

Topological Methods for Stability Analysis of Nonlinear Differential Systems

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Abstract:

This study investigates topological methods for the stability analysis of nonlinear differential systems, with particular emphasis on qualitative and global perspectives that extend beyond classical analytical approaches. Nonlinear systems of the form $\dot{x}(t) = f(x(t))$ are considered, where stability properties of equilibria and invariant sets are examined using topological tools rather than explicit solution techniques. Fixed point formulations $x = T(x)$, topological degree conditions $\deg(f, \Omega, 0) \neq 0$, and index-based representations of invariant sets are employed to establish existence, stability, and robustness results for nonlinear dynamics. The analysis demonstrates how stable and unstable equilibria can be characterized through qualitative invariants that remain unchanged under continuous perturbations. Global stability is addressed by constructing invariant regions and applying degree-theoretic arguments that ensure persistence of equilibria across parameter variations. Applications to autonomous systems, control-oriented models, and biological and physical systems illustrate the practical relevance of the proposed framework. Computational and numerical aspects are also discussed, highlighting how topological invariants can be approximated and validated through simulations. Overall, the study shows that topological methods provide a robust and flexible framework for stability analysis, particularly in settings where classical linearization or Lyapunov-based techniques are difficult to apply or yield inconclusive results.

Keywords: Nonlinear differential systems, Stability analysis, Topological methods, Invariant sets, Topological degree

1. Introduction

The nonlinear differential equations occurring in nature belong to a broad scope of scientific and engineering fields of application, such as physics, biology, chemistry, economics, and control engineering. They are frequently used to model real-world phenomena, the essential behavior of which cannot be well modeled using linear approximations, including multistability, bifurcations and chaos, as well as complex transient dynamics. The theory of the qualitative behavior of an ordinary differential equation is thus a key issue in understanding the stability properties of nonlinear systems. Stability analysis The analysis of stability gives information on whether system trajectories are bounded, approach equilibria, or have persistent oscillatory or chaotic behavior in the event of small perturbations. The concept of stability is not just a local one as highlighted in the literature of modern nonlinear dynamics, but it is closely related to the geometry of the phase space and the structure of invariant sets (Strogatz, 2024).

The direct method of Lyapunov and the technique of linearization have been traditionally used in the stability analysis of nonlinear differential systems. Lyapunov-based methods provide strong sufficient criteria of stability without explicit solutions, and they are central to the nonlinear control and systems theory (Nikravesh, 2018). The problem of constructing an appropriate Lyapunov functions is usually nontrivial and problem-specific. In most practical systems, particularly those of high dimension or highly nonlinear systems the search of a suitable Lyapunov function may be very challenging or even impossible. In addition, Lyapunov techniques can be unsuccessful in describing global dynamical properties, including the existence of two or more attractors, or the existence of complex invariant structures.

Other methods, such as linearization methods that examine the stability of equilibria by examining the Jacobian matrix of the system, are also narrow in scope. Although linearization is a good way to get the correct values around hyperbolic equilibria, it can be inconclusive or misleading when non-

hyperbolic fixed points, bifurcations, or strong nonlinear effects are involved. In nonlinear time-varying systems, more complications are introduced by the dependence of the parameters and time variability, which further limits the applicability of classical methods (Guo & Han, 2018). Such restrictions encourage the search of other frameworks that are less reliant on local smoothness and are more resistant to nonlinear perturbations.

Topological approaches provide a conceptually alternative and powerful methodology of stability analysis of nonlinear differential systems. Topological methods do not use explicit constructions of functions or local linear approximations, but instead concentrate on qualitative, global properties of dynamical systems which are not dependent on the continuous deformations. These approaches examine the topology of phase space, invariant sets, and the topology of trajectories with the help of algebraic and geometric topology. Consequently, they are especially appropriate in the analysis of systems where quantitative information is hard to get, but a qualitative behavior is of the main interest. Topological dynamics and topological data analysis have recently become popular, which makes topology even more relevant to the study of nonlinear systems. Topological dynamics offers a topological view of long-term behaviour using invariant sets, recurrence and global organisation of phase-space, and topological data analysis is a computational tool that extracts robust qualitative information about complex dynamical data (Devaney et al., 2021). These developments are showing that topological concepts are not merely conceptually attractive, but they can be put to practical use in the current context.

