Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 33 No. 1s (2026)

Quasi-Periodicity & Period-Halving bifurcation Routes to Chaos in a Financial Model

Ruma Saha

Department of Applied Sciences & Humanities, MIT School of Computing, MIT Art, Design & Technology University, Pune, Maharashtra, India, Email: rumasaha.2008@gmail.com

Article History:

Received: 08-01-2026

Revised: 15-01-2026

Accepted: 25-01-2026

Abstract: This paper aims to study the complex nonlinear dynamics involved in the financial system, depending on three key parameters: saving amount, the cost associated with unit investment and elasticity of commodity demand. This research explores the formation of a chaotic attractor by altering a parameter within the financial model. We illustrated that a Hopf bifurcation takes place, resulting in the emergence of a stable limit cycle. Previous studies have not addressed the quasi periodicity and period halving bifurcation. Through numerical analysis, we have uncovered a cascade of period halving bifurcation, quasi periodicity which led to the formation of a strange attractor. The existence of chaos in the system has been effectively recognized using various methods, including bifurcation diagrams, lyapunov Dimension, lyapunov Exponents, time series and, two dimensional and three-dimensional phase portraits. The system's sensitivity is estimated utilizing the fourth order Runge-Kutta method.

Keywords: Hopf bifurcation, Period halving bifurcation, Quasi-periodicity, Routh-Hurwitz stability criterion , Lyapunov Exponent ,Lyapunov Dimension.

1. Introduction

A mathematical framework that illustrates the connections among savings, investments, and financial resources within the financial sector is a nonlinear financial system. To aid in decision-making and future predictions, economists and researchers have explored the use of mathematical methods in finance. Predicting financial events has always been a challenge, leading to a growing interest in economic models of nonlinear dynamical systems.

It is known that in some mathematically uncomplicated systems of nonlinear differential equations chaos may develop. The recent paramount progress of AI related neural networks has made it suitable to explore the entire parameter space that exhibit specific characteristics. Uncertain factors have a crucial impact on describing the financial system and emphasize the importance of analyzing financial systems [46-48]. Researchers are presently exploring the core characteristics of economic data, which include unpredictable microeconomic shifts, volatile macroeconomic changes, erratic growth patterns, and changes in syntax.

However, an incorrect combination of parameters in the financial system could cause financial markets to face challenges or drop their stability. Thus, it is vital to perform a comprehensive and structured analysis for this complex financial model. This research will uncover bifurcation phenomena across various parameter settings, inquire the origins of complex nonlinear dynamics, and help in predicting and managing complicated financial systems. Moreover, an essential feature of nonlinear dynamical systems is that multiple attractors can exist simultaneously for identical parameters but with varying initial conditions. This allows for flexibility in the systems' performance without the need for parameter adjustments.

2. Literature review

The research and development of chaos have definitely ranked among the most vital achievements in nonlinear dynamics. In economics and finance the significance of chaos has significantly risen. In 1996, as stated by Serletic [37], chaos signifies a drastic change in the understanding of business cycles. Since the first identification of chaotic phenomena in economics in 1985, numerous phenomena have been illuminated through the use of chaotic theories, and principles. They are continuously uncovering novel discoveries [8,34] (Chen, 1988; Radelet et al., 1998). These theories have altered the way people perceive the economy, and macro control policies in certain economic situations. The manifestation of chaotic behavior in economics implies that, financial systems possess intrinsic uncertainty, emphasizing their importance of study [43, 12].

There are many significant contributions to the field of financial modeling. For instance, in 2005, Ishiyama and Saiki [22] developed a 'macroeconomic growth cycle model' that addressed the descriptive and numerical characteristics of unstable periodic solutions within chaotic attractors. In 2010, Wang et al. [41] investigated bifurcation structures and, overall analysis in a certain category of nonlinear financial systems. In 2011, Zhao et al. [49] applied the Routh-Hurwitz criterion, Lyapunov stability theory and explored the global synchronization phenomenon in a 3-dimensional chaotic financial model. The study of bifurcation and its regulation in a hyperchaotic finance model was carried out by Yu et al. in 2012 [45]. Meanwhile, Cantore and Levine [4] developed a reparametrized model involving evaluation parameters. In 2018, Gao and co-researchers [17] concentrated on the final bounded estimator set along with chaotic synchronization aspects. Liao et al. [28] examined how policy delay affects Hopf bifurcation and chaotic behavior in a macroeconomic framework, and Cao [5] contributed research on chaos control in such systems. For additional insights, one may consult other studies such as those listed in references [9,11,13,16,18,24,38].

3. Financial Model

Economists have validated a nonlinear economic and financial dynamic model based on a financial structure comprising of the key components: money, production, labor force, and stock. [4, 30, 9,10,16, 31,32, 36]. References [31-32] elaborate on the economic concepts and methods that are fundamental to the derivation of this model.

$$\begin{aligned} \dot{x} &= z + (y - a)x \\ \dot{y} &= 1 - by - x^2 \\ \dot{z} &= -x - cz \end{aligned} \tag{1.1}$$

In this model, the variable x represents the interest rate, y stands for the investment demand, z corresponds to the price index, $a > 0$ indicates the saving amount, $b > 0$ signifies unit investment cost, and $c > 0$ reflects the elasticity of commodity demand.

The dynamics of the variable *x* are regulated by the surplus from investment relative to savings and are structurally balanced through pricing mechanisms. The rate of change of variable *y* is directly proportional to the investment rate but inversely affected by the cost of investment and the interest rate. Meanwhile, changes in the variable *z* depend between supply and demand imbalance, in commercial markets and are further influenced by inflation rates. Considerable research works have concentrated on examining bifurcation phenomena, synchronization techniques, and the topological horseshoe behavior within chaotic financial systems. [25,12]

4. Scope and Significance

Previous studies have given significant information for chaotic dynamics and Hopf bifurcations in financial systems, there remains a gap in understanding the detailed mechanisms of bifurcation phenomena for example quasi periodicity, period halving bifurcation, the combined effects of multiple parameters. Further exploration of these areas would provide to a more in-depth understanding of the nonlinear dynamics of financial markets and improve the predictive capabilities of financial models.

In this work, by using bifurcation theory and the Routh-Hurwitz Criterion, we estimate the occurrence of periodic halving bifurcations, Hopf bifurcation, and quasi-periodicity within the model. Also, the bifurcation diagrams, limit cycle, and phase portraits are closely observed to find the route to chaos through the mechanism of bifurcation analysis. The graphical representation is created using Mathematica programming.

The dynamic behavior of this system was analyzed by references [2,5,9,10,12,16,25] using different combinations of parameters. The variable describing the savings amount, denoted as 'a', within the system must be maintained at an appropriate level. A smaller value of 'a' leads to increased variations within the system. If 'a' becomes very small, it may result in chaotic conditions. Conversely, if 'a' is too large, it can lead to a shortfall of dynamism in the economy.

Economic Implications:

Parameter	Economic Meaning	Higher Value Implication	Lower Value Implication
a (Savings Amount)	Determines the available capital for investment	More savings implies lower interest rate volatility	Less savings implies higher interest rate fluctuations
b (Cost of Investment)	Controls sensitivity of investment demand to costs	Investment drops faster implies more responsive business cycles	Investment remains high implies risk of over investment
C (Elasticity of Commodity Demand)	Governs the speed of price adjustments	Prices stabilize quickly implies less inflation persistence	Prices adjust slowly implies longer inflationary periods

Here, the dynamic behavior and the results of the finance system (1) are presented when the parameter a=3 (savings amount), the parameter b=0.06 (cost associated with unit investment), and the parameter c (elasticity of commodity demand) is varied.

The value of a = 3 shows a moderate savings level, that allows interest rates to adjust dynamically while minimizing excessive volatility.

By selecting b = 0.06, the model depend on a moderate level of investment elasticity, indicating that investment affects slowly to changes in costs rather than experiencing a rapid decline.

The variable c regulates the rate at which prices react to alterations in interest rates and economic activity. An appropriate set of c facilitates price stability, avoiding undue volatility.

5. Preliminary Results and Discussion on the Model

In this section we have applied the concept of trace, determinant and Routh-Hurwitz Criterion to find the constraints on the parameters and variables of the system and the corresponding analysis and interpretation of its results.

Taking into account that $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$ in our system (1.1), we derive the subsequent equilibrium points.
The system (1.1) has the only equilibrium point A = $(0, \frac{1}{b}, 0)$ if $c - b - abc \leq 0$; and has three if $-b + c - abc > 0$ i.e. $c \geq \frac{b}{1-a\,b}$.

$$A\left(0, \frac{1}{b}, 0\right), B\left(\frac{\sqrt{-b+c-abc}}{\sqrt{c}}, \frac{1+ac}{c}, -\frac{\sqrt{-b+c-abc}}{c^{3/2}}\right), C\left(-\frac{\sqrt{-b+c-abc}}{\sqrt{c}}, \frac{1+ac}{c}, \frac{\sqrt{-b+c-abc}}{c^{3/2}}\right)$$

The Jacobian matrix is given as $J(x, y, z) = \begin{bmatrix} y-a & x & 1 \\ -2x & -b & 0 \\ -1 & 0 & -c \end{bmatrix}$
The characteristic equation of the system is

$$\lambda^3 + \lambda^2(a + b + c - y) + \lambda(1 + ab + ac + bc + 2x^2 - by - cy) + b + abc + 2cx^2 - bcy = 0$$

At equilibrium $A\left(0, \frac{1}{b}, 0\right)$: the eigen values for Jacobian matrix J applied at A are

$$\lambda_1 = -b, \qquad \lambda_2 = \frac{1-ab-bc-\sqrt{(-1+ab+bc)^2-4b(b-c+abc)}}{2b},$$

$$\lambda_3 = \frac{1 - ab - bc + \sqrt{(-1 + ab + bc)^2 - 4b(b - c + abc)}}{2b}$$

To determine the solution of the characteristic polynomial we compute the trace and determinant (Det) of J.
The determinant and trace of the system (1.1) is given by

$$\text{Det(J)} = -b - abc - 2cx^2 + bcy$$

$$\text{Trace(J)} = -a - b - c + y.$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues,

where $T = Trace = \lambda_1 + \lambda_2 + \lambda_3 = -a - b - c + y$

$K = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 + ab + ac + bc + 2x^2 - by - cy,$

$D = Det(J) = \lambda_1\lambda_2\lambda_3 = \; -b - abc - 2cx^2 + bcy$

At the equilibrium point $A\left(0,\frac{1}{b},0\right)$, T$= -a + \frac{1}{b} - b - c$, K$= ab + ac - \frac{c}{b} + bc$, D$= -b + c - abc;$

“If Det(J) $<$ 0 then the equilibrium point is saddle.

