$$

$$\leq \limsup_{\alpha\to\infty}[\beta d^*\left(\wp\mu_{n(k)}, \wp\mu_{m(k)}\right) \int_0^{J(\wp\mu_{n(k)}, \wp\mu_{m(k)})} \varrho(t)dt]$$

$$= \limsup_{\alpha\to\infty}[\beta d^*\left(\mu_{2n(\alpha)}, \mu_{2m(\alpha)-1}\right)]\limsup_{\alpha\to\infty} \int_0^{J(\wp\mu_{n(k)}, \wp\mu_{m(k)})} \varrho(t)dt]$$

$$< \int_0^{\epsilon} \varrho(t)dt,$$

a contradiction.

Hence $\{v_n\}$ is a Cauchy sequence.

Since $\wp H$ is complete, so there exists a point p in $\wp H$ such that

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \wp\mu_{n+1} = p = \lim_{n\to\infty} \mathbb{Q}\mu_n \tag{2.11}$$

Since $p \in \wp H$, so we can find q in H such that $\wp q = p$.

Now, we claim that $\wp q = \mathbb{Q}q$.

Let, if possible, $\wp q \neq \mathbb{Q}q$.

On putting, $\mu = \mu_{n+1}$, $v = q$ in (2.2), we have

$$J(\wp\mu_{n+1}, \wp q) = \max\left\{d^*(\wp\mu_{n+1}, \wp q), \quad d^*(\mathbb{Q}q, \wp q), \frac{d^*(\mathbb{Q}\mu_{n+1}, \wp\mu_{n+1}).\, d^*(\mathbb{Q}q, \wp q)}{1 + d^*(\mathbb{Q}\mu_{n+1}, \mathbb{Q}q)}, \frac{d^*(\mathbb{Q}\mu_{n+1}, \wp\mu_{n+1}).\, d^*(\mathbb{Q}q, \wp q)}{1 + d^*(\wp\mu_{n+1}, \wp q)}\right\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} J(\wp\mu_{n+1}, \wp q) = max\left\{d^*(\wp q, \wp q), d^*(\mathbb{Q}q, \wp q), \frac{d^*(\wp q, \wp q).d^*(\mathbb{Q}q, \wp q)}{1+d^*(\wp q, \mathbb{Q}q)}, \frac{d^*(\wp q, \wp q).d^*(\mathbb{Q}q, \wp q)}{1+d^*(\wp q, \wp q)}\right\}$$

$$= d^*(\mathbb{Q}q, \wp q).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}q,\wp q)} \varrho(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}q,\mathbb{Q}\mu_{n+1})} \varrho(t)dt$$

$$\leq \limsup_{n\to\infty}[\beta d^*(\wp q,\wp\mu_{n+1}) \int_0^{J(\wp q,\wp\mu_{n+1})} \varrho(t)dt]$$

$$= \limsup_{n\to\infty}[\beta d^*(\wp q,\wp\mu_{n+1})]\limsup_{\alpha\to\infty} \int_0^{d^*(\wp q,\wp\mu_{n+1})} \varrho(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}q,\wp q)} \varrho(t)dt,$$

which is not possible.

Hence, $d^*(\mathbb{Q}q,\wp q) = 0$,which implies that

$$\wp q = \mathbb{Q}q. \tag{2.12}$$

Therefore, q is a coincidence point of $\wp$ and $\mathbb{Q}$.

Now, we show that there exists a common fixed point of $\wp$ and $\mathbb{Q}$.

Since $\wp$ and $\mathbb{Q}$ are weakly compatible, by (2.12), we have

$$\mathbb{Q}\wp q = \wp\mathbb{Q}q \text{ and } \mathbb{Q}p = \mathbb{Q}\wp q = \wp\mathbb{Q}q = \wp p.$$