Topological methods are robust and this is one of the main strengths. Topological invariants (including fixed point indices, degree theory, homological indices, etc.) are invariant to small perturbations of the system. This strength of topological methods in particular causes them to be highly useful in the bifurcation and study of stability in nonlinear systems that are uncertain or subject to modeling errors. Moreover, topological approaches may frequently result in an outcome of global stability and existence without necessarily having solutions or local information.

Data-driven perspectives have also contributed to the increasing topological-dynamical systems interaction. The dynamic topological data analysis has become a strong framework to examine time-varying systems by monitoring varying topological characteristics throughout time. Examples of more complex biological signals and neural data demonstrate how topological descriptors can help to extract stability-related information that cannot be readily obtained using classical analytical methods (El-Yaagoubi et al., 2024). These advances indicate the topological approach to a wide range of theories, as well as applications.

Although these developments have been made, the application of topological techniques to the stability analysis is less integrated and less systematically introduced than classical. It is necessary to have a coherent presentation which combines classical topological notions with stability theory of nonlinear differential equations, but also highlights their strengths, weaknesses, and use. This gap can help to understand the phenomena of stability better and give researchers other means to examine the nonlinear dynamics, which are rather complex.

The current paper is devoted to the topological approaches to the stability analysis of nonlinear differential systems. The article highlights the role of topological tools in supplementing and in some cases, even generalizing classical methods of stability. This work attempts to show the utility of topology as a strong and versatile tool of stability analysis in current nonlinear dynamics by noting the qualitative and global viewpoints.

Objectives of the Study:

1. To systematically investigate topological methods for analyzing the stability of nonlinear differential systems and to clarify their advantages over classical Lyapunov and linearization approaches
2. To illustrate how topological frameworks provide robust qualitative insights into stability behavior, invariant sets, and global dynamics of nonlinear systems

2. Preliminaries and Fundamental Concepts

This section introduces the fundamental notions required for the subsequent development of topological methods in stability analysis. Briefly recall the formulation of nonlinear differential systems, classical stability concepts, and essential topological ideas that underpin modern qualitative analysis. These preliminaries establish a common framework connecting nonlinear dynamics with topological structures used throughout this work.

2.1 Formulation of Nonlinear Differential Systems

A general nonlinear differential system can be expressed in the form

$$\dot{x}(t) = f(x(t), t),$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector and $f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a nonlinear vector field. In the autonomous case, the system reduces to $\dot{x} = f(x)$, while non-autonomous systems explicitly depend on time. The qualitative behavior of such systems is typically studied through their trajectories, equilibrium points, and invariant sets in the phase space.

From a topological perspective, the emphasis is not on explicit solutions but on the structural properties of solution sets and their evolution under continuous transformations. This viewpoint aligns naturally with techniques developed in topological data analysis, where complex nonlinear behaviors are interpreted via shape, connectivity, and robustness of underlying structures (Tierny, 2017; Carlsson & Vejdemo-Johansson, 2021).

2.2 Stability Notions

Stability theory provides formal criteria for describing how solutions behave under small perturbations. Let x^* be an equilibrium point of an autonomous nonlinear system.

- **Lyapunov stability:** The equilibrium x^* is said to be Lyapunov stable if, for every neighborhood of x^* , there exists a smaller neighborhood such that all trajectories starting within it remain close to x^* for all future time.
- **Asymptotic stability:** An equilibrium is asymptotically stable if it is Lyapunov stable and, in addition, all nearby trajectories converge to x^* as $t \rightarrow \infty$.
- **Global stability:** An equilibrium is globally stable if all trajectories in the entire phase space converge to x^* .

