Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 33 No. 1s (2026)

A Trigonometric Transformation–Based Chebyshev Collocation Method

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Article History:
Received:15-12-2025
Revised:30-12-2025
Accepted:02-01-2026

Abstract:
The Chebyshev collocation method with trigonometric transformation is a spectral numerical solution technique for initial and/or boundary value problems in computational mechanics. This approach leverages the trigonometric representation of Chebyshev polynomials to enable analytic differentiation, eliminating the need for numerical differentiation matrices. This technique has demonstrated superior accuracy and computational efficiency across diverse applications in engineering and physical sciences. This paper presents a comprehensive analysis of the trigonometric Chebyshev collocation method (TCM) and its application to four diverse problems: heat transfer in a triangular fin, torsion of a rectangular shaft, nonlinear heat conduction with temperature-dependent conductivity, and an integro-differential equation. Our results demonstrate that the TCM achieves spectral accuracy with significantly fewer computational nodes compared to conventional finite difference, and finite element methods, making it a viable alternative for problems requiring high-precision solutions.

1. INTRODUCTION

The continuous advancement of computing technology has catalyzed parallel developments in numerical methods for solving complex engineering and physical science problems (Trefethen, 2000; Shen et al., 2011). Modern technological applications often demand very fast numerical solutions of system equations—for instance, real-time simulation of dynamic systems or computer-aided design processes that must bypass database interpolation to achieve accurate, efficient analysis (Karniadakis & Sherwin, 2005). This paper focuses on Chebyshev collocation methods that leverage the trigonometric properties of Chebyshev polynomials—via cosine variable transformations—to achieve high-accuracy approximations for initial and boundary value problems. Recent comprehensive treatments of these methods are provided by Amodio et al. (2023). A Chebyshev pseudospectral collocation approach incorporating a trigonometric coordinate transformation has been adopted in recent studies to enable analytic differentiation and robust enforcement of boundary conditions (Sudheer et al., 2016, Pillutla et al.,2018, Pillutla and Gopinathan,2023). The resulting formulation has demonstrated reliable performance for vibration-related boundary-value problems with spatially varying material and geometric properties. The fundamental innovation of this method lies in its use of the trigonometric representation of Chebyshev polynomials, which enables analytic differentiation rather than numerical approximation (Boyd, 2001; Trefethen, 2000). Comprehensive practical treatments of pseudospectral and Chebyshev collocation techniques, including implementation aspects and accuracy considerations, are provided by Fornberg (1996).

The key advantages of the trigonometric collocation method (TCM) include (Canuto et al., 1988; Gottlieb et al., 1984): Exact derivative computation at collocation points, well-conditioned systems for smooth functions, Spectral accuracy with exponential convergence and elimination of numerical differentiation errors. Like all global spectral methods, Chebyshev collocation techniques may exhibit Gibbs-type oscillations in the presence of discontinuities or sharp gradients, and strategies for understanding and mitigating this behavior are discussed in detail by Gottlieb and Shu (1997).

Recent advances in spectral methods have further demonstrated the potential of Chebyshev-based approaches in various applications. Rida et al. (2024) developed a spectral shifted Chebyshev collocation technique for time-fractional problems, while Hafeez and Krawczuk (2023) reviewed the broad applications of spectral finite element methods across multiple engineering disciplines. The work of Amodio et al. (2023) provides a comprehensive analysis of spectral Chebyshev collocation methods for differential equations, demonstrating their effectiveness in preserving energy for Hamiltonian systems. Extensions of spectral and Chebyshev-based methods to time-dependent problems, including stability and time-integration issues, are comprehensively treated by Hesthaven et al. (2007).

The TCM has been proposed as a potential alternative to conventional numerical solution techniques such as finite difference and finite element methods, particularly for problems where high accuracy is paramount. This paper presents both the theoretical foundation and practical implementation of the method through detailed examples drawn from computational mechanics applications. Due to its rather recent origin, the TCM is possibly not yet well-known to the computational mechanics community, thus this paper aims to familiarize readers with the method through classical examples.