Now, consider

$$J(\wp q,\wp p) = max\{d^*(\wp q,\wp p), d^*(\mathbb{Q}p,\wp p), \frac{d^*(\mathbb{Q}q,\wp q).d^*(\mathbb{Q}p,\wp p)}{1+d^*(\mathbb{Q}q,\mathbb{Q}p)}, \frac{d^*(\mathbb{Q}q,\wp q).d^*(\mathbb{Q}p,\wp p)}{1+d^*(\wp q,\wp p)}\}$$

$$= max\{d^*(p,\mathbb{Q}p),\ 0\ ,\ 0\ ,0\ \}$$

$$= d^*(p,\mathbb{Q}p).$$

Now,

$$0 < \int_0^{d^*(p,\mathbb{Q}p)} \varrho(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}q,\mathbb{Q}p)} \varrho(t)dt$$

$$\leq [\beta d^*(\wp q,\wp p) \int_0^{J(\wp q,\wp p)} \varrho(t)dt]$$

$$< \int_0^{d^*(p,\mathbb{Q}q)} \varrho(t)dt,$$

which is again a contradiction.

Hence $\wp p = \mathbb{Q}p = p$.

This implies p is common fixed point of $\wp$ and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then from (2.2), we have

$$J(\wp r, \wp s) = max\{d^*(\wp r, \wp s), d^*(\mathbb{Q}r, \wp r),$$

$$\frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\mathbb{Q}p,\mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\wp s,\wp r)}\}$$

$$= d^*(r,s).$$

And

$$0 < \int_0^{d^*(r,s)} \varrho(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \varrho(t)dt$$

$$\leq \beta d^*(\wp r, \wp s) \int_0^{J(\wp r,\wp s)} \varrho(t)dt$$

$$= \beta d^*(r,s) \int_0^{d^*(r,s)} \varrho(t)dt$$

$$< \int_0^{d^*(r,s)} \varrho(t)dt,$$

which is a contradiction, hence $r = s$.

This proves the uniqueness of the common fixed point.

Hence completes the proof of the theorem.

Corollary 2.2. Let Tbe self - map on a metric space (H, d^*) satisfying the followings: There exists a continuous mapping $\varrho: [0,\infty) \to [0,\infty)$ with $\varrho(0) = 0$ and $\varrho(\alpha) > \alpha$ for all $\alpha > 0$ such that:

$$\int_0^{d^*(T\mu,T\upsilon)} \varrho(t)dt \leq \beta\big(d^*(\mu,\upsilon)\big) \int_0^{J(\mu,\upsilon)} \varrho(t)dt,$$

where $(\varrho, \beta) \in \varrho_1 \times \varrho_3$ and for all μ, υ in H.

where

$$J(\mu,v) = max\left\{d^*(\mu,v),\ d^*(Tv,v), \frac{d^*(T\mu,\mu).d^*(Tv,v)}{1+d^*(T\mu,Tv)}, \frac{d^*(T\mu,\mu).d^*(Tv,v)}{1+d^*(\mu,v)}\right\}.$$

If TH is complete, then T has a unique fixed point.

Theorem 2.3. Let $\wp$ *and* $\mathbb{Q}$ be self-maps of a metric space (H, d^*)satisfying (2.2) and the followings:

(2.13) $\wp$ *and* $\mathbb{Q}$ are weakly compatible,

(2.14) $\wp$ and $\mathbb{Q}$ satisfy the E.A. property.

If either $\wp H$ or $\mathbb{Q}H$ is a complete subspace of H, then $\wp$ and $\mathbb{Q}$ have a unique common fixed point in H.

Proof. Since $\wp$ and $\mathbb{Q}$ satisfy the E.A. property, there exists a sequence $\{\mu_n\}$ in H such that

$$\lim_{n\to\infty} \wp\, \mu_n = \lim_{n\to\infty} \mathbb{Q}\, \mu_n = \mu, \tag{2.15}$$

for some μ in H.

Now, suppose that $\wp H$ is complete subspace of H. Then, there exists z in H such that $\mu = \wp z$.