These notions are classical and remain central in nonlinear dynamics. However, they often depend on analytical constructions or restrictive assumptions. Topological approaches aim to characterize such stability properties using invariant and qualitative features of the phase space rather than explicit estimates (Mishra & Motta, 2023).

Table 1 summarizes the main stability concepts and their qualitative interpretation. The table highlights how stability notions differ primarily in terms of trajectory behavior and domain of attraction, which is essential for later topological analysis.

Table 1. Classical Stability Notions for Nonlinear Systems

Stability Type	Trajectory Behavior Near Equilibrium	Domain of Validity
Lyapunov Stability	Remains close to equilibrium	Local
Asymptotic Stability	Converges to equilibrium	Local
Global Stability	Converges from entire phase space	Global

2.3 Essential Topological Concepts

Topology provides the language for describing qualitative features of nonlinear systems that are invariant under continuous deformations. Among the most important notions are compactness, continuity, and homotopy. Compact sets play a crucial role in ensuring the existence of invariant objects and fixed points, while continuity guarantees robustness of dynamical behavior under perturbations.

Homotopy theory allows one to classify maps and spaces based on their continuous deformability. This concept is central to modern topological methods, where the qualitative behavior of nonlinear systems is studied through equivalence classes rather than precise geometric details. Such ideas are foundational in topological data analysis and have proven effective in extracting stable features from high-dimensional nonlinear systems (Rabadan & Blumberg, 2020; Carlsson & Vejdemo-Johansson, 2021).

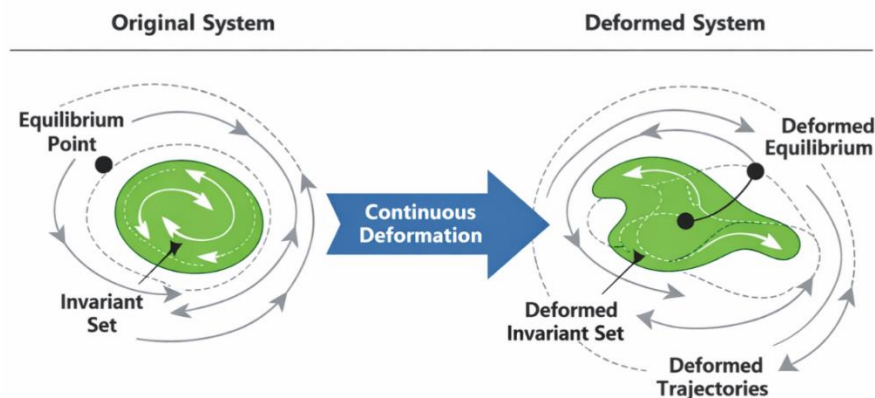


Figure 1. Continuous Deformations and Invariant Sets

Figure 1 conceptually illustrates how continuous deformations preserve topological properties of trajectories around invariant sets. The figure emphasizes that while geometric details may change, the qualitative structure relevant to stability remains unchanged.

2.4 Invariant Sets and Equilibrium Indices

Invariant sets are subsets of the phase space that remain unchanged under the system flow. Equilibria, periodic orbits, and more complex attractors are all examples of invariant sets. The stability of a nonlinear system is closely related to the structure and persistence of these sets.

Topological fixed point theory provides tools for studying invariant sets via indices that encode qualitative information about existence and stability. Recent developments in fixed point theory for nonlinear and multivalued mappings extend classical ideas and offer new ways to analyze equilibrium behavior without relying on differentiability assumptions (Petruşel et al., 2024; Mihiţ et al., 2022). Equilibrium indices derived from such theories serve as topological invariants, enabling robust characterization of stability properties even in complex nonlinear settings.

