Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 10s (2025)

A 2-D Diffusion-Based Population Dynamics Model Using Bicubic B-Spline Interpolation

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Article History:

Received:10-01-2025

Revised:27-04-2025

Accepted:10-05-2025

Abstract:
Understanding how urban populations change over time and across space is important for studying city growth. Traditional models often focus only on population growth over time and ignore spatial effects, which limits their accuracy. In this study, a 2-D diffusion-based model is developed to describe population changes in both time and space using an analogy with the 2-D heat equation. The model is solved numerically using bicubic B-spline interpolation for space and the Crank–Nicolson scheme for time. This approach provides smooth spatial results, stable calculations and accurate time integration. The model is applied to Surat city, India, divided into 25 subregions, using census-based initial conditions and regression-based boundary conditions. Predicted populations for 2011 closely match actual census data, and error measures such as Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) show good predictive performance. The results show that the model can capture patterns of population movement and growth, making it useful for urban planning. Future work may include socio-economic factors and variable spatial coefficients to improve local accuracy.

Keywords: Mathematical Modelling, Population Growth, 2-D region, 2-D heat equation, Bicubic B-Spline Interpolation Method, Numerical Approach

1. Introduction:

Population growth and its spatial distribution are essential aspects in urban planning, resource management and socio-economic policy development. The rapid pace of urbanization and the heterogeneous distribution of populations across geographic regions have created a pressing need for reliable mathematical models that can accurately predict population dynamics both temporally and spatially. Classical population growth models often formulated using ordinary differential equations or statistical extrapolation methods, predominantly focus on temporal evolution and typically neglect spatial diffusion effects. Consequently, such models are limited in their ability to account for migration, interregional interactions and local variations in population density.[1][2][3]

Partial differential equation (PDE) based models provide a rigorous framework for integrating both temporal dynamics and spatial transport in population studies. In particular, diffusion-type PDEs have been extensively employed to model migration, dispersal processes and spatial interactions between subregions. The mathematical analogy between population diffusion and heat conduction is particularly effective as both phenomena are governed by similar conservation principles.

Accordingly, the 2-D heat equation offers a natural basis for modeling population propagation over a planar domain.[4][5][6][7][8]

Despite their theoretical robustness, obtaining analytical solutions for PDE-based population models is often intractable due to complex boundary conditions, spatial heterogeneities and constraints imposed by real-world data. This necessitates the use of numerical methods. While classical finite difference and finite element methods are widely applied, they may exhibit numerical instability or reduced accuracy when addressing smooth spatial distributions or higher-order derivatives.[9][10]

Spline-based numerical techniques, especially cubic and bicubic B-spline interpolation methods, provide distinct advantages for the solution of 2-D PDEs. These methods afford higher smoothness, local support and superior approximation accuracy compared to conventional discretization schemes. In particular, bicubic B-splines ensure continuity of both first and second derivatives, which is highly desirable when modeling diffusion-driven population dynamics.[11][12][13][19]

In the present study, a 2-D population growth model is formulated by drawing an analogy with the 2-D heat equation, and the governing dynamics are expressed as a partial differential equation. The proposed model captures both spatial diffusion and temporal evolution of population density over a bounded domain. A bicubic B-spline interpolation scheme, combined with Crank–Nicolson time discretization, is employed to obtain a stable and accurate numerical solution. The validity and effectiveness of the model are demonstrated through a case study using historical population data from selected subregions of Surat city, Gujarat, India. The predicted population values are systematically compared with actual census data to evaluate the accuracy and practical applicability of the proposed approach.

2. Literature Review:

Population dynamics models based on partial differential equations (PDEs) have been widely used to describe the temporal and spatial distribution of biological and socio-ecological populations. Classical reaction–diffusion frameworks, such as the Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) equation have long been applied to model biological invasions and spatial dispersal patterns, emphasizing the interaction between growth and spatial spread mechanisms. Recent studies extend these frameworks to incorporate structured environments and nonlinear diffusion behaviours, providing deeper insights into complex spatial dynamics of populations.