Subsequently, we have

$$\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \mathbb{Q}\mu_n = \mu = \wp z. \tag{2.16}$$

Now, we show that $\wp z = \mathbb{Q}z$.

$$\begin{aligned}\lim_{n\to\infty} J(\wp\mu_n, \wp z) &= \lim_{n\to\infty} max\{d^*(\wp\mu_n, \wp z), d^*(\mathbb{Q}z, \wp z), \\ &\qquad \frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n).\, d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\mathbb{Q}\mu_n, \mathbb{Q}z)}, \frac{d^*(\mathbb{Q}\mu_n, \wp\mu_n).\, d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp\mu_n, \wp z)}\Bigg\} \\ &= max\{d^*(\wp z, \wp z), d^*(\mathbb{Q}z, \wp z), \\ &\qquad \frac{d^*(\wp z, \wp z).\, d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp z, \mathbb{Q}z)}, \frac{d^*(\wp z, \wp z).\, d^*(\mathbb{Q}z, \wp z)}{1 + d^*(\wp z, \wp z)}\Bigg\} \\ &= \max\{0, d^*(\wp z, \mathbb{Q}z), 0, 0\}. \\ &= d^*(\mathbb{Q}z, \wp z).\end{aligned}$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}z,\wp z)} \varrho(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}z,\mathbb{Q}\mu_n)} \varrho(t)dt$$

$$\leq \limsup_{n\to\infty}[\beta d^*(\wp z,\wp\mu_n) \int_0^{J(\wp z,\wp\mu_n)} \varrho(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}z,\wp z)} \varrho(t)dt,$$

which is impossible.

Hence, $d^*(\mathbb{Q}z,\wp z) = 0$.

Which implies that

$$\wp z = \mathbb{Q}z$$

Since $\wp$and $\mathbb{Q}$ are weakly compatible. Therefore, $\mathbb{Q}\wp z = \wp\mathbb{Q}z$, implies that,

$$\wp\wp z = \wp\mathbb{Q}z = \mathbb{Q}\wp Z = \mathbb{Q}\mathbb{Q}z.$$

Now, we claim that $\mathbb{Q}z$ is the common fixed point of $\wp$ and$\mathbb{Q}$.

$$\lim_{n\to\infty} J(\wp z,\wp\mathbb{Q}z) = max\{d^*(\wp z,\wp\mathbb{Q}z), d^*(\mathbb{Q}\mathbb{Q}z,\wp\mathbb{Q}z),$$

$$\frac{d^*(\mathbb{Q}z,\wp z).\, d^*(\mathbb{Q}\mathbb{Q}z,\wp\mathbb{Q}z)}{1+d^*(\mathbb{Q}z,\mathbb{Q}\mathbb{Q}z)}, \frac{d^*(\mathbb{Q}z,\wp z).\, d^*(\mathbb{Q}\mathbb{Q}z,\wp\mathbb{Q}z)}{1+d^*(\wp z,\wp\mathbb{Q}z)}\Bigg\}.$$

$$= max\{d^*(\mathbb{Q}z,\mathbb{Q}\mathbb{Q}z), 0, 0\,, 0\}.$$

$$= d^*(\mathbb{Q}z,\mathbb{Q}\mathbb{Q}z).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}z,\mathbb{Q}\mathbb{Q}z)} \varrho(t)dt$$

$$\leq [\beta d^*(\wp z,\wp\mathbb{Q}z) \int_0^{J(\wp z,\wp\mathbb{Q}z)} \varrho(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}z,\mathbb{Q}\mathbb{Q}z)} \varrho(t)dt,$$

which is not possible.

Which implies that

$$\mathbb{Q}z = \mathbb{Q}\mathbb{Q}z = \wp\mathbb{Q}\mathbf{z}.$$

Hence $\mathbb{Q}z$ is common fixed point of $\wp$ and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then from (2.2), we have

$$J(\wp r, \wp s) = max\{d^*(\wp r, \wp s), d^*(\mathbb{Q}r, \wp r), \frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\mathbb{Q}p,\mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\wp s,\wp r)}\}$$

$$= d^*(r, s).$$

And

$$0 < \int_0^{d^*(r,s)} \varrho(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \varrho(t)dt$$

$$\leq \beta d^*(\wp r, \wp s) \int_0^{J(\wp r,\wp s)} \varrho(t)dt$$

$$= \beta d^*(r, s) \int_0^{d^*(r,s)} \varrho(t)dt$$

$$< \int_0^{d^*(r,s)} \varrho(t)dt,$$

which is a contradiction, hence $r = s$.

This proves the uniqueness of the common fixed point.