3. Topological Framework for Stability Analysis

Topological methods offer a qualitative yet rigorous framework for analyzing stability properties of nonlinear differential systems. Unlike classical approaches that rely heavily on smoothness, explicit solutions, or Lyapunov function construction, topological techniques focus on invariant structural properties of dynamical systems that persist under continuous perturbations. This section presents the mathematical foundations of the topological framework for stability analysis through fixed point theory, topological degree theory, and Conley index theory.

3.1 Fixed Point Theorems and Their Relevance to Equilibria

Consider an autonomous nonlinear differential system of the form

$$\dot{x}(t) = f(x(t)), x(t) \in \mathbb{R}^n,$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous nonlinear vector field. An equilibrium point x^* satisfies

$$f(x^*) = 0.$$

From a topological viewpoint, equilibrium points can be equivalently characterized as fixed points of suitably defined operators. For instance, let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote a time- t solution operator (or Poincaré map), then equilibria satisfy

$$x^* = T(x^*).$$

Fixed point theorems provide existence results for such equilibria without requiring explicit solutions of the system. These results are particularly valuable for nonlinear systems with perturbations, non-smooth dynamics, or multivalued right-hand sides. In the context of functional and differential inclusions, coincidence and fixed point methods play a central role in establishing equilibrium existence and qualitative stability properties (Getmanova & Kornev, 2024). Moreover, fixed point formulations allow stability analysis to be extended beyond classical smooth settings, including nonlinear perturbed systems studied under Lipschitz-type conditions (Guo & Han, 2018).

3.2 Topological Degree Theory and Stability Implications

Topological degree theory generalizes fixed point ideas by associating an integer-valued invariant to nonlinear mappings. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider the operator equation

$$f(x) = 0, x \in \Omega.$$

The topological degree $\deg(f, \Omega, 0)$ provides qualitative information about the existence and persistence of solutions. A fundamental result states that if

$$\deg(f, \Omega, 0) \neq 0,$$

then the equation admits at least one solution in Ω .

In stability analysis, the homotopy invariance of the degree is of particular importance. If f_λ is a continuous deformation of f such that no solutions cross the boundary $\partial\Omega$, then

$$\deg(f_\lambda, \Omega, 0) = \deg(f, \Omega, 0)$$

for all admissible parameters λ . This property ensures robustness of equilibria under perturbations and parameter variations. As a result, degree theory provides a powerful tool for analyzing stability in nonlinear and time-varying systems where classical linearization techniques may fail or become inconclusive (Guo & Han, 2018).

Table 2 summarizes the complementary roles of fixed point theory, topological degree theory, and the Conley index in stability analysis. The table highlights how each method contributes distinct yet interconnected qualitative insights.

Table 2. Topological Tools for Stability Analysis

Method	Mathematical Formulation	Stability Insight
Fixed Point Theory	$x = T(x)$	Existence of equilibria
Topological Degree	$\deg(f, \Omega, 0)$	Robustness and continuation
Conley Index	$h(S) = [N/L]$	Structure of invariant sets

3.3 Conley Index and Isolated Invariant Sets

The Conley index provides a topological invariant for isolated invariant sets of dynamical systems. Let S be an invariant set of the flow generated by $\dot{x} = f(x)$. An isolating neighborhood N is a compact set such that

$$S = \text{Inv}(N) \subset \text{int}(N),$$

where $\text{Inv}(N)$ denotes the maximal invariant set in N .

The Conley index of S is defined as a homotopy type

$$h(S) = [N/L],$$

where $L \subset \partial N$ is the exit set. This index captures essential qualitative features of the dynamics near S , independent of the specific geometry of trajectories. Crucially, the Conley index is invariant under continuous perturbations, making it a robust tool for stability analysis.