PDE-based population growth models have been enhanced in recent research to include both diffusive movement and various non-linear processes. Perusquía-Cortés and Padilla (2025) developed diffusion and discrete temporal models for urban animal populations, capturing the interplay between spatial movement and temporal recurrence in free-roaming cat populations, demonstrating the effective use of diffusion equations in urban ecological modelling. In mathematical biology, density-dependent diffusion and aggregation mechanisms have been rigorously analysed, with recent work highlighting the role of heterogeneous environments in shaping population fronts and spatio-temporal patterns, thereby advancing traditional reaction–diffusion theory. Similarly, the dynamics of population fronts under nonlinear diffusion mechanisms reveal critical effects on wave propagation speed and spatial structure, reinforcing the importance of diffusion terms in continuous population models.[6][15][16]

From a numerical modelling perspective, several recent studies focus on advanced numerical schemes to approximate solutions of diffusion and reaction–diffusion PDEs with improved accuracy and stability. Ahn *et al.* (2025) compared spline-based Galerkin methods using quadratic and cubic B-splines for solving nonlinear diffusion equations and demonstrated that spline basis functions provide accurate approximations with quantifiable error norms. Spline collocation and other B-spline based numerical methods have also been applied to advection–diffusion and related problems, with unconditional stability and high convergence properties reported. These studies underscore the suitability of B-spline approaches for smoothly approximating spatial derivatives in diffusion problems, which aligns with the numerical methods adopted in the present work.[11][13][17]

In parallel, meshless and collocation techniques have been explored for population-related diffusion models, such as thin plate spline (TPS) radial basis function (RBF) methods for logistic diffusion models, which avoid complex mesh generation and enhance flexibility in non-uniform spatial resolutions. Incorporating sophisticated numerical schemes in population models enables robust simulation of spatial dynamics under a variety of environmental and demographic influences.[18]

Despite these advances, a gap remains in integrating B-spline based numerical techniques specifically for 2-D population diffusion models with census-based validation in urban contexts. Previous work has largely focused on either one-dimensional test problems or abstract ecological settings. Therefore, the application of bicubic B-spline spectral approximations combined with Crank–Nicolson time stepping in a realistic urban population prediction framework enriches the numerical ecology literature and bridges methodological development with applied population studies.

3. Proposed Methodology:

This section presents the mathematical formulation and numerical methodology adopted for predicting population growth in a 2-D spatial domain. The proposed framework models population diffusion and temporal evolution by drawing an analogy with the 2-D heat equation and solves the resulting partial differential equation using a bicubic B-spline interpolation scheme combined with Crank–Nicolson time discretization.

The system architecture of the proposed population growth model is illustrated in Figure 1

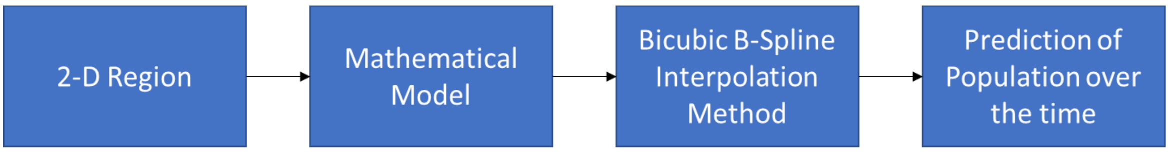

Figure 1: Proposed Methodology

2-D Region

P1	P2	P3	P4	P5
P6	P7	P8	P9	P10
P11	P12	P13	P14	P15
P16	P17	P18	P19	P20
P21	P22	P23	P24	P25

The computational domain is a two-dimensional region subdivided into 25 subregions denoted by P1,P2,…P25, representing selected localities of Surat city, Gujarat, India. An initial population distribution is assumed to exist in all subregions, and population growth and spatial diffusion are considered over a specified time interval.

3.1 Diffusion-Based Population Growth Model:

Population migration and spatial interaction processes often exhibit diffusion-like behavior similar to heat conduction. Based on this analogy, population density evolution is modeled using a two-dimensional diffusion equation.

Let $p = p(x, y, t)$ represent the population density at location $(x, y)$and time t. The governing equation is formulated as

$$\frac{\partial p}{\partial t} = m\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \text{ where } 0 < x < a, 0 < y < b, t \geq 0. \quad (3.1)$$

where m denotes the population diffusion or growth coefficient, which may be constant or spatially and temporally dependent.

The model is supplemented with Dirichlet boundary conditions,

$$p(0, y, t) = p_1(y, t), \quad p(a, y, t) = p_2(y, t), \quad 0 \leq y \leq b, \quad t \geq 0$$
$$p(x, 0, t) = p_3(x, t), \quad p(x, b, t) = p_4(x, t), \quad 0 \leq x \leq a, \quad t \geq 0 \quad (3.2)$$

and an initial condition,

$$p(x, y, 0) = p(x, y). \;\; 0 \leq x \leq a, 0 \leq y \leq b \quad (3.3)$$

Due to the absence of closed-form solutions under realistic boundary conditions, a numerical approximation strategy is employed.