If Det(J) $>$ 0 and Trace(J) $<$ 0, then the equilibrium point is stable.

If Det(J) $>$ 0 and Trace(J) $>$ 0, then the equilibrium point is unstable”.

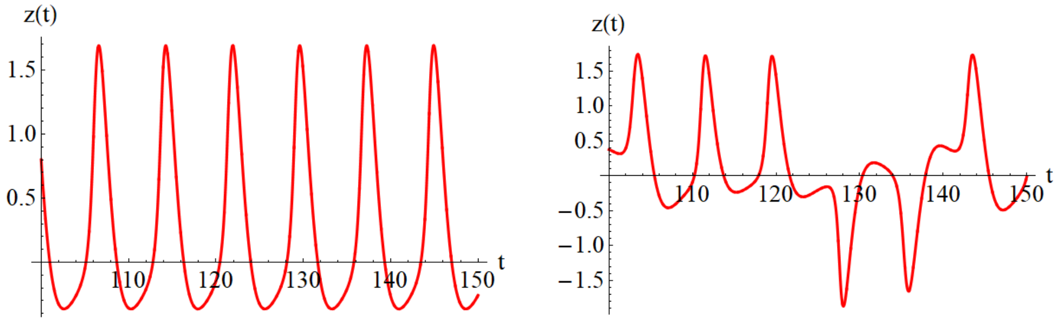
For $c < \frac{b}{1-ab}$, the equilibrium point A is a saddle point. Since for the determinant of the Jacobian matrix less than zero, the equilibrium point is a saddle point.

For $a = 3, b = 0.06$, we get A (0, 16.6667, 0). The eigen values for Jacobian matrix J applied at A are given by $\lambda_1 = -0.06, \lambda_2 = 6.83333 - \frac{c}{2} - 0.5\surd(11.6667 + c)\surd(15.6667 + c)$

$\lambda_3 = 6.83333 - \frac{c}{2} + 0.5\surd(11.6667 + c)\surd(15.6667 + c)$

If $6.83333 < \frac{c}{2} + 0.5\surd(11.6667 + c)\surd(15.6667 + c)$ then $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 > 0$.

A is a (unstable) saddle point for c=0.73170.



(a) (b)

Fig. 1 The time series plot for (a) z values at c=0.7 (b) z values at c=0.74 showing A is saddle point at the parameter value $c = 0.73170$ with initial point $x_0 = 1, y_0 = 1$, $z_0 = 1$. (Similarly, a time series plot can also be created for the x values and y values)

For $c > \frac{b}{1-a\,b}$, $-a + \frac{1}{b} - b < c$ this implies that A is always stable. Since for the determinant of the Jacobian matrix greater than zero and Trace(J) $<$ 0, the equilibrium point is stable.

For $c > \frac{b}{1-a\,b}$, $-a + \frac{1}{b} - b > c$, this implies that A is always unstable as at least one eigenvalue is positive. Since for the determinant of the Jacobian matrix greater than zero and Trace(J) $>$ 0, the equilibrium point is unstable.

At the fixed point $B\left(\frac{\sqrt{-b+c-abc}}{\sqrt{c}}, \frac{1+ac}{c}, -\frac{\sqrt{-b+c-abc}}{c^{3/2}}\right)$ and $C\left(-\frac{\sqrt{-b+c-abc}}{\sqrt{c}}, \frac{1+ac}{c}, \frac{\sqrt{-b+c-abc}}{c^{3/2}}\right)$

T= $-b+\frac{1}{c}-c$, K= $2-2ab+b(-\frac{3}{c}+c)$, D= $2(b-c+abc)$

We observe that for $c > \frac{b}{1-ab}$, D < 0, This implies that at least one negative eigenvalue exists. Depending on the values of the other two eigenvalues, B may be a node, or a spiral, stable or unstable. Therefore $c=\frac{b}{1-ab}$ is a bifurcation value of the type known as saddle node bifurcation [1,19,20].

Study of the stability of the system with Routh-Hurwitz Criterion:

Routh-Hurwitz Criterion: "Let us consider the cubic polynomial $p(\lambda)=\lambda^3+p_1\lambda^2+p_2\lambda+p_3$, with real coefficients. Then

(i) The equilibrium point is stable (i.e all its roots have the negative real part) if and only if $p_1, p_2 > 0, p_1p_2-p_3>0$.

(ii)If $p_3>0$ and $p_1p_2-p_3<0$, then p has two roots with a positive real part and one negative real root.

(iii)If $p_1, p_2, p_3>0$ and $p_1p_2-p_3=0$, then p has two complex conjugate roots with zero real part and one negative real root."

The characteristic equation of the system is
$\lambda^3+\lambda^2(a+b+c-y)+\lambda(1+ab+ac+bc+2x^2-by-cy)+b+abc+2cx^2-bcy=0$
At the equilibrium point $\left(0,\frac{1}{b},0\right)$, $p_1=a+b+c-\frac{1}{b}, p_2=ab+ac-\frac{c}{b}+bc, p_3=b-c+abc.$
Then by Routh-Hurwitz Criterion,
The characteristic equation has two complex conjugate roots with zero real part and one negative real root
if $a+b+c-\frac{1}{b}>0, ab+ac-\frac{c}{b}+bc>0, b-c+abc>0$
and $\left(a+b+c-\frac{1}{b}\right)\left(1+ab+ac+bc+-1-\frac{c}{b}\right)-(b+abc-c)=0$

At the equilibrium point $\left(0,\frac{1}{b},0\right)$, for $a=3, b=0.06$; the above equation becomes

$c>13.6067, c<\ 0.0132288, c<\ 0.0731707$

$0.82c+(0.18\ -13.6067\ \mathrm{c})\ (-13.6067+\mathrm{c})-0.06=0$

$=> c=13.6667 \quad or \quad c=\ 0.0134934$

As c = 0.0134934 fails to satisfy the above inequality, we will not consider this value for the bifurcation analysis.

6. Bifurcation Analysis and Graphical Representation

If the dissipative system progresses over time, its trajectory within the state space will converge towards a specific point, curve, or area. This endpoint or curve is referred to as the attractor of the system, as multiple distinct trajectories will tend to approach this collection of points in the state space.

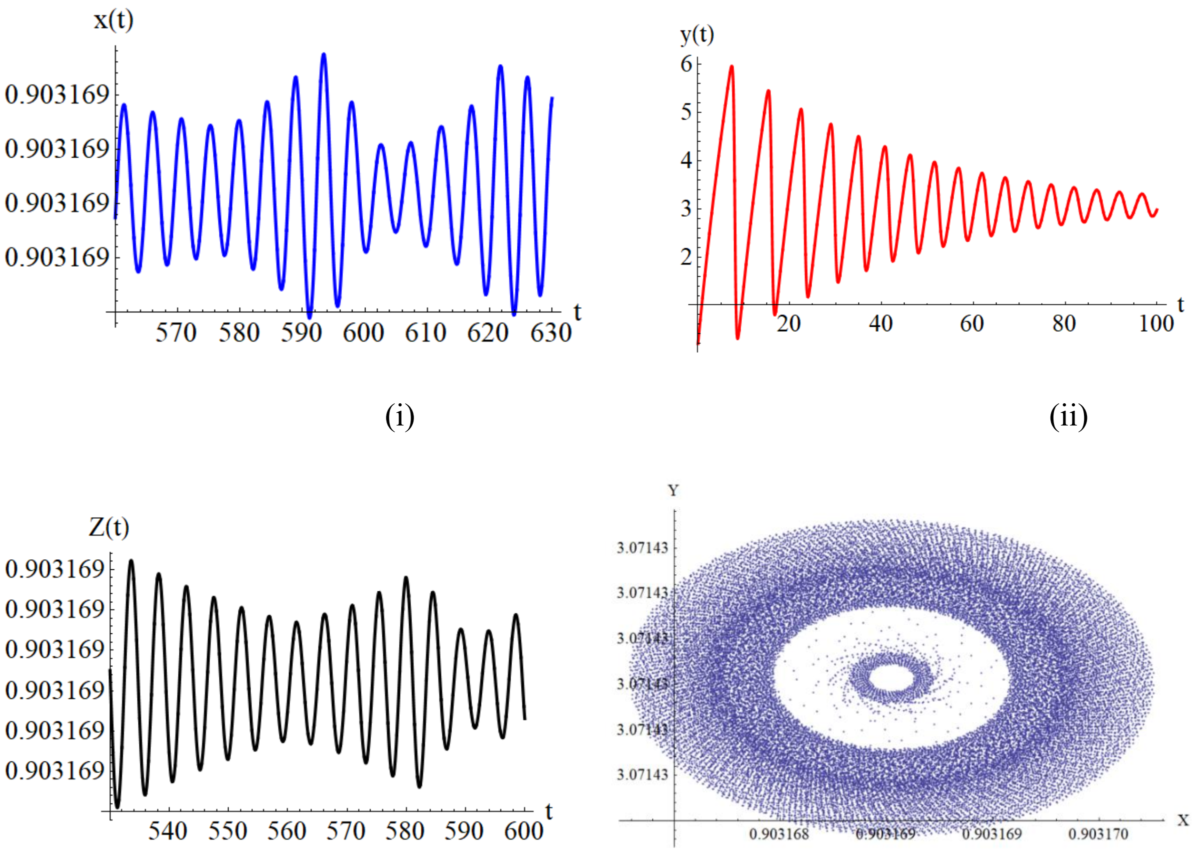
If a dissipative system possesses an attractor with a non-integer dimension, it is referred to as having a strange attractor.