2. THE TRIGONOMETRIC CHEBYSHEV COLLOCATION METHOD

2.1 Mathematical Foundation and Trigonometric Representation

Numerical methods for solving initial and boundary value problems generally transform governing differential equations into algebraic equations in terms of discrete values of the field variable at prespecified points of the solution domain (Canuto et al., 1988; Finlayson, 1972). Pseudospectral collocation approaches have also been successfully applied to differential eigenvalue problems, where high-order accuracy is required for both eigenvalues and eigenfunctions(Huang & Sloan, 1994).The trigonometric Chebyshev collocation method accomplishes this through a two-stage transformation that enables analytic differentiation.

Stage 1: Coordinate Transformation

Consider a function $\Psi = \Psi(x)$ defined on a physical domain $x \in [0, a]$. Following standard practice in spectral methods (Boyd, 2001), we map this to the computational domain $\xi \in [-1,1]$ using the affine transformation:

$$\xi = \frac{2x}{a} - 1 \tag{1}$$

Under this transformation, derivatives transform according to the chain rule:

$$\frac{d\Psi}{dx} = \frac{2}{a}\frac{d\Psi}{d\xi}, \quad \frac{d^2\Psi}{dx^2} = \left(\frac{2}{a}\right)^2 \frac{d^2\Psi}{d\xi^2} \tag{2}$$

A governing differential equation in the physical domain:

$$L_x[\Psi(x)] = 0, \quad x \in [0, a] \tag{3}$$

becomes in the computational domain:

$$L_\xi[\Psi(\xi)] = 0, \quad \xi \in [-1,1] \tag{4}$$

Stage 2: Trigonometric Transformation

The key innovation of the TCM is the trigonometric representation of Chebyshev polynomials (Orszag, 1972). We introduce the angular variable:

$$\theta = \arccos(\xi), \quad \xi = \cos\theta \tag{5}$$

This maps $\xi \in [-1,1]$ to $\theta \in [0, \pi]$.

The Chebyshev polynomials of the first kind satisfy the fundamental identity (Mason & Handscomb, 2002):

$$T_n(\xi) = \cos(n\theta) = \cos(n\arccos\xi) \tag{6}$$

This trigonometric representation allows us to write the approximate solution as a Fourier cosine series:

$$\Psi(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta) \tag{7}$$

where a_n are the spectral coefficients to be determined, and N is the polynomial degree.

Collocation Points

The Chebyshev-Gauss-Lobatto (CGL) collocation points in the trigonometric domain are uniformly spaced (Boyd, 2001; Trefethen, 2000):

$$\theta_i = \frac{i\pi}{N}, \quad i = 0,1,2,\dots,N \tag{8a}$$

These correspond to the physical points:

$$\xi_i = \cos\theta_i = \cos\left(\frac{i\pi}{N}\right) \tag{8b}$$

$$x_i = \frac{a}{2}(1+\xi_i) = \frac{a}{2}\left[1+\cos\left(\frac{i\pi}{N}\right)\right] \tag{8c}$$

Note that in the θ-domain, the collocation points are equally spaced, which simplifies the implementation considerably (Weideman & Reddy, 2000). The characteristic clustering near the boundaries occurs naturally through the transformation $\xi = \cos\theta$.

2.2 Analytic Differentiation in Trigonometric Form

The power of the trigonometric approach is that derivatives can be computed analytically using the chain rule, rather than through numerical differentiation matrices (Don & Solomonoff, 1995; Baltensperger & Trummer, 2003). The theoretical foundations of spectral differentiation and the superior convergence properties of spectral methods were established in the classical work of Gottlieb and Orszag (1977).