3.2 Bicubic B-Spline Spatial Approximation:

To obtain a smooth and accurate spatial approximation of the population density, a bicubic B-spline interpolation method is adopted. Bicubic B-splines are constructed as tensor products of one-dimensional cubic B-spline basis functions and possess C^2-continuity, making them well suited for diffusion-type partial differential equations.

Let $\{x_i\}$ and $\{y_i\}$ be uniform partitions of the spatial domain with step sizes $\Delta x$and Δy, respectively. The approximate population surface is expressed as

$$S_B(x, y) = \sum_{i=-3}^{m-1} \sum_{j=-3}^{n-1} C_{ij} B_j^4(x) B_j^4(y), \quad x \in [x_0, x_m], \;\; y \in [y_0, y_n], \;\; m.n \geq 1 .$$
(3.4)

where $C_{ij}$are unknown spline coefficients and $B_i^{(4)}$denotes the cubic B-spline basis function.

The properties and analytical expressions of cubic B-spline basis functions and their derivatives follow standard formulations available in the literature and are therefore omitted here for brevity. [11][12][19]

3.3 Discrete Spatial Derivatives:

Evaluating the bicubic B-spline surface and its derivatives at the grid points (x_i, y_i) yields simplified expressions involving only neighboring spline coefficients. These expressions significantly reduce computational complexity and enable efficient evaluation of spatial derivatives.

At grid points, the approximate solution and its second-order spatial derivatives required in the diffusion equation can be written as

$$p(x_i, y_i, t) \approx S_B(x_i, y_i) \tag{3.5}$$

$$\frac{\partial^2 p}{\partial x^2}(x_i, y_i, t) \approx \frac{\partial^2 S_B}{\partial x^2}(x_i, y_i),\ \frac{\partial^2 p}{\partial y^2}(x_i, y_i, t) \approx \frac{\partial^2 S_B}{\partial y^2}(x_i, y_i) \tag{3.6}$$

The resulting discrete expressions involve a compact stencil of spline coefficients due to the local support of cubic B-splines, ensuring numerical efficiency and stability.

3.5 Time Discretization and Numerical Scheme:

The spatial and temporal variables are discretized as

$$x_i = i\Delta x,\quad \Delta x = \frac{a}{h},\quad h > 1,\ i \in Z$$
$$y = i\Delta y,\quad \Delta y = \frac{b}{n},\quad n > 1,\ i \in Z$$
$$t_k = k\Delta t,\quad \Delta t = time\ step,\ k \in N$$

Equation (3.4) is discretized using a θ-weighted time integration scheme:

$$(p_t)_{i,j}^k = (1-\theta)u_{i,j}^k + \theta u_{i,j}^{k+1} \tag{3.7}$$

where $u_{i,j}^k = m\left[(p_{xx})_{i,j}^k + (p_{yy})_{i,j}^k\right]$

Setting $\theta = \frac{1}{2}$ yields the Crank–Nicolson scheme, which is second-order accurate in time and unconditionally stable. Substituting the forward difference approximation for the time derivative leads to equation (3.8).[14]

$$p_{i.j}^{k+1} - \theta\Delta t u_{i,j}^{k+1} = p_{i.j}^k + (1-\theta)\Delta t u_{i,j}^k \tag{3.8}$$

The discretized boundary and initial conditions are given by equations (3.9) and (3.10), respectively. At each time step, the resulting underdetermined system of linear equations is solved using the least-squares method implemented in MATLAB to obtain the spline coefficients C_{ij}^k.

$$p_{0,j}^k = p_1(y_j, t_k),\quad p_{m,j}^k = p_2(y_j, t_k),\quad j = 0,1,2 \dots\dots n$$
$$p_{i,0}^k = p_3(x_i, t_k),\quad p_{i,n}^k = p_4(x_i, t_k),\quad i = 0,1,2 \dots\dots h-1 \tag{3.9}$$
$$p_{i.j}^0 = p(x_i, y_j),\quad i = 0,1,2 \dots h,\quad j = 1,2,3, \dots . n \tag{3.10}$$

The reconstructed bicubic spline surface $S_B(x, y)$provides a smooth and accurate numerical approximation of the population density at each time level.