Consider the system with these parameter values $a = 3, b = 0.06$;.

Proposition 1: If $c < 13.6667$, then the equilibrium point A is a stable focus of the system.

Proposition 2: If c $\cong 13.6667$ the equilibrium point is a weak stable focus.

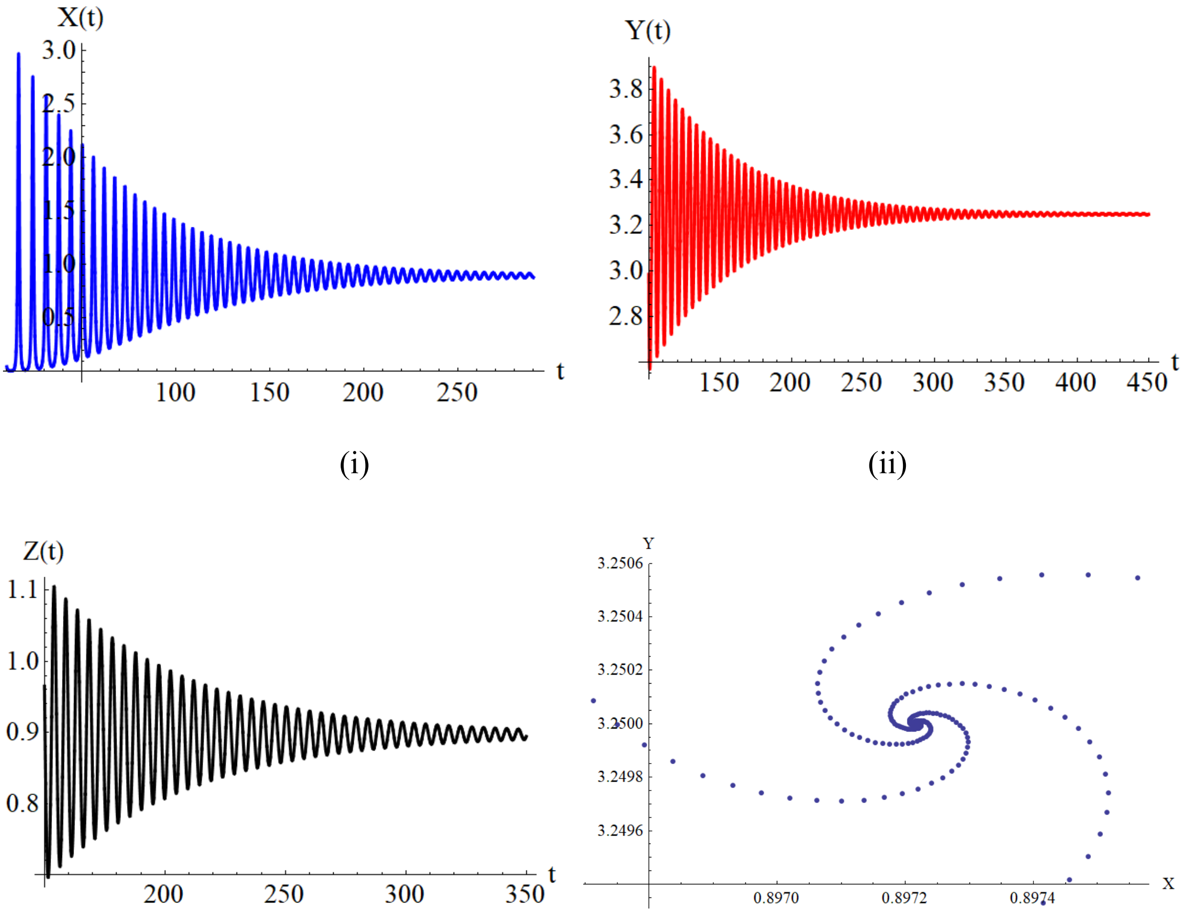
Proposition 3: When approaching this value $c > 13.6667$, the equilibrium point turns into an unstable focus, resulting in the formation of a small stable limit cycle around the equilibrium point. That is to say, a Hopf bifurcation occurs at c=13.6667.



(i) (ii)

(iii) (iv)

Fig. 2 The time series plot for (i) x (ii) y (iii) z respectively (iv) phase portrait after the hopf bifurcation at $c = 14$ with initial point $x_0 = 0.1, y_0 = 0.2$, $z_0 = 0$. The equilibrium point A transitions into an unstable focus, resulting in the emergence of small stable limit cycle surrounding the equilibrium point.

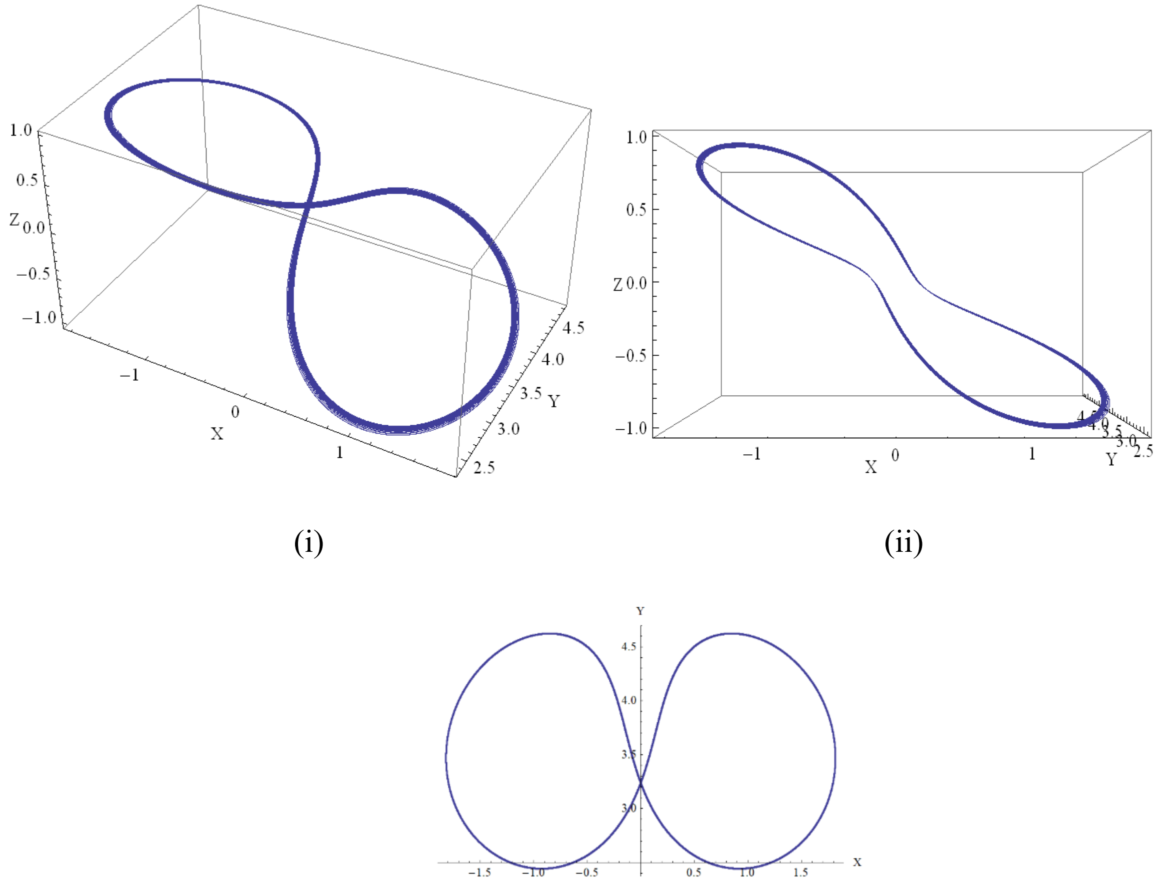


(i) (ii)

(iii) (iv)

Fig. 3 The time series plot for (i) x (ii) y (iii) z respectively and (iv) phase portrait before the Hopf bifurcation at $c = 4$ with initial point $(x_0, y_0, z_0) = (0.001, 0.2, 0.4)$ the equilibrium point A is a stable focus.

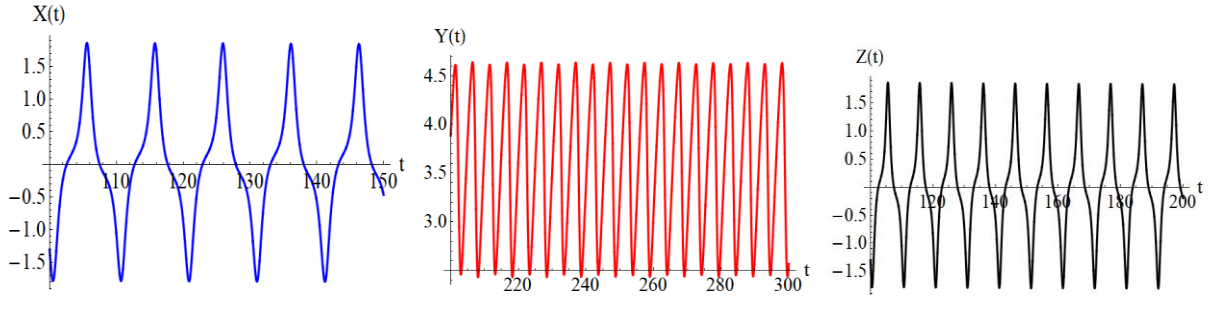
For c values that are near or below c=13.6667, where the Hopf bifurcation takes place at the equilibrium point A, we find a stable limit cycle with a period of 1. By decreasing c to 1.19, we have numerically identified that this limit cycle bifurcates into a periodic orbit of period 2. When c is further decreased to 1.18, a periodic orbit of period 4 is noted. As c continues to decrease, we see quasi-periodicity, as well as cycles of period 5 and period 3. This process persists until a chaotic attractor is established at c=1.12. This bifurcation is termed quasi-periodicity and period halving bifurcation [6,7,15,33]. The cascade of period halving bifurcation indicates a seamless transition from chaos to a periodic solution. Please see the figures below.