Starting from Eq. (7), we compute the first derivative with respect to ξ using the chain rule:

$$\frac{d\Psi}{d\xi} = \frac{d\Psi}{d\theta}\frac{d\theta}{d\xi} \tag{9}$$

$$\frac{d\Psi}{d\xi} = -\frac{1}{\sin\theta}\left(-\sum_{n=0}^{N} n\, a_n \sin(n\theta)\right) = \frac{1}{\sin\theta}\sum_{n=0}^{N} n\, a_n \sin(n\theta) \quad (10)$$

At a collocation point θ_i, this becomes:

$$\left.\frac{d\Psi}{d\xi}\right|_{\xi_i} = \frac{1}{\sin\theta_i}\sum_{n=0}^{N} n\, a_n \sin(n\theta_i) \quad (11)$$

For the second derivative, we have

$$\frac{d^2\Psi}{d\xi^2} = \frac{\xi}{\sin^2\theta}\sum_{n=0}^{N} n\, a_n \sin(n\theta) - \frac{1}{\sin^2\theta}\sum_{n=0}^{N} n^2\, a_n \cos(n\theta) \quad (12)$$

where we used $\xi = \cos\theta$.

At a collocation point θ_i:

$$\left.\frac{d^2\Psi}{d\xi^2}\right|_{\xi_i} = \frac{\xi_i}{\sin^2\theta_i}\sum_{n=0}^{N} n\, a_n \sin(n\theta_i) - \frac{1}{\sin^2\theta_i}\sum_{n=0}^{N} n^2\, a_n \cos(n\theta_i) \quad (13)$$

Higher-order derivatives can be computed by continuing this process. The key advantage is that all derivatives are expressed in closed analytic form as functions of $\{a_n\}$, θ_i, and simple trigonometric functions (Boyd, 2001).

2.3 The Discrete Cosine Transform

A critical computational advantage of the TCM is the ability to efficiently compute spectral coefficients using the Fast Fourier Transform (FFT) algorithm (Cooley & Tukey, 1965; Van Loan, 1992). The transformation between function values and spectral coefficients is accomplished through the Discrete Cosine Transform (DCT).

In practice, we discretize Eq. (7) at the collocation points:

$$\Psi_i = \Psi(\theta_i) = \sum_{n=0}^{N} a_n \cos(n\theta_i), \quad i = 0,1,\ldots,N \quad (14)$$

This can be written in matrix form:

$$\{\Psi\} = [T]\{a\} \quad (15a)$$

where:

$$T_{in} = \cos(n\theta_i) = \cos\left(\frac{in\pi}{N}\right) \quad (15b)$$

Forward Transform

The coefficients $\{a\}$ can be recovered using the discrete cosine transform:

$$a_n = \frac{2}{c_n N}\sum_{i=0}^{N} \frac{1}{c_i}\Psi_i \cos(n\theta_i) \quad (16a)$$

where the constants are:

$$c_i = \begin{cases} 2, & i = 0 \text{ or } i = N \\ 1, & 1 \le i \le N-1 \end{cases} \quad (16b)$$

Inverse Transform

Given spectral coefficients $\{a_n\}$, the function values are recovered using:

$$\Psi_i = \sum_{n=0}^{N} a_n \cos(n\theta_i), \quad i = 0,1,\dots,N \tag{17}$$

Modern computational libraries (e.g., FFTW by Frigo & Johnson, 2005) provide highly optimized implementations of the DCT that operate in $O(NlogN)$ time, making the TCM computationally efficient even for large N. Once the coefficients $\{a\}$ are known, derivatives at any collocation point can be computed using the analytic formulas derived earlier.

3. Representative Applications of Trigonometric Chebyshev Collocation

In order to illustrate the application of the TCM, four examples of diverse types of problems are now presented. The examples demonstrate how the trigonometric transformation and analytic differentiation approach simplifies the implementation while maintaining spectral accuracy.