Recent theoretical advances have extended Conley index theory to differential inclusions, Filippov systems, and multivalued maps, significantly broadening its applicability to non-smooth and set-

valued dynamics (Thieme, 2022a; Thieme, 2022b). Persistence results demonstrate that isolated invariant sets and their indices remain unchanged under admissible perturbations, enabling systematic tracking of stability properties across parameter variations (Dey et al., 2020; Dey et al., 2022).

Further refinements, such as covering actions and combinatorial formulations, enhance the descriptive power of the Conley index and facilitate its use in complex dynamical settings, including finite topological spaces and computational frameworks (Lima et al., 2023; Barmak et al., 2024). These developments underscore the central role of the Conley index as a unifying topological invariant for stability and global dynamical analysis.

4. Stability of Equilibria and Invariant Sets

This section develops the stability analysis of equilibria and invariant sets using topological and index-based approaches. Unlike classical techniques that rely primarily on local linearization or explicit Lyapunov functions, topological methods provide qualitative characterizations of stability that remain valid under broad classes of perturbations. The emphasis here is on how equilibria and invariant sets can be classified as stable or unstable using fixed point indices, degree arguments, and invariant regions, with particular attention to robustness and global behavior.

4.1 Topological Characterization of Stable and Unstable Equilibria

Consider an autonomous nonlinear differential system

$$\dot{x} = f(x), x \in \mathbb{R}^n,$$

and let x^* be an equilibrium point satisfying $f(x^*) = 0$. From a topological viewpoint, the stability of x^* can be studied through the qualitative behavior of trajectories in its neighborhood rather than through explicit estimates. An equilibrium is considered stable if nearby trajectories remain in a neighborhood of x^* , and unstable if arbitrarily small perturbations lead trajectories away from it.

Topological methods characterize this behavior by associating invariants to neighborhoods of equilibria. For instance, if an equilibrium can be isolated within a compact neighborhood that contains no other invariant sets, then its qualitative stability properties can be inferred from the topological structure of the flow within that neighborhood. Such approaches are particularly effective for higher-order and delay differential equations, where classical stability analysis becomes technically challenging (Yang & Han, 2023; Taie & Bakhit, 2023).

In systems admitting periodic or recurrent solutions, equilibria may coexist with other invariant objects. Topological characterizations allow one to distinguish stable equilibria from unstable ones by examining how trajectories enter or exit isolating neighborhoods, providing insight that goes beyond purely local analysis.

4.2 Index-Based Stability Criteria

Index theory plays a central role in connecting topological invariants with stability properties. Let x^* be an isolated equilibrium of the system $\dot{x} = f(x)$. One can associate an index, often derived from fixed point or degree-theoretic constructions, that encodes information about the local behavior of the vector field near x^* .

Formally, consider a bounded open set $\Omega \subset \mathbb{R}^n$ such that x^* is the only solution of

$$f(x) = 0 \text{ in } \Omega.$$

The index of x^* can be defined via the topological degree $\deg(f, \Omega, 0)$. A nonzero index implies the persistence of the equilibrium under small perturbations, while changes in the index signal possible instability or bifurcation.

Index-based criteria are particularly useful in nonlinear and non-self mappings, where classical derivative-based methods are not applicable. Fixed point results developed for nonlinear contractions and relational metric spaces provide general conditions under which equilibria exist and are stable in a qualitative sense (Hasanuzzaman & Imdad, 2023; Chifu et al., 2020). These results highlight how index theory extends stability analysis to broader functional and metric settings.

Table 3 summarizes how index values are qualitatively associated with stability properties. The table provides an intuitive link between abstract index notions and dynamical behavior, which will be useful in later applications.

Table 3. Qualitative Interpretation of Index-Based Stability

Index Property	Stability Interpretation
Nonzero index	Existence and persistence of equilibrium
Index invariant under perturbation	Robust stability
Index change	Possible instability or bifurcation

4.3 Global Stability via Invariant Regions and Degree Arguments

While local stability concerns behavior near equilibria, global stability addresses the behavior of trajectories over the entire phase space. Topological degree theory provides a natural framework for establishing global stability results by constructing invariant regions that trap trajectories.