4. Experimental results:

To demonstrate the applicability and effectiveness of the proposed population growth model, numerical simulations were carried out for a selected two-dimensional region of Surat city, Gujarat, India. The study area was discretized into 25 subregions arranged in a 5 × 5 spatial grid, as listed in Tables 1 and 2.

Table 1: Area of Surat				
Rander	Saiyadpura	Singanpor	Mahidharpura	Karanj
Adajan	Shahpor	Haripura	Begumpura	Umarwada
Pal	Sonifalia	Gopipura	Wadifalia	Anjana
Athwa	Nanpura	Sagrampura	Salabatpura	Dabholi
Bharthana-Vesu	Piplod	Bhatar	Udhana	Bhedvad

Table 2: Area wise Population of Surat in 2001				
86047	55179	7215	19817	198482
152274	23265	12654	45830	61170
11165	14426	19310	9552	78344
7726	51749	77316	55675	7968
1920	8871	28622	186860	8219

Table 2 shows the initial population of 2-d subregion of the Surat city (Gujarat, India) in 2001.

The selection of the study area, spatial discretization, initial conditions, boundary conditions, and population growth rate were based on historical census data and urban growth characteristics of Surat city. The formulation of the initial and boundary conditions with respect to both space and time was guided by past population data spanning the period 1961–2001, using a multipolynomial regression approach

4.1 Initial Population Data and Model Parameters:

Table 1 presents the names of the selected subregions of Surat city, while Table 2 reports the corresponding population data from the 2001 census, which serve as the initial condition for the numerical simulations. [26]

The average population growth rate of Surat city in 2001 was taken as 5.95%, which is incorporated into the diffusion-based population model. Accordingly, the governing equation for population growth is expressed as

$$\frac{\partial p}{\partial t} = 0.0595\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \; where \; 0 < x < 8, 0 < y < 8, t \geq 0 \tag{4.1}$$

where $\Delta x = 2, \Delta y = 2 \; and \; \Delta t = 1.$

The continuous initial population distribution and the associated boundary conditions were obtained via multipolynomial regression and are defined as follows:

$$P(x,y,t) = 3179.55t - 12690.67y - 930.83yt - 6317.03x + 1641.74xt + 1731.7xy + 28726.72 + 468.84x^2 + 1682.2y^2 + 1093.88t^2$$

$$P(x,y,0) = -12690.67y + 6317.03x + 1731.78xy + 28726.71 + 468.84x^2 + 1682.23y^2$$

$$P(0,y,t) = 3179.55t - 12690.67y - 930.83yt + 28726.7 + 1682.23y^2 + 1093.88t^2$$

$$P(8,y,t)) = 3179.55t - 12690.67y - 930.83yt - 50536.24 + 13133.92t + 13854.24y + 28726.71 + 300005.76 + 1682.23y^2 + 1093.88t^2$$

$$P(x,0,t) = 3179.55t - 6317.03x + 1641.74xt + 28726.71 + 468.84x^2 + 1093.88t^2$$

$$P(x,8,t) = 3179.55257t - 101525.43 - 7446.64t + -6317.03x + 1641.74xt + 13854.24x + 28726.7 + 468.84x^2 + 107662.72 + 1093.88t^2$$

4.2 Numerical Simulation and Predicted Population:

The proposed diffusion-based population growth model was solved numerically using the bicubic B-spline interpolation method combined with Crank–Nicolson time discretization, as described in Chapter 3.

The predicted population distribution for the year 2011 across the 25 subregions is reported in Table 3.

Table 3: Predicted Population if subregion of Surat in year 2011				
62819	25957	13770	25970	62767
35355	13539	16430	43791	95808
21047	14265	32227	74717	141932
19637	27992	61094	118645	200959
31507	55044	103328	175925	273319

4.3 Comparison with Actual Census Data:

A graphical comparison between the actual census population data and the model-predicted population values for the year 2011 is presented in Figure 2.