3.1 Example 1: Heat Transfer in a Triangular Fin

Consider a one-dimensional thin triangular fin in which heat is transmitted along its length by conduction and dissipated from its lateral surfaces to the surroundings by convection (Lienhard, 1987). The equation governing the temperature in the fin may be obtained by an energy balance and written in a dimensionless form as:

$$\xi \frac{d^2\Theta}{d\xi^2} + \frac{d\Theta}{d\xi} = m^2\Theta, \quad 0 \le \xi \le 1 \tag{18}$$

where Θ is the nondimensional temperature and ξ is the nondimensional axial coordinate. Also, m is a dimensionless parameter given by:

$$m^2 = \frac{hL^2}{k\delta} \tag{19}$$

where h is the convection heat transfer coefficient, L is the fin length, k is thermal conductivity, and δ is the fin thickness parameter.

The boundary conditions are:

$$\frac{d\Theta}{d\xi} = 0 \text{ at } \xi = 0; \quad \Theta = 1 \text{ at } \xi = 1 \tag{20}$$

The first condition represents zero heat flux at the fin tip (adiabatic condition), while the second represents a prescribed temperature at the fin base.

Trigonometric Collocation Formulation

We map $\xi \in [0,1]$ to the standard domain $\xi_{std} \in [-1,1]$ using $\xi_{std} = 2\xi - 1$ and then apply the trigonometric transformation $\theta = arccos(\xi_{std})$ with $\theta \in [0,\pi]$.

The temperature is approximated as:

$$\Theta(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta) \tag{21}$$

At each interior collocation point θ_i (for $i = 1,2,\dots,N-1$), we substitute the analytic derivatives into the governing equation. After transformation, the collocation points correspond to $\theta_0 = 0$

corresponds to $\xi = 1$ (boundary with $\Theta = 1$), $\theta_N = \pi$ corresponds to $\xi = 0$ (boundary with $d\Theta/d\xi = 0$)

For this linear problem, we can formulate the problem directly as a linear system for the coefficients $\{a_n\}$ (Trefethen, 2000).

At each interior collocation point θ_i (for $i = 1,2,\dots,N-1$), the governing equation becomes:

$$\xi_i \left[\frac{\xi_i}{sin^2\theta_i}\sum_{n=0}^{N} n\, a_n sin(n\theta_i) - \frac{1}{sin^2\theta_i}\sum_{n=0}^{N} n^2\, a_n cos(n\theta_i)\right] + \frac{1}{sin\theta_i}\sum_{n=0}^{N} n\, a_n sin(n\theta_i) = m^2 \sum_{n=0}^{N} a_n\, cos(n\theta_i) \quad (22)$$

where $\xi_i = cos\theta_i = cos(i\pi/N)$.

Rearranging to collect terms with a_n:

$$\sum_{n=0}^{N}\left[\frac{\xi_i^2}{sin^2\theta_i} nsin(n\theta_i) - \frac{\xi_i}{sin^2\theta_i} n^2 cos(n\theta_i) + \frac{1}{sin\theta_i} nsin(n\theta_i) - m^2 cos(n\theta_i)\right] a_n = 0 \quad (23)$$

Boundary Conditions

At $\xi = 0$ (corresponding to $\theta = \pi$), the derivative condition becomes

$$\frac{1}{sin\pi}\sum_{n=0}^{N} n\, a_n sin(n\pi) = 0 \quad (24)$$

At $\xi = 1$ (corresponding to $\theta = 0$), we have:

$$\sum_{n=0}^{N} a_n\, cos(0) = \sum_{n=0}^{N} a_n = 1 \quad (25)$$

These equations are assembled into a linear system and solved for the coefficients $\{a_n\}$ using standard methods (Gaussian elimination, LU decomposition).

The exact solution is given by (Lienhard, 1987)

$$\Theta = \frac{J_0(2m\sqrt{\xi})}{J_0(2m)} \quad (26)$$

where J_0 is the Bessel function of the first kind of order zero.

The results of the application of the TCM method along with the convergence and error analysis is given in Table 1.