(i) (ii)

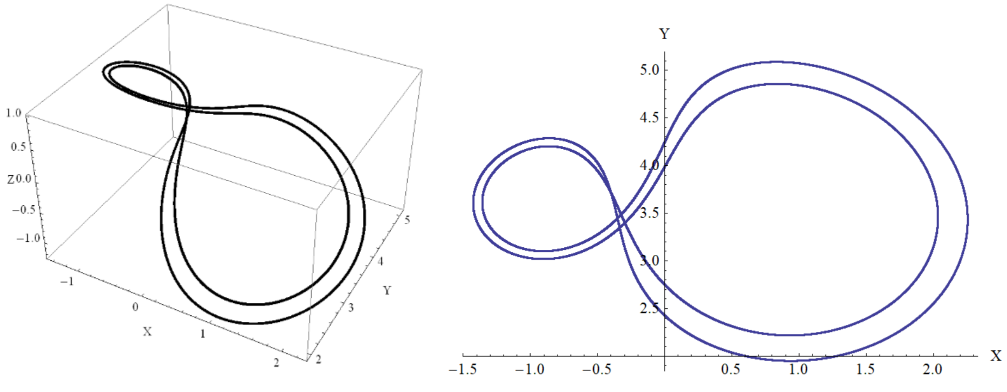
(iii)

Fig. 4 (i) Phase portraits of the financial model in the phase space when $c = 1.4$ with initial point $(x_0, y_0, z_{0)} = (1,1,1)$ (ii) Front view (iii) Projection in the XY plane, 40,000 iterations.



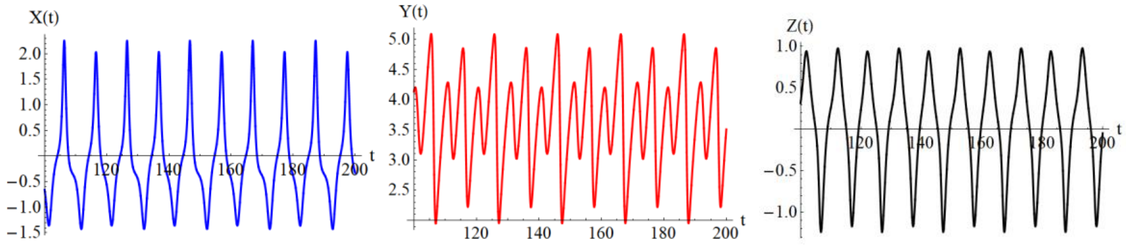
(i) (ii) (iii)

Fig. 5 Time series plot for (i) x, (ii) y and (iii) z values respectively which shows period one behavior. when $c = 1.4$ with initial point $(x_0, y_0, z_0) = (1,1,1)$.



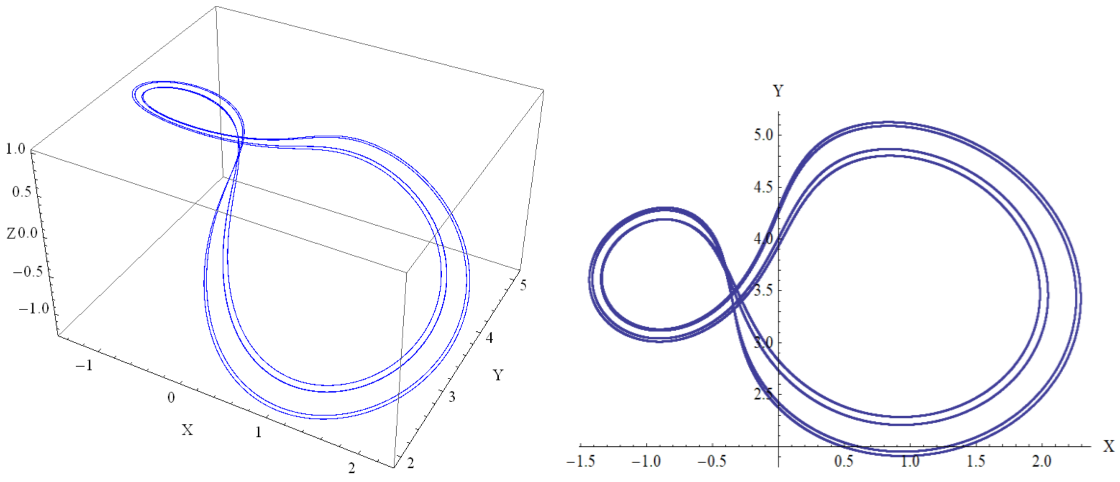
(i) (ii)

Fig. 6 (i) Phase portraits of the financial model in the phase space when $c = 1.19$ with initial point $(x_0, y_0, z_0) = (1, 0.1, 1)$ (ii) Projection in XY plane, 50,000 iterations and limit cycle splits into a period 2 periodic orbit.



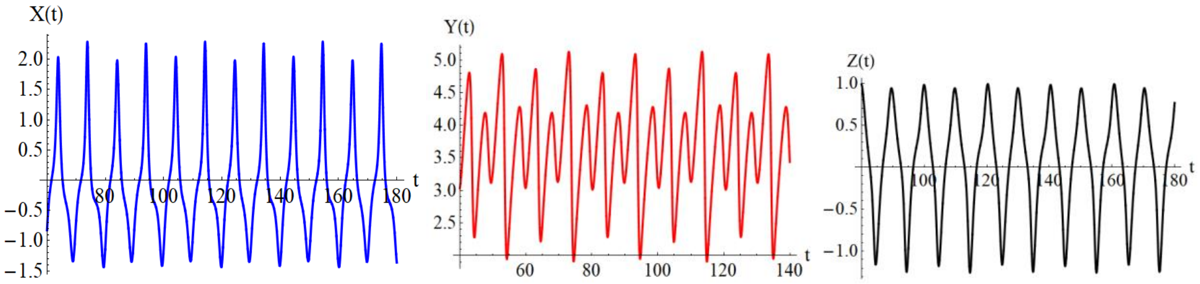
(i) (ii) (iii)

Fig. 7 Time series plot for x, y and z values respectively which shows period two behavior. when $c = 1.19$ with initial point $(x_0, y_0, z_0) = (1, 0.1, 1)$.



(i) (ii)

Fig. 8 (i) Phase portraits of the financial model in the phase space when $a = 3, b = 0.06, c = 1.18$ with initial point $x_0 = 2, y_0 = 1, z_0 = 1$ (ii) Projection in XY plane, 50,000 iterations (periodic orbit of period 4 is observed).



(i) (ii) (iii)

Fig. 9 Time series plot for x, y, and z values respectively which shows period four behavior. when $c = 1.18$ with initial point $(x_0, y_0, z_0) = (2,1,1)$.

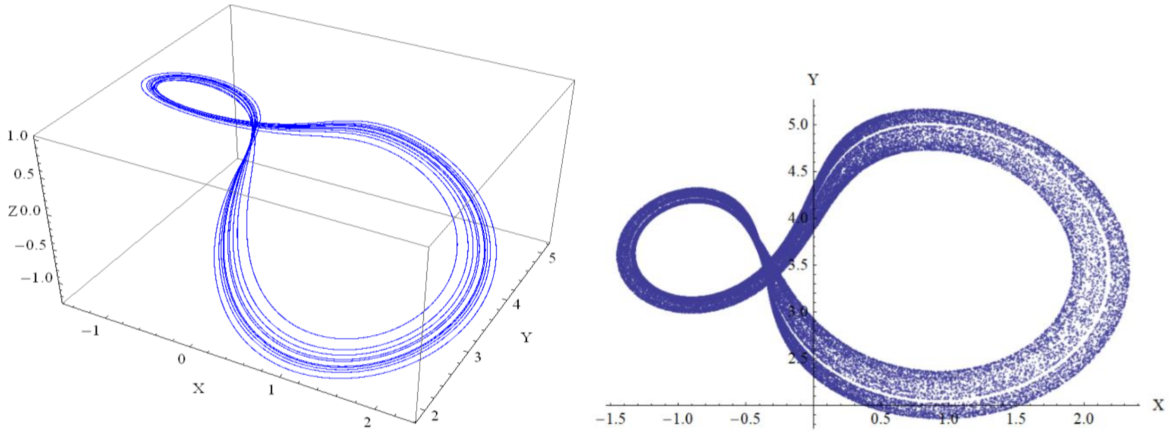
7. Quasi-Periodic behaviors

For the purpose to detect the quasi-periodic and the chaotic [21,22,23,27,35,50] behaviors of the financial system (1), the three-dimensional and two-dimensional phase portraits, as well as time series analysis, are displayed in Figs. 10(i)-(ii) and 11(i)-(iii), respectively, under the specific parameter selection of a=3, b=0.06, c=1.17 for the initial conditions x=2, y=0.01, z=0.02. It is acknowledged that the system (1) shows patterns of quasi-periodicity.

When c=1.13 is chosen and the other parameters remain unchanged, the system (1) presents an irregular pattern, indicating the presence of chaotic dynamics, as shown in Fig 14. The chaotic behavior of the model is verified by its time series analysis (Figs. 15(i)-(iii)).

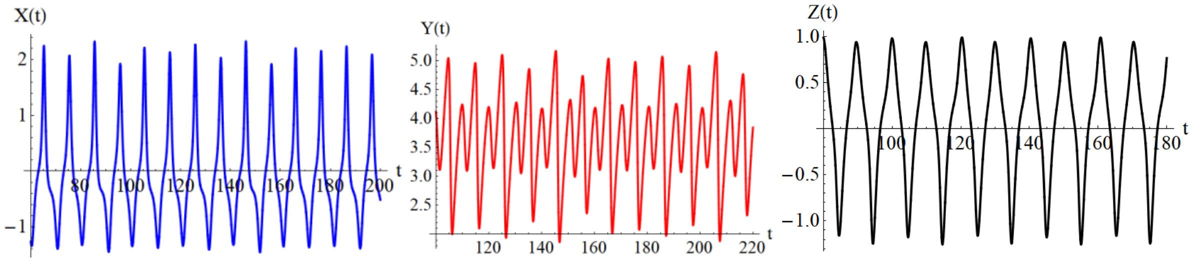
For a specific choice of c=1.12, while keeping the other parameters constant, system (1) exhibits a strange attractor, indicative of chaotic behavior, as shown in Figs. 16(i)-(ii) and 17(i)-(iii).