Let $\Omega \subset \mathbb{R}^n$ be a positively invariant region, meaning that trajectories starting in Ω remain in Ω for all future time. If the degree condition

$$\deg(f, \Omega, 0) \neq 0$$

holds and $f(x) \neq 0$ on $\partial\Omega$, then at least one equilibrium exists in Ω . Under additional qualitative assumptions on the flow, this equilibrium may be shown to attract all trajectories in Ω , yielding global stability.

Such degree-based arguments have been successfully applied to higher-order autonomous systems and nonlinear models admitting periodic solutions (Bayat & Asadi, 2023; Yang & Han, 2023). In population dynamics and delay systems, invariant regions are often constructed using biological or physical constraints, allowing topological methods to establish global stability without explicit solution formulas (Ossandón & Sepúlveda, 2024).

4.4 Robustness Under Perturbations

A key advantage of topological approaches is their robustness. Stability properties derived from indices, degree, or invariant regions remain valid under continuous perturbations of the system. Consider a perturbed system.

$$\dot{x} = f(x) + \varepsilon g(x),$$

where ε is a small parameter. If no equilibria cross the boundary of an isolating region as ε varies, then topological invariants such as degree or fixed point index remain unchanged.

This invariance ensures that qualitative stability properties persist even when the system is subject to modeling errors, parameter uncertainty, or external disturbances. Robustness is particularly important in delay and neutral differential equations, where small changes can significantly alter local dynamics but may leave global topological features intact (Taie & Bakhit, 2023; Ossandón & Sepúlveda, 2024). Overall, the stability of equilibria and invariant sets can be effectively analyzed through topological characterizations, index-based criteria, and global degree arguments. These methods provide a unifying qualitative framework that complements classical stability theory and offers robust tools for analyzing complex nonlinear differential systems.

5. Applications to Nonlinear Differential Systems

Topological methods are not only of theoretical interest but also provide effective tools for analyzing concrete nonlinear differential systems arising in applied sciences. This section illustrates how the topological framework developed earlier can be applied to autonomous nonlinear systems, control-oriented stabilization problems, and representative models from physics and biology. The emphasis is on how qualitative topological tools yield existence, stability, and robustness results that are difficult to obtain using purely classical techniques.

5.1 Autonomous Nonlinear Systems

Autonomous nonlinear systems of the form

$$\dot{x} = f(x)$$

serve as a fundamental class of models in nonlinear dynamics. For such systems, topological methods are particularly effective because the qualitative structure of trajectories is entirely determined by the vector field and its invariant sets. Fixed point and degree-theoretic approaches allow one to establish the existence of equilibria and periodic solutions without explicit integration of the system.

In higher-order and integral differential equations, classical stability analysis often becomes technically cumbersome. Topological degree methods have been successfully applied to demonstrate the existence of periodic solutions and to analyze their qualitative stability properties in such systems. These approaches rely on invariant regions and continuation arguments rather than explicit solution formulas, making them well suited for strongly nonlinear autonomous systems (Amster & Bondorevsky, 2021). The robustness of degree-based results ensures that qualitative behavior persists under admissible perturbations.

5.2 Control-Oriented Systems and Stabilization Problems

In control theory, stabilization issues are often nonlinear, uncertain or set-valued feedback laws. Topological techniques offer a loose circuit of dealing with such difficulties, concentrating on the invariant properties of the closed-loop system as opposed to precise paths. Controlled systems also have equilibria which can be defined as fixed points of appropriately defined operators, allowing fixed point and contraction principles to be used to study stability.

Multi-valued mappings occur naturally in a discontinuous or switching controlled system. Multi-valued and non-self operators fixed point results enable the determination of existence and stability of controlled equilibria in the case of violations of classical conditions, like smoothness or uniqueness (Petruşel et al., 2020). The domain invariance theorems also provide that the trajectories stay inside given safety or performance areas, a major demand in the stabilization issues (Lazăr et al., 2009).