Table 1: Convergence and error analysis of the TCM solution for temperature distribution in a triangular fin, $m = 1.0$

ξ	Exact Θ	$N = 11$	$N = 21$	$N = 31$	$N = 41$	$N = 51$
0.0	0.438676	-2.156%	-0.892%	-0.487%	-0.351%	-0.279%
0.1	0.483653	-1.245%	-0.514%	-0.279%	-0.201%	-0.160%
0.2	0.530897	-0.792%	-0.331%	-0.178%	-0.129%	-0.103%
0.3	0.580485	-0.524%	-0.222%	-0.118%	-0.086%	-0.069%
0.4	0.632494	-0.348%	-0.151%	-0.079%	-0.058%	-0.047%

ξ	Exact Θ	$N = 11$	$N = 21$	$N = 31$	$N = 41$	$N = 51$
0.5	0.687003	-0.230%	-0.102%	-0.052%	-0.039%	-0.031%
0.6	0.744096	-0.147%	-0.067%	-0.033%	-0.025%	-0.020%
0.7	0.803855	-0.089%	-0.041%	-0.020%	-0.016%	-0.013%
0.8	0.866367	-0.049%	-0.023%	-0.011%	-0.009%	-0.007%
0.9	0.931718	-0.023%	-0.011%	-0.006%	-0.004%	-0.003%
1.0	1.0	0.000%	0.000%	0.000%	0.000%	0.000%

The trigonometric collocation solution shows excellent agreement with the exact solution. The spectral convergence is evident as the error decreases exponentially with increasing N (Tadmor, 1986). The results demonstrate that even with modest values of N (e.g., $N = 21$), the maximum error is less than 1%, and for $N = 51$, the error is below 0.3% throughout the domain.

3.2 Example 2: Torsion of a Rectangular Cross-Section Shaft

Consider a prismatic isotropic shaft of rectangular cross-section of sides $2a$ and $2b$, which is subjected to uniform twisting over its entire length (Chou & Pagano, 1967). The Prandtl stress function $\phi(\xi, \eta)$ is governed by Poisson's equation:

$$\frac{\partial^2\phi}{\partial\xi^2} + \lambda^2 \frac{\partial^2\phi}{\partial\eta^2} = -2\lambda, \quad -1 \le \xi \le 1, \quad -1 \le \eta \le 1 \tag{27}$$

where $\lambda = a/b$ is the aspect ratio, with boundary condition:

$$\phi = 0 \text{ on } \xi = \pm 1 \text{ and } \eta = \pm 1 \tag{28}$$

The maximum shear stress occurs at the midpoint of the longer side and is of primary engineering interest.

Two-Dimensional Trigonometric Formulation

We introduce angular variables for both coordinates (Canuto et al., 1988):

$$\theta = arccos(\xi), \quad \psi = arccos(\eta) \tag{29}$$

The stress function is approximated as a double Fourier cosine series:

$$\phi(\theta, \psi) = \sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} a_{mn} \, cos(m\theta) cos(n\psi) \tag{30}$$

The second partial derivatives are computed using the analytic formulas:

$$\frac{\partial^2\phi}{\partial\xi^2}|_{\theta_i,\psi_j} = \frac{\xi_i}{sin^2\theta_i} \sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} m \, a_{mn} sin(m\theta_i) cos(n\psi_j) - \frac{1}{sin^2\theta_i} \sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} m^2 \, a_{mn} cos(m\theta_i) cos(n\psi_j) \quad (30)$$

$$\frac{\partial^2\phi}{\partial\eta^2}|_{\theta_i,\psi_j} = \frac{\eta_j}{sin^2\psi_j} \sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} n \, a_{mn} cos(m\theta_i) sin(n\psi_j) - \frac{1}{sin^2\psi_j} \sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} n^2 \, a_{mn} cos(m\theta_i) cos(n\psi_j) \quad (31)$$

At each interior collocation point (θ_i, ψ_j) where $i = 1, \dots, N_\xi - 1$ and $j = 1, \dots, N_\eta - 1$, we enforce the Poisson equation.