This is determined that the system (1.1) displays both quasi-periodic and chaotic motion. These findings are further elucidated by calculating the Lyapunov exponents and Lyapunov dimension in sections 8 and 9



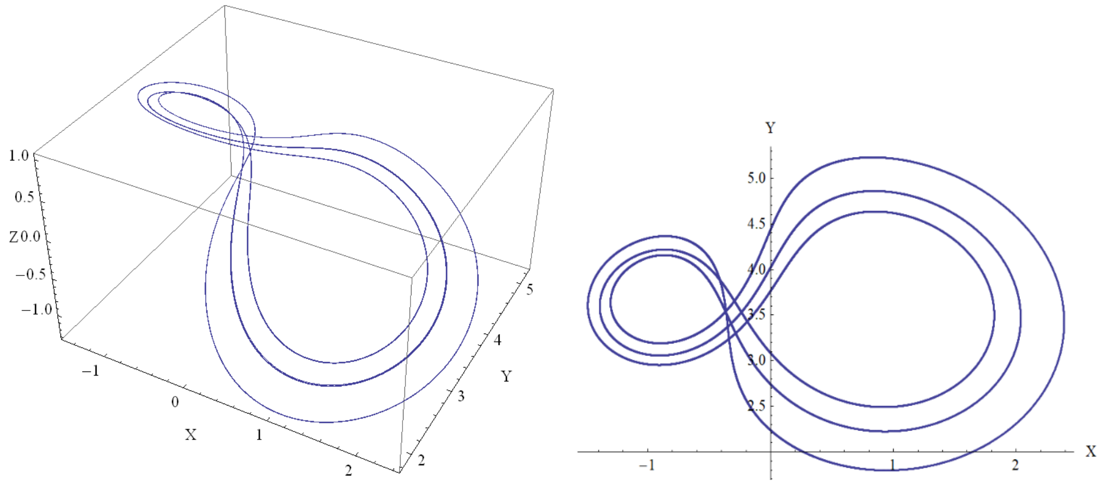
(i) (ii)

Fig. 10 (i) Phase portraits of the financial model in the phase space when $c = 1.17$ with initial point $(x_0,, y_0,, z_0) = (2, 0.01, 0.02)$ (ii) Projection in XY plane, 50,000 iterations.



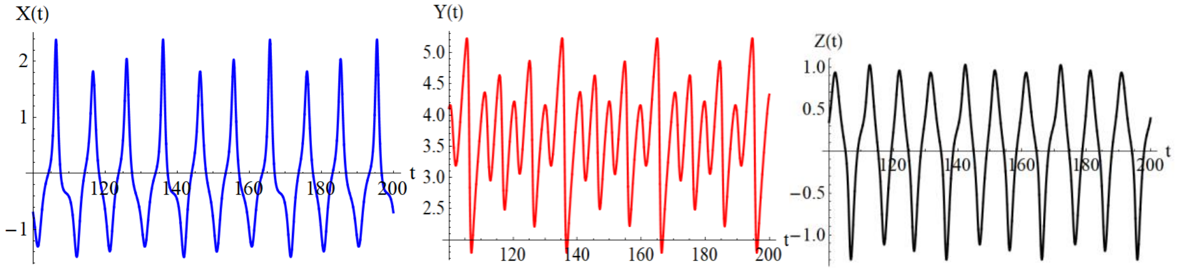
(i) (ii) (iii)

Fig. 11 Time series plot for x, y and z values respectively which shows quasi-periodic behavior. when c $= 1.17$ with initial point $(x_0,, y_0,, z_0) = (2, 0.01, 0.02)$.



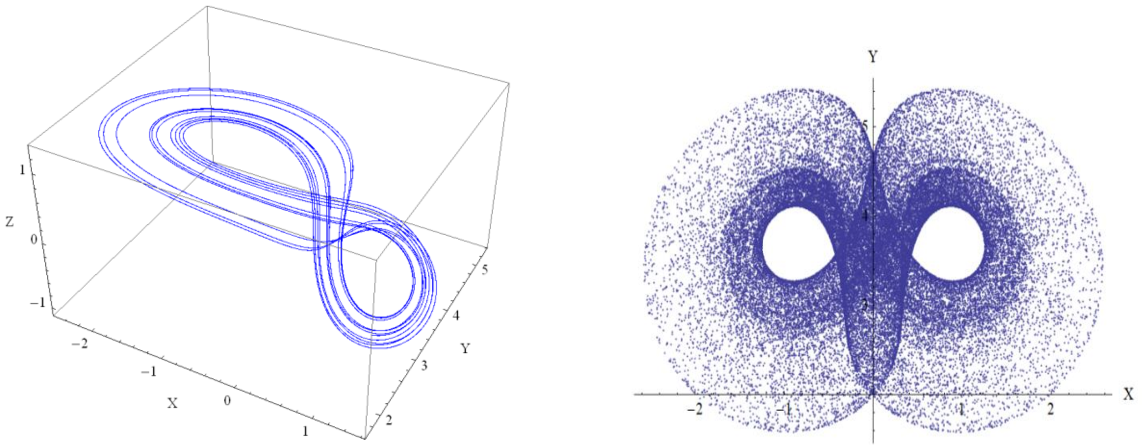
(i) (ii)

Fig. 12 (i) Phase portraits of financial model in the phase space when $a = 3, b = 0.06, c = 1.158$ with initial point $x_0 = 0.1, y_0 = 0.1, z_0 = 1$ (ii) Projection in the XY plane, 30,000 iterations.



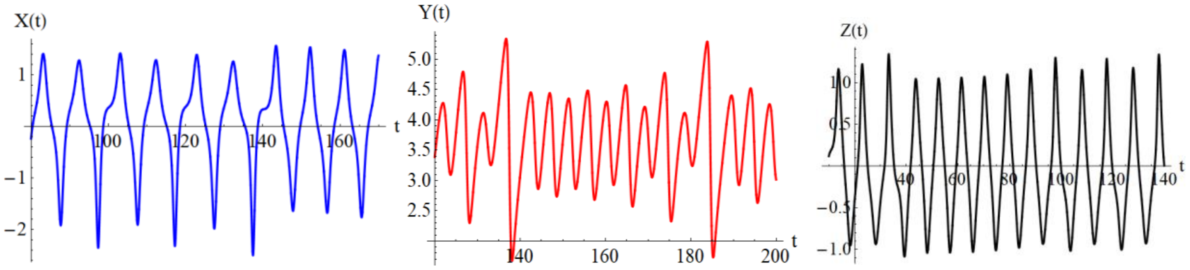
(i) (ii) (iii)

Fig. 13 Time series plot for x, y and z values respectively which shows period three behavior. when $a = 3, b = 0.06, c = 1.158$ with initial point $x_0 = 0.1, y_0 = 0.1, z_0 = 1$.



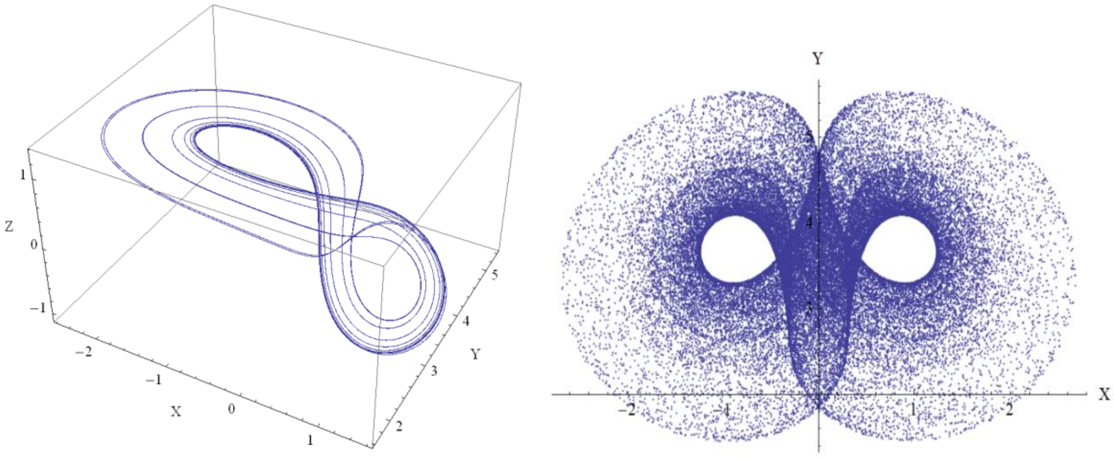
(i) (ii)

Fig. 14 (i) Phase portraits of the financial model in the phase space when $a = 3, b = 0.06, c = 1.13$ with initial point $x_0 = 0.2, y_0 = 1, z_0 = 1$ (ii) Projection in the XY plane, 50,000 iterations.



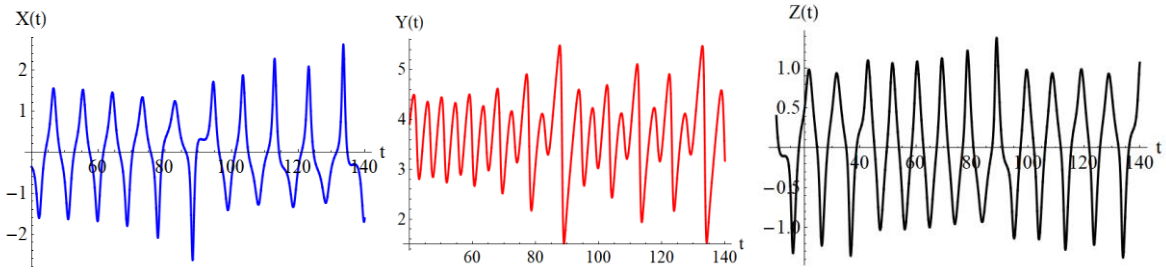
(i) (ii) (iii)

Fig. 15 Time series plot for x, y and z values respectively which shows chaos when $a = 3, b = 0.06, c = 1.13$ with initial point $x_0 = 0.2, y_0 = 1, z_0 = 1$.