Topologically, stabilization can be understood to be the creation of invariant regions, which draw the trajectories to where they want to be. The index-based arguments ensure that stability persists when controllers are perturbed, which gives robustness that is very important in real-world control.

5.3 Illustrative Examples from Physical and Biological Models

Topological methods have been useful in physical and biological modeling, where delays and nonlinearities are inherent. Delay terms, neutral terms or integral terms are commonly present in models in population dynamics and epidemiology due to the effect of memory and interaction. Such properties make the classical analysis of stability difficult, but they are automatically addressed in a topological context.

As an example, the dynamic behaviour of neutral-type integral differential equations that occur in epidemic models has complex behaviour such as oscillations and persistence. The existence of positive periodic solutions and their qualitative stability has been established using topology, and it has been used to study their qualitative stability, which is biologically relevant like population persistence (Yang et al., 2023). The construction of such results is based on invariant cones and degree arguments as opposed to explicit solution construction.

The Conley index has also been used to realize global dynamical properties of nonlinear models such as the structure of invariant sets and the phase changes between one dynamical regime and another. Covering actions are some of the refinements, which can increase the descriptive power of the Conley index and enable one to see subtle stability-related features in complex systems (Lima et al., 2023). These are especially useful in biological and physical systems where the uncertainties of experimental procedures and inaccuracies in models are inevitable. Table 4 briefly summarises the ways in which various types of nonlinear systems can be useful for topological methods. The fact that the table shows

correspondence between the type of system, topological tool, and the stability insight derived by it further supports the fact that topology is unifying across applications.

Table 4. Applications of Topological Methods in Nonlinear Systems

System Type	Topological Tool Used	Main Outcome
Autonomous nonlinear systems	Degree theory	Existence of equilibria/periodic solutions
Control-oriented systems	Fixed point theory	Robust stabilization
Biological and physical models	Conley index	Global stability structure

Overall, these applications demonstrate that topological methods provide a versatile and robust framework for analyzing nonlinear differential systems across diverse domains. By emphasizing qualitative invariants and global structures, topology complements classical analytical techniques and offers powerful tools for understanding stability in complex real-world systems.

6. Computational and Numerical Considerations

While topological methods provide powerful theoretical tools for stability analysis, their practical applicability relies heavily on effective computational and numerical techniques. This section discusses how topological invariants can be approximated numerically, how degree and Conley index computations are implemented in practice, and how simulations support and validate theoretical stability results for nonlinear differential systems.

6.1 Approximation of Topological Invariants

Topological data like fixed point indices, degree and homological measures are qualitative by their nature but can be approximated numerically with discretization and data based methods. Topological data analysis (TDA) has in recent years become a computational framework, which can be used to extract robust topological structures of sampled or simulated dynamical data.

Persistent homology is key to this situation as it keeps track of topological features births and deaths across multiple scales. These properties are important in reliable numerical analysis of nonlinear systems by the stability of these properties to perturbations. Quantitative tools Existing distance measures between persistence diagrams are used to compare dynamical behaviors and understand robustness especially in systems that change over time (Ye et al., 2023). These methods enable a person to estimate the structures of invariance that are related to stability without having to make explicit analytic descriptions. Topological approaches that are based on time series can also be used to analyze nonlinear dynamics of a numerical simulation or experiment. Qualitative stability information can be directly inferred based on data by embedding trajectories into appropriate phase spaces and computing the topological signatures of the trajectories (Umeda, 2017).

6.2 Numerical Methods for Degree and Conley Index Computation

Numerical computation of topological degree and the Conley index typically relies on discretization of the phase space and approximation of the underlying vector field or multivalued map. For degree theory, numerical schemes often involve subdividing a bounded region $\Omega \subset \mathbb{R}^n$ and verifying sign or orientation conditions on the boundary to ensure that

$$\deg(f, \Omega, 0) \neq 0.$$

Such methods provide constructive ways to verify existence and stability of equilibria in complex nonlinear systems, including fractional and nonlocal models (Asmaa et al., 2024).