Theorem 2.4. Let (H, d^*)be metric space, let $\wp$ and $\mathbb{Q}$ be self maps on H satisfying (2.2), (2.13) and if $\wp$ and $\mathbb{Q}$ satisfy $(CLR_{\wp})$ property.

Then $\wp$ and $\mathbb{Q}$ have a unique common fixed point in H.

Proof: Since $\wp$ and $\mathbb{Q}$ satisfy the $(CLR_{\wp})$ property, there exits a sequence $\{\mu_n\}$ in H such that

$$\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \mathbb{Q}\mu_n = \wp\mu, \qquad (2.17)$$

for some μ in H.

First we prove that $\wp\mu = \mathbb{Q}\mu$.

Let, if possible, $\wp\mu \neq \mathbb{Q}\mu$.

On putting $\mu = \mu_n$ and $\nu = \mu$ in (2.2), we have

$$J(\wp\mu_n, \wp\mu) = max\{d^*(\wp\mu_n, \wp\mu), d^*(\mathbb{Q}c, \wp c), \frac{d^*(\mathbb{Q}c_n, \wp c_n). d^*(\mathbb{Q}c, \wp c)}{1 + d^*(\mathbb{Q}c_n, \mathbb{Q}c)}, \frac{d^*(\mathbb{Q}c_n, \wp c_n). d^*(\mathbb{Q}c, \wp c)}{1 + d^*(\wp\mu_n, \wp\mu)}\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} J(\wp\mu_n, \wp\mu) = max\{d^*(\wp\mu, \wp\mu), d^*(\mathbb{Q}\mu, \wp\mu), \frac{d^*(\wp\mu, \wp\mu). d^*(\mathbb{Q}\mu, \wp\mu)}{1 + d^*(\wp\mu, \mathbb{Q}\mu)}, \frac{d^*(\wp\mu, \wp\mu). d^*(\mathbb{Q}\mu, \wp\mu)}{1 + d^*(\wp\mu, \wp\mu)}\}$$

$$= \max\{0, \quad d^*(\wp\mu, \mathbb{Q}\mu), 0, 0\}$$

$$= d^*(\mathbb{Q}\mu, \wp\mu).$$

Now,

$$0 < \int_0^{d^*(\mathbb{Q}\mu, \wp\mu)} \varrho(t)dt$$

$$= \limsup_{n\to\infty} \int_0^{d^*(\mathbb{Q}\mu, \mathbb{Q}\mu_n)} \varrho(t)dt$$

$$\leq \limsup_{n\to\infty} [\beta d^*(\wp\mu, \wp\mu_n) \int_0^{J(\wp\mu, \wp\mu_n)} \varrho(t)dt]$$

$$< \int_0^{d^*(\mathbb{Q}\mu, \wp\mu)} \varrho(t)dt,$$

this is possible only when $d^*(\wp\mu, \mathbb{Q}\mu) = 0$.

Hence $\wp\mu = \mathbb{Q}\mu$.

Now, let $d = \wp\mu = \mathbb{Q}\mu$.

Since $\wp\mathbb{Q}\mu = \mathbb{Q}\wp\mu$,implies that,

$$\wp d = \wp\mathbb{Q}\mu = \mathbb{Q}\wp\mu = \mathbb{Q}d.$$

Now, we claim that $\mathbb{Q}d = d$.

$$J(\wp\mu, \wp d) = max\{d^*(\wp\mu, \wp d), d^*(\mathbb{Q}d, \wp d), \frac{d^*(\mathbb{Q}\mu, \wp\mu).d^*(\mathbb{Q}d, \wp d)}{1+d^*(\mathbb{Q}\mu, \mathbb{Q}d)}, \frac{d^*(\mathbb{Q}\mu, \wp\mu).d^*(\mathbb{Q}d, \wp d)}{1+d^*(\wp\mu, \wp d)}\}$$

$$= max\{d^*(d, \mathbb{Q}d), 0, 0, 0\}$$

$$= d^*(d, \mathbb{Q}d).$$

and

$$0 < \int_0^{d^*(d,\mathbb{Q}d)} \varrho(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}\mu,\mathbb{Q}d)} \varrho(t)dt$$

$$\leq [\beta d^*(\wp\mu, \wp d) \int_0^{J(\wp\mu,\wp d)} \varrho(t)dt]$$

$$< \int_0^{d^*(d,\mathbb{Q}d)} \varrho(t)dt.$$

This is possible only when $\mathbb{Q}d = d$.