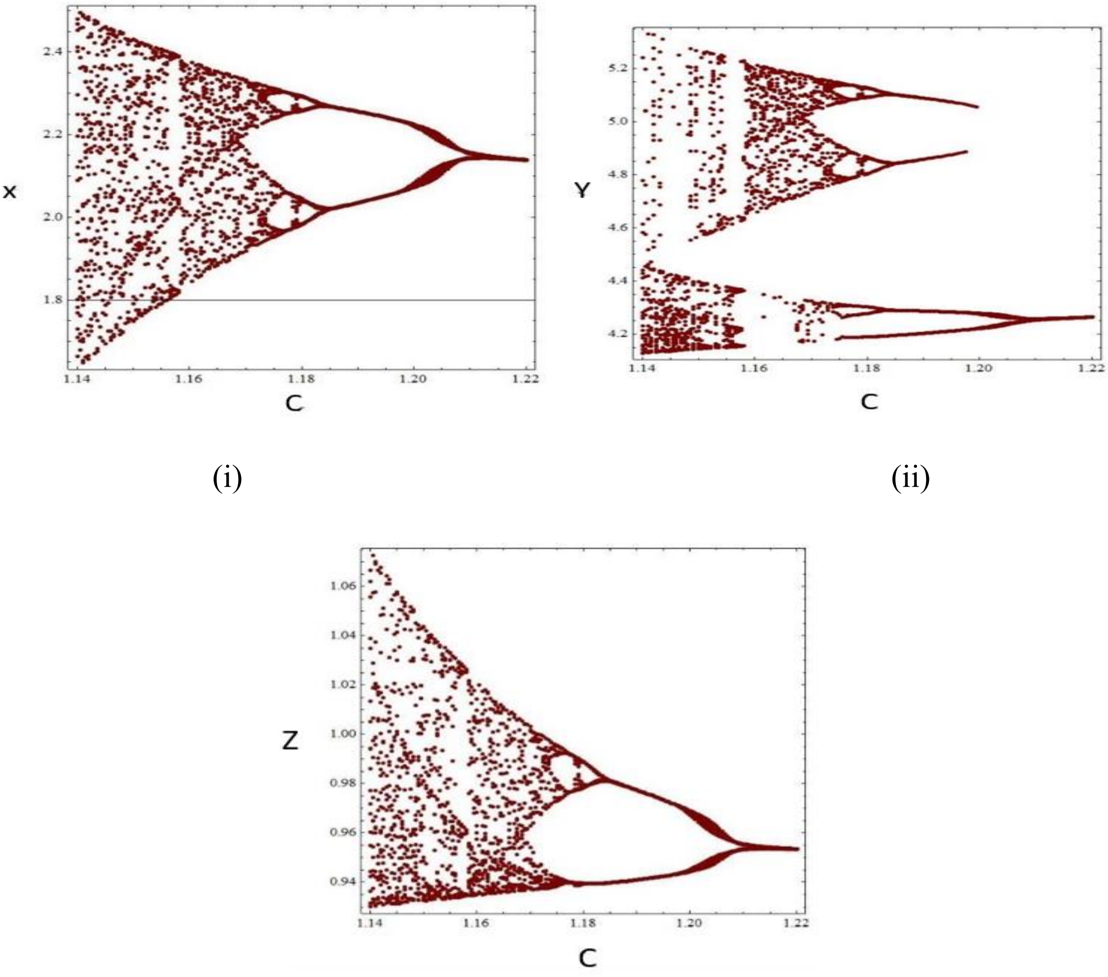
(i) (ii)

Fig. 16 (i) Phase portraits of the financial model in the phase space when $a = 3, b = 0.06, c = 1.12$ with initial point $x_0 = 0.1, y_0 = 0.1, z_0 = 1$ (ii) Projection in the XY plane, 50,000 iterations.



(i) (ii) (iii)

Fig. 17 Time series plot for x , y and z values respectively which shows chaotic attractor when $a = 3, b = 0.06, c = 1.12$ with initial point $x_0 = 0.1, y_0 = 0.1, z_0 = 1$.



(i) (ii)

(iii)

Fig. 18 Bifurcation diagram for the financial system (1.1). Chaotic behavior to periodic behavior when we increase the parameter ‘c’ from 1.14 to 1.22. Where the ordinate depicts the maximum value of periodic solutions in the (a) x-axis (b) y-axis (c) z-axis respectively.

8. Lyapunov Exponent

Another notable estimation involves the Lyapunov exponents [3,26,40], which approximate the exponential rates of convergence among proximate trajectories in state space. A Lyapunov exponent that is either positive or zero indicates chaotic dynamics, the existence of two zero Lyapunov exponents serves as evidence of a bifurcation. Additionally, a zero and a negative Lyapunov exponent recommended periodic behavior. It is crucial to identify that the total of the Lyapunov exponents must give in a negative value. In the context of three-dimensional continuous dissipative systems, the combinations $(\lambda_1, \lambda_2, \lambda_3)$ can lead to various phenomena: (+, 0, -) designates a strange attractor;

(0, 0, -) corresponds to a two-torus; (0, -, -) indicates a limit cycle; and (-, -, -) regarded as a fixed point.

9. Lyapunov Dimension

Kaplan and Yorke [1979] have introduced a conjecture that links the fractal dimension of the attractor to the Lyapunov spectrum.

$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{\lambda_{j+1}}$, where the LCEs are ordered in the usual way as $\lambda_1 \geq \cdots \geq \lambda_n$ and where j is the largest integer such that $\lambda_1 + \cdots + \lambda_j > 0$.

The Lyapunov dimension [14,26,40] and quasi-periodicity are related concepts in the study of dynamical systems, particularly when analysing the behaviour of such systems over time. The Lyapunov dimension provides a quantitative assessment of the complexity associated with a dynamical system, while quasi-periodicity signifies a behavioural type that can be analysed through the concepts of Lyapunov exponents and dimensions. Exploring these connections enhances our understanding of the dynamics within complex systems.

The Lyapunov exponents and their associated dimensions were calculated along selected trajectories, with the results being validated for consistency through the examination of various initial conditions and extended integration periods. We perform a numerical approximation of the Lyapunov exponents and dimensions over the time interval [0, T]. (see Mathematica code in [29])

The time interval is considered as [0, T = 1000], k = 10000, time step = 0.1.

Presented in the table below is a summary of the numerical computation that illustrates the relationship between the Lyapunov dimension and the Lyapunov exponents associated with the financial system for 1.12 < c <14 for three different initial conditions (x, y, z) = (1,5, 0.1), (x, y, z) = (4, 2, 5), (x, y, z) = (0, 15, 5) respectively.

	Lyapunov Dimension	t =10000 (1, 5, 0.1)	t = 10000 (4, 2, 5)	t = 10000 (0 ,15, 5)
c = 14	$D_L =$	0	0	0
c = 4	$D_L =$	0	0	0
c = 1.4	$D_L =$	1.17198	1.17067	1.02274
c = 1.19	$D_L =$	1.00504	0	1.05465
c = 1.18	$D_L =$	1.03933	1.00769	1.05603
c = 1.17	$D_L =$	2.04211	2.04188	2.04625
c = 1.165	$D_L =$	1.76686	1.06939	2.02412

c = 1.164	$D_L =$	2.06042	2.06268	2.04922
c = 1.158	$D_L =$	1.16171	1.22605	1.21942
c = 1.156	$D_L =$	2.05456	2.04958	2.05369
c = 1.13	$D_L =$	2.11315	2.12054	2.12045
c = 1.12	$D_L =$	2.13876	2.13115	2.13091

Table 1 Summery of the relationship between Lyapunov Dimensions at (x, y, z) = (1,5, 0.1), (x, y, z) = (4, 2, 5), (x, y, z) = (0, 15, 5)

Parameter (c)	Lyapunov Exponents	D_L
c = 14	{-0.0293513, -0.0300431, -13.9283}	0
c = 4	{-0.0168354, -0.0173573, -3.77581}	0
c = 1.4	{0.00188585, -0.0109655, -0.859718}	1.17198
c = 1.19	{0.000251879, -0.049927, -0.565094}	1.00504
c = 1.18	{0.00140318, -0.0356778, -0.568199}	1.03933
c = 1.17	{0.0243445,0.00164797, -0.617223}	2.04211
c = 1.165	{0.00168174, -0.00219302, -0.583925}	1.76686
c = 1.164	{0.0385545, -0.00110851, -0.619716}	2.06042
c = 1.158	{0.001648, -0.010191, -0.569444}	1.16171
c = 1.156	{0.0343979, -0.00128756, -0.606889}	2.05456
c = 1.13	{0.0695701,0.000245019, -0.617028}	2.11315
c = 1.12	{0.0879927, -0.00170136, -0.62189}	2.13876