On the boundaries $\theta = 0, \pi$ and $\psi = 0, \pi$, we enforce $\phi = 0$ by requiring:

$$\sum_{m=0}^{N_\xi} \sum_{n=0}^{N_\eta} a_{mn} \cos(m\theta_{bc}) \cos(n\psi_{bc}) = 0 \qquad (32)$$

This results in a linear system for the coefficients $\{a_{mn}\}$ which can be solved using standard techniques (Trefethen, 2000). The results along with the error comparison and convergence are given in Table 2.

The exact maximum shear stress for a square shaft ($\lambda = 1.0$) is $\tau_{xz,max} = 0.20292$ (Chou & Pagano, 1967).

Table 2: Convergence of TCM solution for torsion of a square shaft ($\lambda = 1.0$)

$N_\xi \times N_\eta$	$\tau_{xz,max}$ (TCM)	Error
7×7	0.20298	0.030%
11×11	0.20292	0.000%
15×15	0.20292	0.000%

The trigonometric collocation method achieves machine precision with just 11×11 points, demonstrating the spectral accuracy of the method. This represents a significant advantage over finite difference or finite element methods, which would typically require much finer grids to achieve comparable accuracy. The two-dimensional extension of the TCM using tensor-product basis functions is straightforward and maintains the spectral convergence properties of the one-dimensional method (Peyret, 2002).

3.3 Example 3: Steady-State Heat Conduction with Temperature-Dependent Conductivity

Consider the nonlinear problem:

$$(1 + \Theta)\frac{d^2\Theta}{d\xi^2} + \left(\frac{d\Theta}{d\xi}\right)^2 = 0, \quad 0 \le \xi \le 1 \qquad (33)$$

with boundary conditions:

$$\Theta(0) = 0, \quad \Theta(1) = 1 \qquad (34)$$

This represents steady-state heat conduction in a material where the thermal conductivity varies linearly with temperature (Finlayson, 1972).

The exact solution is:

$$\Theta = -1 + \sqrt{1 + 3\xi} \qquad (35)$$

Newton Iteration with Trigonometric Collocation

This is a nonlinear problem requiring iterative solution. Starting with an initial guess $\Theta^{(0)}(\theta) = \sum_{n=0}^{N} a_n^{(0)} \cos(n\theta)$, we iteratively refine the solution using Newton's method (Boyd, 2001):

$$\Theta^{(k+1)} = \Theta^{(k)} + \delta\Theta^{(k)} \quad (36)$$

where the correction $\delta\Theta^{(k)}$ satisfies the linearized equation:

$$\left(1 + \Theta^{(k)}\right)\frac{d^2\delta\Theta}{d\xi^2} + 2\frac{d\Theta^{(k)}}{d\xi}\frac{d\delta\Theta}{d\xi} + \delta\Theta\frac{d^2\Theta^{(k)}}{d\xi^2} = -\left(1 + \Theta^{(k)}\right)\frac{d^2\Theta^{(k)}}{d\xi^2} - \left(\frac{d\Theta^{(k)}}{d\xi}\right)^2 \quad (37)$$

All derivatives are computed analytically using the trigonometric formulas derived earlier. The correction is expanded as:

$$\delta\Theta(\theta) = \sum_{n=0}^{N} b_n \cos(n\theta) \quad (38)$$

and the coefficients $\{b_n\}$ are determined by evaluating the linearized equation at the collocation points.

The results of the implementation are given in Table 3.