Hence $\wp d = \mathbb{Q}d = d$.

So, d is the common fixed point of $\wp$ and $\mathbb{Q}$.

For the uniqueness, let r and s be two common fixed points of $\wp$ and $\mathbb{Q}$, such that $r \neq s$, then from (2.2), we have

$$J(\wp r, \wp s) = max\{d^*(\wp r, \wp s), d^*(\mathbb{Q}r, \wp r), \frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\mathbb{Q}p,\mathbb{Q}s)}, \frac{d^*(\mathbb{Q}s,\wp s).d^*(\mathbb{Q}r,\wp r)}{1+d^*(\wp s,\wp r)}\}$$

$$= d^*(r, s).$$

And

$$0 < \int_0^{d^*(r,s)} \varrho(t)dt$$

$$= \int_0^{d^*(\mathbb{Q}r,\mathbb{Q}s)} \varrho(t)dt$$

$$\leq \beta d^*(\wp r, \wp s) \int_0^{J(\wp r,\wp s)} \varrho(t)dt$$

$$= \beta d^*(r, s) \int_0^{d^*(r,s)} \varrho(t)dt$$

$< \int_0^{d^*(r,s)} \varrho(t)dt,$

which is a contradiction.

Hence $r = s$.

This proves the uniqueness of the common fixed point.

Example 2.5. Let $H = \mathbb{R}^+$ be equipped with the metric space and $d^*(u,v) = |u-v|$ for all $u, v \in H$.

Define $\wp, \mathbb{Q} : H \to H$ by

$$\wp u = u$$

$$\mathbb{Q}u = \frac{u}{3} + \frac{2}{3}.$$

Clearly, $\mathbb{Q}H \subset \wp H$.

Let $\{\mu_n\}$be a sequence in H such that $\{\mu_n\} = \frac{n+1}{n}$ for each n.

Also, let $\varrho: [0,\infty) \to [0,\infty)$ be defined by:

$$\varrho(t) = 2t$$

Clearly, $\wp(1) = \mathbb{Q}(1) = 1$ and $\wp\mathbb{Q}(1) = \mathbb{Q}\wp(1) = 1$, this shows that $\wp$ and $\mathbb{Q}$ are weakly compatible and let $u, v \in H$.

Now, we shall prove the inequality (2.2) of the Theorem 2.1.

$$d^*(\mathbb{Q}u, \mathbb{Q}v) = \frac{1}{3}|u-v|$$

Clearly,

$$\int_0^{d^*(\mathbb{Q}u,\mathbb{Q}v)} \varrho(t)dt = \frac{1}{9}(u-v)^2 \tag{2.18}$$

Now,

$$J(\wp u, \wp v) \geq d^*(\wp u, \wp v)$$

And

$$d^*(\wp u, \wp v) = |u-v|$$

This implies

$$\int_0^{d^*(\wp u,\wp v)} \varrho(t)dt = (u-v)^2 \tag{2.19}$$

From (2.16) and (2.17), we can conclude that

$$\int_0^{d^*(\mathbb{Q}u,\mathbb{Q}v)} \varrho(t)dt \leq \beta\big(d^*(\wp u, \wp v)\big) \int_0^{J(\wp u,\wp v)} \varrho(t)dt,$$

since $\xi(t) < 1$ for all t.

Now, $\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \mathbb{Q}\mu_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right) + \frac{2}{3} = 1$, where $1 \in H$.

This implies $\wp$ and $\mathbb{Q}$ satisfies E.A. property.

Also, $\lim_{n\to\infty} \wp\mu_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \mathbb{Q}\mu_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right) + \frac{2}{3} = 1 = \wp(1)$,

where $1 \in H$.

This implies $\wp$ and $\mathbb{Q}$satisfies $(CLR_{\wp})$ property.

Hence all the properties of Theorems 2.1, 2.3 and 2.4 are satisfied.

Here 1 is the common fixed point of $\wp$ and $\mathbb{Q}$.

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