For the Conley index, computational approaches focus on identifying isolating neighborhoods and approximating exit sets using combinatorial or grid-based representations. These methods preserve the homotopy type of invariant sets while allowing numerical implementation. In multivalued and non-smooth settings, recent fixed point results for Feng–Liu type operators provide a theoretical

foundation for algorithmic treatments of stability problems involving set-valued dynamics (Petruşel et al., 2024).

Table 5 summarizes common computational approaches used for approximating topological invariants in stability analysis. The table highlights the connection between numerical technique and the corresponding topological concept, illustrating how theory is translated into computation.

Table 5. Numerical Approximation of Topological Invariants

Topological Concept	Numerical Approach	Main Purpose
Fixed point index	Discretized operator iteration	Detection of equilibria
Topological degree	Boundary subdivision methods	Existence and robustness analysis
Conley index	Grid-based isolating neighborhoods	Global invariant set structure
Persistent homology	Filtration and persistence diagrams	Data-driven stability features

6.3 Simulation Results Supporting Theoretical Findings

Simulations are also important in the illustration and confirmation of theoretical stability findings made using topological techniques. Simulations give one the visual intuition of invariant regions, attractors, and the passage of abstract topological concepts between stable and unstable regimes.

In more complicated dynamical systems like particle ensembles or nonlinear models on a high dimensional space, TDA-based simulations have been employed to describe stability by tracking topological features which are persistent in time. These characteristics are numerical values of steady and unsteady behavior and are resilient to noise and perturbations (Sudhan Kumar, 2024). These findings indicate that topological measures can be used to supplement conventional numerical measures such as Lyapunov exponents.

Numerical analysis of coupled and fractional differential equations and systems has shown that the existence and stability results of degree are not only theoretical but also can be realized in dynamical simulations. Simulations of topological degree approaches to nonlinear systems in the real world by checking invariant regions, equilibria persistence and numerically support the validity of topological degree approaches in practice (Asmaa et al., 2024).

On the whole, computational and numerical methods fill the gap between conceptual topological theory and good old-fashioned stability analysis. These techniques make the topological methods much more applicable to nonlinear differential equations by making it possible to approximate topological invariants, and to make predictions about the theory using simulations.

7. Conclusions

The analysis of the topological approach to the stability of nonlinear differential equations has been thoroughly conducted in this article with the focus on the fact that the topological approaches have been used as robust and qualitative alternatives to classical analytical methods. The work has combined fixed point theory, topological degree theory and Conley index theory to show how stability properties of equilibria and invariant sets can be defined without the need to know any explicit solutions, any smoothness, or anything about the construction of Lyapunov functions. It was noted in the discussion that topological invariants give useful information on local and global stability, equilibria persistence, and structure of nonlinear dynamics. It was applied to autonomous systems, control-oriented models, and physical and biological systems to demonstrate the practical applicability of these methods, and computational concerns demonstrated how abstract topological models could be successfully approximated and verified using numerical methods and simulations. Topological methods have the benefits of being robust to perturbations, being applicable to non-smooth and high-dimensional systems, and are able to capture global qualitative behaviour that classical local methods can fail to capture. Simultaneously, there are still some restrictions such as the computational complexity of the invariant approximation in large-scale systems and the abstract character of topological tools, which might not be easily interpreted directly into physical terms.

These restrictions indicate a number of open questions and future directions of research, including the creation of more efficient numerical algorithms, applying a topological theory of stability to hybrid and switched systems, as well as adding stochastic effects and time delays to the topological theory. Overcoming such issues will help to make topological techniques more applicable and their contribution to the study and management of nonlinear dynamical systems more significant.

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