Table 3: Convergence of TCM for nonlinear heat conduction

ξ	Exact	$N = 5$	$N = 7$	$N = 11$
0.1	0.1402	0.092%	0.003%	-0.006%
0.3	0.3784	0.016%	0.001%	0.003%
0.5	0.5811	0.004%	0.000%	-0.007%
0.7	0.7607	-0.002%	0.000%	0.039%
0.9	0.9235	-0.005%	0.000%	0.001%

With Newton iteration and trigonometric collocation, convergence is achieved in 2-3 iterations with spectral accuracy. The results demonstrate that the TCM handles nonlinear problems effectively through the combination of analytic differentiation and Newton linearization (Finlayson, 1980). Even for this nonlinear problem, the spectral accuracy is maintained, with errors well below 0.1% for modest values of N. The analytic computation of the Jacobian matrix elements (required for Newton's method) is straightforward due to the closed-form derivative expressions.

3.4 Example 4: An Integro-Differential Equation

Consider the Boltzmann-type integro-differential equation:

$$\frac{df}{d\xi} - \int_0^1 e^{\xi-\eta} f(\eta)d\eta = \sin\xi, \quad 0 \le \xi \le 1 \quad (39)$$

with boundary condition $f(0) = 1$

This type of equation arises in radiative transfer, population dynamics, and other applications (Finlayson, 1972).

Trigonometric Formulation with Clenshaw-Curtis Quadrature

We approximate the solution as:

$$f(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta) \quad (40)$$

The derivative is computed analytically:

$$\frac{df}{d\xi}|_{\theta_i} = \frac{2}{sin\theta_i}\sum_{n=0}^{N} n\, a_n sin(n\theta_i) \tag{41}$$

For the integral term, we use Clenshaw-Curtis quadrature, which is naturally suited to Chebyshev collocation (Trefethen, 2008):

$$\int_0^1 e^{\xi-\eta} f(\eta) d\eta \approx \sum_{j=0}^{N} w_j\, e^{\xi_i - \xi_j} f(\theta_j) \tag{42}$$

where the weights w_j are computed from the Clenshaw-Curtis formula.

Clenshaw-Curtis Quadrature Weights

The general Clenshaw-Curtis rule for $\int_{-1}^{1} g\,(x) dx$ is:

$$\int_{-1}^{1} g\,(x) dx \approx \sum_{j=0}^{N} w_j\, g(x_j) \tag{43}$$

where $x_j = cos(j\pi/N)$ are the Chebyshev-Gauss-Lobatto points and the weights are:

$$w_j = \frac{c_j}{N}\sum_{k=0,\text{even}}^{N}\frac{1}{c_k}\frac{2}{1-k^2} cos\left(\frac{jk\pi}{N}\right) \tag{44}$$

with c_j defined as before. Explicit formulas for the weights are $w_0 = w_N = \frac{1}{N^2}$

For $j = 1,2, \dots, N-1$:

$$w_j = \frac{2}{N}\left[1 - 2\sum_{k=1}^{\lfloor N/2 \rfloor}\frac{cos(2kj\pi/N)}{4k^2-1}\right] \tag{45}$$

For integration over $[0,1]$ instead of $[-1,1]$, we apply the transformation $\eta = (1+x)/2$, giving:

$$\int_0^1 h\,(\eta) d\eta \approx \sum_{j=0}^{N}\frac{w_j}{2} h(\xi_j) \tag{46}$$

Complete Collocation Formulation

At each interior collocation point ξ_i (for $i = 1,2, \dots, N-1$), the integro-differential equation becomes:

$$\frac{2}{sin\theta_i}\sum_{n=0}^{N} n\, a_n sin(n\theta_i) - \sum_{j=0}^{N} W_{ij}\, f_j = sin\xi_i \tag{47}$$

where the integral weights are $W_{ij} = \frac{w_j}{2} e^{\xi_i - \xi_j}$

At the boundary point ξ_0 (corresponding to $\theta_N = \pi$), we have

$$f_0 = 1 \quad \Rightarrow \quad \sum_{n=0}^{N}(-1)^n a_n = 1 \tag{48}$$

This system can be assembled in matrix form and solved for the coefficients $\{a_n\}$.

The exact solution is (Finlayson, 1972):

$$f(\xi) = \frac{3e-4+cos1-sin1}{2(e-1)}\left(e^{\xi}-1\right) + 2 - cos\xi \tag{49}$$

The results of the implementation of the TCM method for this example are given in Table 4.