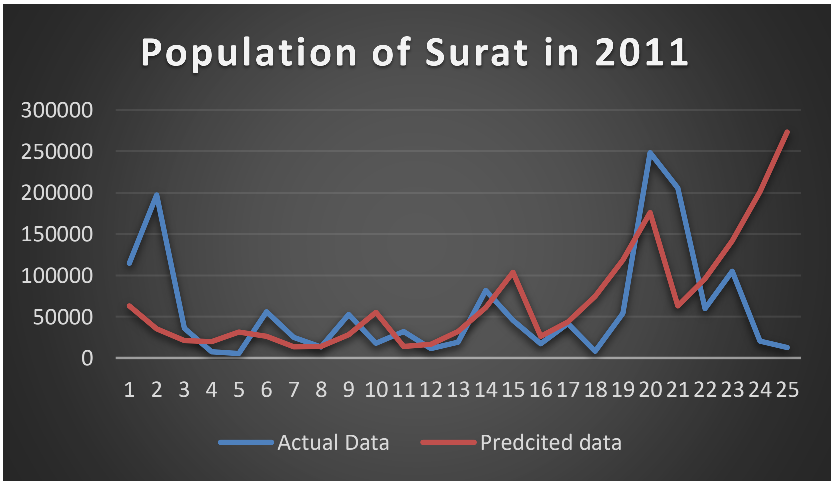

Figure 2: Comparison between Actual Data and Predicted data

The comparison reveals a close agreement between the predicted and observed population values across most subregions. Variations in population growth trends are evident, with some subregions experiencing population increases, while others exhibit stagnation or decline. These variations can be attributed to migration patterns, urban expansion, economic activity, and infrastructure development, which are implicitly captured by the diffusion-based formulation.

4.4 Discussion:

The numerical results demonstrate that the proposed two-dimensional diffusion model is capable of capturing the spatial heterogeneity of population growth within an urban environment. The use of bicubic B-spline interpolation ensures smooth population surfaces and enhances numerical accuracy, particularly for regions with gradual spatial variation.

The observed discrepancies between predicted and actual values in certain subregions highlight the influence of non-diffusive socio-economic factors, which may be incorporated in future extensions of the model.

4.5 Error Analysis:

To quantitatively evaluate the performance of the proposed population growth model, statistical error metrics were employed to compare the predicted population values with the actual census-based population data across the selected 25 subregions of Surat city.
Let A_i and P_i denote the actual and predicted population values, respectively, for the i-th subregion. The Mean Absolute Error (MAE) and Root Mean Square Error (RMSE) are defined as

MAE is computed as: MAE = $\left(\frac{1}{n}\right) \sum \left|\frac{A_{\mathrm{i}} - P_{\mathrm{i}}}{A_{\mathrm{i}}}\right| \times 100$

RMSE is computed as: RMSE = $\sqrt{\left(\frac{1}{\mathrm{n}}\right) \sum (A_i - P_i)^2}$

where $N = 25$ represents the total number of subregions.

For the proposed bicubic B-spline based diffusion model, the computed error values are:

- MAE = 50,161.39
- RMSE = 81,140.92

The MAE value reflects the average absolute deviation between predicted and observed population values, while the RMSE emphasizes larger deviations and provides insight into regional variability. Considering the spatial heterogeneity of urban population dynamics and the decade-long prediction horizon, the obtained error magnitudes indicate acceptable predictive accuracy. The results confirm that the proposed model effectively captures large-scale spatial population trends, while localized discrepancies may arise due to migration, infrastructural development, and socio-economic factors not explicitly included in the diffusion formulation.

5. Conclusion:

A 2-D diffusion-based population growth model was formulated and numerically implemented to study the spatial–temporal evolution of urban population in Surat city, India. The model employs census-based initial conditions and regression-derived boundary conditions, ensuring consistency with historical population trends. Spatial discretization was performed using bicubic B-spline interpolation, while temporal integration was carried out using the Crank–Nicolson scheme.
The combined numerical approach is unconditionally stable and second-order accurate in time, and the smoothness properties of bicubic B-splines contribute to enhanced spatial consistency and reduced

numerical oscillations. The resulting discrete system provides a reliable approximation of the continuous diffusion equation on the structured grid.

Model predictions for the year 2011 show good agreement with observed census data across most subregions. Error metrics, including MAE and RMSE, indicate acceptable predictive accuracy over a ten-year horizon, confirming the model's capability to capture large-scale spatial population trends. Discrepancies in certain subregions suggest the presence of non-diffusive influences beyond the scope of the current formulation.

Overall, the results demonstrate that diffusion-type partial differential equations, when coupled with stable and consistent numerical schemes, provide an effective framework for modeling urban population dynamics. Future extensions may incorporate reaction terms or spatially varying coefficients to improve local accuracy.

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