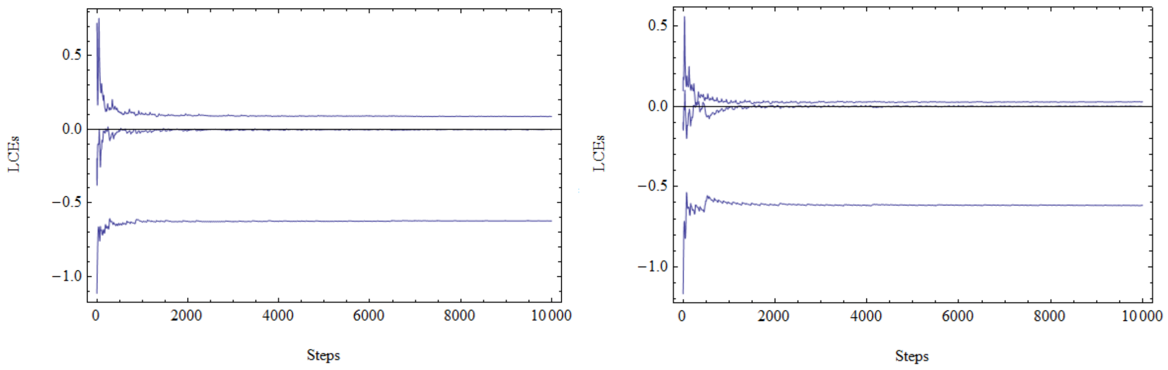
Table 2 Summery of the relationship between Lyapunov Exponents and the Lyapunov Dimension at (x, y, z) = (1, 5, 0.1)

Parameter (c)	Lyapunov Exponents	D_L
c = 14	{-0.0293513, -0.0300431, -13.9283}	0
c = 4	{-0.0168354, -0.0173573, -3.77581}	0
c = 1.4	{0.00187317, -0.0109756, -0.859705}	1.17067
c = 1.19	{-0.00338319, -0.0417506, -0.569089}	0
c = 1.18	{0.000271084, -0.0352505, -0.567386}	1.00769

c = 1.17	{0.0268758, -0.00107775, -0.61598}	2.04188
c = 1.165	{0.000274619 0.00395738, -0.583}	1.06939
c = 1.164	{0.0405023, -0.00156846, -0.621141}	2.06268
c = 1.158	{0.00164107, -0.0072599, -0.572355}	1.22605
c = 1.156	{0.0330788, -0.00297316, -0.607236}	2.04958
c = 1.13	{0.0746443, 0.000515602, -0.623551}	2.12054
c = 1.12	{0.0813732, -0.000555021, -0.616241}	2.13115

Table 3 Summery of the relationship between Lyapunov Exponents and the Lyapunov Dimension at (x, y, z) = (4, 2, 5)

Different quasi-periodic attractors [44] may show different counts of zero Lyapunov exponents. Therefore, the Lyapunov dimension serves as a means to specify these quasi-periodic attractors, calculated from the entirety of the Lyapunov exponents.



(i) (ii)

Fig. 19 Relationship of the Lyapunov Exponents

For this system it is evident that, one Lyapunov exponent is positive, another is zero and the third is negative, which implies chaotic nature. The graphical representation of LEs of the system is presented in Fig.19-(i) under the consideration of parameters of a=3, b=.06, c=1.12 and initial condition x=1, y=5, z=0.1. Therefore, the system (1.1) has a chaotic attractor at c=1.12 as shown in Fig. 16(i)-(ii) and 19 (i).

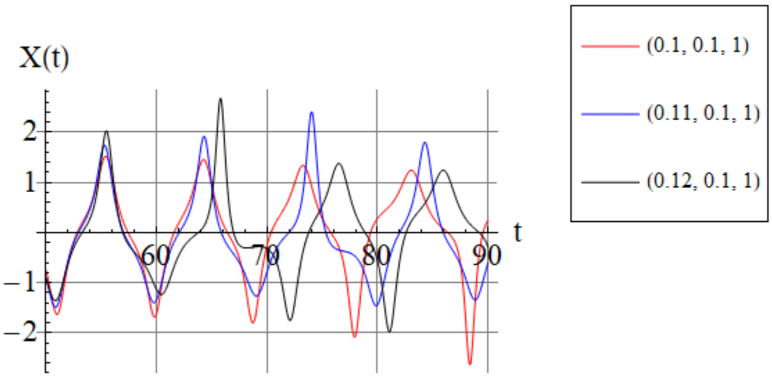
It is seen from Fig.19-(ii) for the parameters of a=3, b=.06, c=1.17 for initial condition x=4, y=2, z=5 that the system verifies the quasiperiodicity and chaotic behavior because the values of LEs maintain the chaotic condition $(\lambda_1, \lambda_2, \lambda_3)$ = (+, 0, -).

Figures 18 and 19 illustrate that as the elasticity of commodity demand, denoted as c, diminishes, economic fluctuations become increasingly pronounced, leading to greater instability within the system that may ultimately result in chaos. Therefore, maintaining the elasticity of commodity demand at an appropriate level is crucial for fostering rapid and sustainable economic growth. As the derived

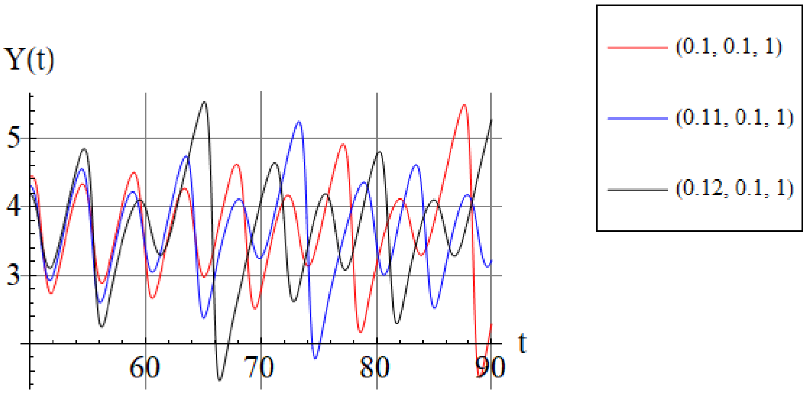
Lyapunov dimensions gives significant insights into the dynamics of the system, we acknowledge the method's limitations and recommend further investigation employing alternative approaches.

10. Sensitivity test of the system:

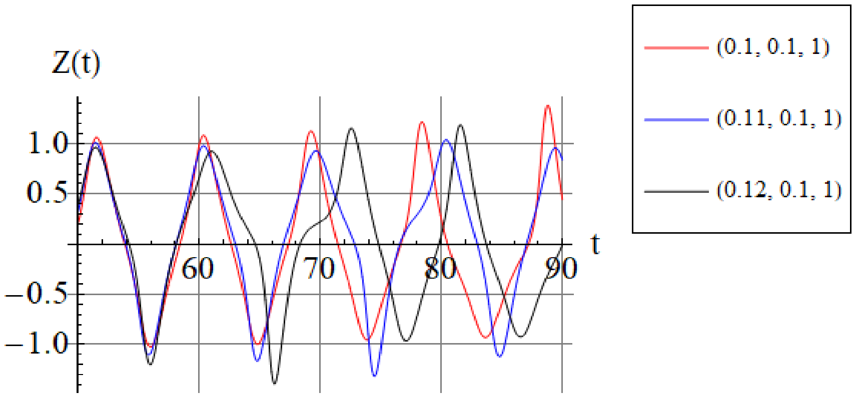
This section focuses on estimating the system's sensitivity, which effects the model's performance based on its initial conditions. A model is classified as less responsive if it shows minimal variation when initial conditions are slightly modified. Conversely, a system is considered highly sensitive if even small changes in initial conditions lead to significant differences. Sensitivity to initial conditions can be computed using Lyapunov Exponents (LEs). Thus, by examining various initial conditions, we will find the sensitivity of system (1). Figures 20 (i)-(iii) provide in-depth investigation of system (1) under these varying initial conditions. The sensitivity analysis applies the Runge-Kutta method of 4th order, taking into account three distinct initial conditions. Figure 20 illustrates the sensitivity of system (1) with parameters set as $a = 3$, $b = 0.06$, $c = 1.12$, where the initial conditions (0.1,0.1,1), (0.11,0.1,1), (0.12,0.1,1) are depicted by a red line, blue line, black line. The analysis shown in Figures 20(i)–(iii) reveals that even small changes in the initial values result in notable alterations in the solution of the model, which represents strong sensitivity in the model. The outcomes implies that the system exhibits a high degree of sensitivity to initial conditions, leading to chaotic behaviour.



(i)



(ii)



(iii)

Fig. 20 Sensitivity test of the system (i)x-values (ii) y-values (iii) z-values, parameters set as a = 3, b = 0.06, c = 1.12, where the initial conditions (x, y, z) = (0.1,0.1,1) are represented by a red line, (x, y, z) = (0.11,0.1,1) by a blue line, and (x, y, z) = (0.12,0.1,1) by a black line.

11. Conclusion

In the Financial model (1.1) by decreasing the parameter c and keeping a and b fixed at 3 and 0.06 respectively, it exhibits quasiperiodicity & a period-halving route to chaos. We obtained a chaotic attractor at 1.12 of the value of c. This work utilizes bifurcation diagram to asssess the results, which highlights about the behavior for all value of the parameter. A range of graphical representations is pictured to recognize the quasi-periodic and the chaotic characteristics of the system. This involves the incorporation of visual representations in three-dimensional and two-dimensional phase portraits, time series graphs, and Lyapunov exponents. Our analysis of bifurcation in financial system shows that the system displays dynamic behaviors and responses.

Based on the outcomes of the simulation discussed above, it is evident that the inappropriate combination of system parameters is the root cause of chaos within the economic system. Such a combination can predispose the system to chaotic behavior, leading to a loss of control and a subsequent descent into stagnation and inflexibility. Therefore, regardless of whether the economy is experiencing significant inflation or a period of slack, the transformation of mechanisms and structural adjustments should always be regarded as the primary focus in the reform of the financial system.

References

[1] Akgul, A., Calgan, H., Koyuncu, I., Pehlivan, I. and Istanbullu, A. (2016), Chaos-based engineering applications with a 3D chaotic system without equilibrium points. Nonlinear Dynamics 84(2), 481–495.

[2] Alzaid, S.S., Kumar, A., Kumar, S. and Alkahtani, B.S.T. (2023), Chaotic behavior of financial dynamical system with generalized fractional operator. Fractals 31(4), 2340056. https://doi.org/10.1142/S0218348X2340056X.

[3] Alsafasfeh, H.Q. and Mohammad, S.A. (2011), A new chaotic behavior from Lorenz and Rossler systems and its electronic circuit implementation. Circuits and Systems 2, 101–105.