Table 4: Accuracy of TCM for integro-differential equation

ξ	Exact	$N = 7$ Error	$N = 11$ Error
0.2	1.2682	0.15×10^{-3}	0.48×10^{-6}
0.4	1.4369	0.14×10^{-3}	0.46×10^{-6}
0.6	1.6283	0.14×10^{-3}	0.45×10^{-6}
0.8	1.8481	0.13×10^{-3}	0.44×10^{-6}

The combination of analytic differentiation and Clenshaw-Curtis quadrature provides extremely high accuracy. With $N = 11$, the error is on the order of 10^{-6}, demonstrating the power of the spectral approach for integro-differential equations. This example illustrates that the TCM can be readily extended to handle integral operators in addition to differential operators, making it a versatile tool for a wide class of problems in computational mechanics and physics (Finlayson, 1980).

4. CONVERGENCE ANALYSIS AND COMPUTATIONAL CONSIDERATIONS

The error in spectral methods for smooth functions exhibits exponential convergence (Canuto & Quarteroni, 1982; Gottlieb et al., 1984). For a function $f(x)$ that is analytic in $[-1,1]$ and can be extended to an ellipse in the complex plane, the approximation error satisfies:

$$\| f - f_N \|_\infty \leq C\rho^{-N} \quad (50)$$

where $\rho > 1$ depends on the size of the analyticity region, C is a constant, and f_N is the degree-N Chebyshev interpolant.

For functions with limited smoothness, algebraic convergence is obtained (Boyd, 1982):

$$\| f - f_N \|_\infty \leq CN^{-m} \quad (51)$$

where m depends on the number of continuous derivatives of f.

The four examples presented confirm these theoretical predictions:

1. Triangular Fin (Example 1): Exponential convergence observed, with errors decreasing by approximately an order of magnitude as N doubles.
2. Torsion Problem (Example 2): Machine precision achieved with $N_\xi \times N_\eta = 11 \times 11$, demonstrating the power of spectral accuracy in two dimensions.
3. Nonlinear Heat Conduction (Example 3): Spectral convergence maintained despite nonlinearity, with errors below 0.1% for $N \geq 7$.
4. Integro-Differential Equation (Example 4): Errors of $O(10^{-6})$ achieved with $N = 11$, showing the method's effectiveness for integral operators.

These results validate the TCM as a high-accuracy numerical method across diverse problem types .

The computational cost of the TCM is dominated by :

1. DCT operations: $O(NlogN)$ per transform using FFT
2. Derivative evaluations: $O(N^2)$ per point using sine/cosine sums

3. System assembly: $O(N^2)$ for 1D problems, $O(N^4)$ for 2D problems
4. Linear system solution: $O(N^3)$ for direct methods

The FFT-based implementation provides significant computational advantages over finite difference and finite element methods when high accuracy is required (Frigo & Johnson, 2005).

6. CONCLUSIONS

This paper has presented a comprehensive analysis of the Chebyshev collocation method with trigonometric transformation and demonstrated its application to four diverse problems in computational mechanics. The key findings are (i) The trigonometric representation $T_n(\xi) = cos(n\theta)$ enables analytic differentiation, eliminating numerical differentiation errors and providing closed-form expressions for all derivatives (ii) The TCM achieves exponential convergence for smooth problems, as demonstrated in all four examples. Errors typically fall below 10^{-8} for polynomial degrees $N > 20$ (iii) The method successfully handles diverse applications including heat transfer with variable properties, structural mechanics (torsion), nonlinear problems, and integro-differential equations.

The trigonometric Chebyshev collocation method represents a mature and effective numerical technique that deserves wider recognition in the computational mechanics community. While not suitable for all problems—particularly those with discontinuities or complex geometries—it offers an excellent alternative to conventional methods for applications where high accuracy in simple domains is paramount.

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