[4] Cantore, C. and Levine, P. (2012), Getting normalization right: dealing with 'dimensional constants' in macroeconomics. Journal of Economic Dynamics and Control 36(2), 1931–1949.

[5] Cao, L. (2018), A four-dimensional hyperchaotic finance system and its control problems. Journal of Control Science and Engineering 2018, Article ID 4976380, 12 pages. https://doi.org/10.1155/2018/4976380

[6] Chang, S.C. (2020), Controlling chaos through period-doubling bifurcations in attitude dynamics for power systems. Mathematical Problems in Engineering 2020, Article ID 8853459, 10 pages. https://doi.org/10.1155/2020/8853459

[7] Changjin, X., Chaouki, A., Zixin, L., Qiwen, Q. and Lingyun, Y. (2022), Bifurcation control strategy for a fractional-order delayed financial crises contagions model. AIMS Mathematics 7(2), 2102–2122.

[8] Chen, P. (1988), Empirical and theoretical evidence of economic chaos. System Dynamics Review 4, 1–38.

[9] Chen, W.C. (2008), Dynamics and control of a financial system with time-delayed feedbacks. Chaos, Solitons & Fractals 37(4), 1198–1207.

[10] Chen, W.C. (2008), Nonlinear dynamics and chaos in a fractional-order financial system. Chaos, Solitons & Fractals 36(5), 1305–1314.

[11] Chian, A.C.L., Rempel, E.L. and Rogers, C. (2006), Complex economic dynamics: Chaotic saddle, crisis and intermittency. Chaos, Solitons & Fractals 29, 1194–1218.

[12] Ding, Y. and Cao, J. (2015), Bifurcation analysis and chaos switchover phenomenon in a nonlinear financial system with delay feedback. International Journal of Bifurcation and Chaos 25(12), 1550165. https://doi.org/10.1142/S0218127415501655

[13] Fanti, L. and Manfredi, P. (2007), Chaotic business cycles and fiscal policy: An IS-LM model with distributed tax collection lags. Chaos, Solitons & Fractals 32, 736–744. https://doi.org/10.1016/j.chaos.2005.11.024

[14] Farmer, J.D., Ott, E. and Yorke, J.A. (1983), The dimension of chaotic attractors. Physica D: Nonlinear Phenomena 7(1–3), 153–180. https://doi.org/10.1016/0167-2789(83)90125-2

[15] Galindo, M.C., Nespoli, C. and Messias, M. (2015), Hopf bifurcation, cascade of period doubling, chaos, and the possibility of cure in a 3D cancer model. Advances in Mathematical Physics 2015, Article ID 354918. https://doi.org/10.1155/2015/354918

[16] Gao, Q. and Ma, J.H. (2009), Chaos and Hopf bifurcation of a finance system. Nonlinear Dynamics 58(1–2), 209–216.

[17] Gao, W., Yan, L., Saeedi, M., et al. (2018), Ultimate bound estimation set and chaos synchronization for a financial risk system. Mathematics and Computers in Simulation 154, 19–33.

[18] Goodwin, R.M. (1951), The nonlinear accelerator and the persistence of business cycles. Econometrica 19, 1–17.

[19] Gu, E.G., Ni, J. and He, Z.H. (2023), Bifurcation, chaos and multi-stability regions in an asset pricing model with three subsystems. Journal of Difference Equations and Applications 30(4), 475–503. https://doi.org/10.1080/10236198.2023.2295404

[20] Gu, E. (2024), The inherent law of the unpredictability of financial asset price fluctuations: Multistability and chaos. Journal of Systems Science and Complexity 37(2), 776–804.

[21] Hassan, T.S., Elmandouh, A.A., Attiya, A.A. and Khedr, A.Y. (2022), Bifurcation analysis and exact wave solutions for the double-chain model of DNA. Journal of Mathematics 2022(1), 7188118.

[22] Ishiyama, K. and Saiki, Y. (2005), Unstable periodic orbits and chaotic economic growth. Chaos, Solitons & Fractals 26(8), 33–42.

[23] Jhangeer, A., Hussain, A., Junaid-U-Rehman, M., Baleanu, D. and Riaz, M.B. (2021), Quasi-periodic, chaotic and traveling wave structures of modified Gardner equation. Chaos, Solitons & Fractals 143, 110578. https://doi.org/10.1016/j.chaos.2020.110578

[24] Jiang, Z.C., Guo, Y.F. and Zhang, T.Q. (2019), Double delayed feedback control of a nonlinear finance system. Discrete Dynamics in Nature and Society 2019, Article ID 7254121, 17 pages. https://doi.org/10.1155/2019/7254121

[25] Kai, G., Zhang, W., Jin, Z. and Wang, C.Z. (2020), Hopf bifurcation and dynamic analysis of an improved financial system with two delays. Complexity 2020, Article ID 3734125, 13 pages. https://doi.org/10.1155/2020/3734125

[26] Kaplan, J.L. and Yorke, J.A. (1979), Functional differential equations and approximations of fixed points. In: Lecture Notes in Mathematics, Vol. 730, Eds. Peitgen, H.-O. and Walther, H.-O., 204–227. Springer, Berlin.

[27] Kumar, D. (2024), Bifurcations of phase portraits and chaotic behaviors of the (2+1)-dimensional double-chain DNA system with beta derivative: A qualitative approach. Heliyon 10(14), e34421. https://doi.org/10.1016/j.heliyon.2024.e34421

[28] Liao, X.F., Li, C.D. and Zhou, S.B. (2005), Hopf bifurcation and chaos in macroeconomic models with policy lag. Chaos, Solitons & Fractals 15, 91–108. https://doi.org/10.1016/j.chaos.2004.09.075

[29] Lynch, S. (2007), Dynamical Systems with Applications using Mathematica. Birkhäuser, Boston.

[30] Ma, C. and Wang, X. (2012), Hopf bifurcation and topological horseshoe of a novel finance chaotic system. Communications in Nonlinear Science and Numerical Simulation 17(2), 721–730.

[31] Ma, J.H. and Chen, Y.S. (2001), Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system. I. Applied Mathematics and Mechanics 22(11), 1240–1251.

[32] Ma, J.H. and Chen, Y.S. (2001), Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system. II. Applied Mathematics and Mechanics 22(12), 1375–1382.

[33] Ma, J.H., Cui, Y.Q. and Liu, L.X. (2008), Hopf bifurcation and chaos of financial system on condition of specific combination of parameters. Journal of Systems Science and Complexity 21(2), 250–259.

[34] Radelet, S. and Sachs, J. (1998), The East Asian financial crisis: Diagnosis, remedies, prospects. Brookings Papers on Economic Activity 1, 1–74.

[35] Rongyan, Z. (2012), Bifurcation analysis for a kind of nonlinear finance system with delayed feedback and its application to control of chaos. Journal of Applied Mathematics 2012, Article ID 316390, 18 pages. https://doi.org/10.1155/2012/316390

[36] Saberi Nik, W. (2018), Integrability analysis of chaotic and hyperchaotic finance systems. Nonlinear Dynamics 94(1), 443–459.

[37] Serletic, A. (1996), Is there chaos in economic series? Canadian Journal of Economics 29, 210–212.

[38] Son, W.K. and Park, Y.J. (2011), Delayed feedback on the dynamical model of a financial system. Chaos, Solitons & Fractals 44(4–5), 208–217.

[39] Wiggins, S. (2003), Introduction to Applied Nonlinear Dynamical Systems and Chaos. Vol. 2 of Texts in Applied Mathematics, Springer, New York, NY, USA, 2nd edition.

[40] Wolf, A., Swift, J.B., Swinney, H.L. and Vastano, J.A. (1985), Determining Lyapunov exponents from a time series. Physica D: Nonlinear Phenomena 16(3), 285–317. https://doi.org/10.1016/0167-2789(85)90011-9

[41] Wang, Y., Jiang, W. and Wang, H. (2010), Delayed feedback control and bifurcation analysis of Rossler chaotic system. Nonlinear Dynamics 61(4), 707–715.

[42] Wu, W. and Chen, Z. (2010), Hopf bifurcation and intermittent transition to hyperchaos in a novel strong four-dimensional hyperchaotic system. Nonlinear Dynamics 60(4), 615–630.

[43] Yang, J., Zhang, E. and Liu, M. (2016), Bifurcation analysis and chaos control in a modified finance system with delayed feedback. International Journal of Bifurcation and Chaos 26(6), 1650105. https://doi.org/10.1142/S0218127416501054

[44] Yue, Y. (2016), Bifurcations of the symmetric quasi-periodic motion and Lyapunov dimension of a vibro-impact system. Nonlinear Dynamics 2016(3), 1697–1713. https://doi.org/10.1007/s11071-016-2598-3

[45] Yu, H.J., Cai, G.L. and Li, Y.X. (2012), Dynamic analysis and control of a new hyperchaotic finance system. Nonlinear Dynamics 67(3), 2171–2182.

[46] Zhang, W., Cao, J., Alsaedi, A., et al. (2017), Synchronization of time-delayed fractional-order chaotic financial system. Discrete Dynamics in Nature and Society 2017(1), 29–32.

[47] Zhang, X.D., Liu, X.D., Zheng, Y. and Liu, C. (2013), Chaotic dynamic behavior analysis and control for a financial risk system. Chinese Physics B 22(3), Article ID 030509.

[48] Zhang, X. and Zhu, H. (2019), Hopf bifurcation and chaos of a delayed finance system. Complexity 2019, Article ID 6715036. https://doi.org/10.1155/2019/6715036

[49] Zhao, X.S., Li, Z.B. and Li, S. (2011), Synchronization of a chaotic finance system. Applied Mathematics and Computation 217(2), 6031–6039.

[50] Zhu, C., Al-Dossari, M., Rezapour, S., Alsallami, S.A.M. and Gunay, B. (2024), Bifurcations, chaotic behavior, and optical solutions for the complex Ginzburg–Landau equation. Results in Physics 59, 107601. https://doi.org/10.1016/j.rinp.2024.107601