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Two Barriers Reflected Backward Doubly Stochastic Differential Equation with generalized coefficients

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Abstract:
We study double reflected backward doubly stochastic differential equations (DRBDSDEs) with a generator fff of generalized growth and a square-integrable terminal condition. We introduce new local conditions on the generator f, assuming that it is left-Lipschitz continuous in the variable y and Lipschitz continuous in z. Under these assumptions, we establish the existence of solutions. Our results extend previous work on reflected backward doubly stochastic differential equations with two barriers. Although our main focus is on the double-reflected case, we also derive a new comparison theorem for DRBDSDEs. In particular, when the generator f is Lipschitz continuous in both y and z, we show that one can compare not only the Y-components of the solutions, but also the associated non-decreasing processes K+
As an application, we show that DRBDSDEs with two reflecting barriers are closely related to obstacle problems for semilinear stochastic partial differential equations (SPDEs) whose generators exhibit generalized growth.

Keywords: Reflected BDSDEs, double barrier, existence of a solution, left-Lipschitz coefficients, comparison theorem.

1 INTRODUCTION

A new kind of backward stochastic differential equations was introduced by Pardoux and Peng [11] in 1994, that is a class of backward doubly stochastic differential equations (BDSDEs) with two different directions of stochastic integrals, i.e., equations involving both a standard (forward) stochastic integral and a backward stochastic integral. Roughly speaking, BDSDE are stochastic differential equation of the forme :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s; \quad 0 \leq t \leq T. \qquad (1)$$

where the dW is a forward Itô integral, and the dB is a backward Itô integral. The terminal value ξ, the coefficient f and g are the data of the problem. In [11], the existence and uniqueness of solutions are established under uniformly Lipschitz condition on the coefficients. It is worth noting that the definition of a solution of this type of equations is slightly different from that of classical backward stochastic differential equations.

From the beginning, many authors attempted to improve the result of [11] by weaking the Lipschitz continuity of coefficient f, see e.g [1,4,5,7,8,9,14,15,16,17].

Bahlali et al 2010 [2], studed the case where the solution is forced to stay above a given stochastic process, called the obstacle. He obtained the real valued reflected backward doubly stochastic differential equation :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s + K_T - K_t - \int_t^T Z_s dW_s; \quad 0 \le t \le T. \qquad (2)$$

Bahlali et al established the existence and uniqueness of solutions for equation (2) under uniformly Lipschitz condition on the coefficients, and in the case where the coefficient *f* is only continuous, he established the existence of a maximal and a minimal solutions. Carrying on this work, Marzougue M (2022) [10], introduced the notion of Two-barriers reflected backward doubly SDEs (DRBDSDE) :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dW_s \ , \quad 0 \le t \le T \qquad (3)$$

In this equation in order to force the solution remain between two prescribed continuous processes *L* and *U*, called lower and upper reflecting obstacle, respectively. Merzougue [10] proved the existence and uniqueness of solution of this equation, under the Lipschitz assumption.

The theory of Backward Doubly Stochastic Differential Equations as well as their relation with the stochastic optimal control problems see([3], [6]), and stochastic partial differential equations (SPDEs) see ([1], [4], [5], [7], [9], [11], [12], [14], [15], [16]). Our motivation is seeing equation (3) as an solution of obstacle probleme to a class of stochastic quasilinear partial differential equations (SPDEs), and their probabilistic interpretation of solution. In this paper, our aim is to generalize the result established in Merzougue [10] with generalized assumption, *f* are left-Lipschitz in *y* and Lipschitz in *z*. Before this result, we give comparison theorem of DRBDSDR under Lipischitz condition in the coefficient *f*.

The paper is organized as follows. In section 2, we state the notations, assumptions, definition and prove the comparaison theorem. In section 3, we give the main result of paper.

2 Notation, assumptions and Definitions

Let (Ω,F,*P*) be a complete probability space, and *T>0*. Let $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ be two independent standard Brownian motion defined on (Ω,F,*P*) with values in R^d and R, respectively. For $t \in [0,T]$, we put, $\boldsymbol{F}_t = \boldsymbol{F}_t^W \vee \boldsymbol{F}_{t,T}^B$, and $\boldsymbol{G}_t = \boldsymbol{F}_t^W \vee \boldsymbol{F}_T^B$, where $\boldsymbol{F}_t^W = \boldsymbol{\sigma}(\boldsymbol{W}_s; 0 \le s \le t)$ and $\boldsymbol{F}_{t,T}^B = \boldsymbol{\sigma}(\boldsymbol{B}_s - \boldsymbol{B}_t; t \le s \le T)$, completed with *P*-null sets. It should be noted that ($\boldsymbol{F}_t$) is not an increasing family of sub σ-fields, and hence it is not a filtration. However ($\boldsymbol{G}_t$) is a filtration.

In this work, we will use the following notations,

*) $M_T^2(0, T, R^d)$ denote the set of *d*-dimensional, jointly measurable stochastic processes $\{\varphi_t; t \in [0,T]\}$, which satisfy :

a) $\int_0^T |\varphi_t|^2 dt < \infty.$

b) is $\boldsymbol{F}_t$-measurable, for any $\in [0,T]$

*) $S_T^2([0,T],R)$, the set of continuous stochastic processes φ_t, which satisfy :

a') $(sup_{0<t<T}|\varphi_t|^2) < \infty$

b') For every $\in [0,T]$, . is $\boldsymbol{F_t}$-measurable.

*) $L^2(\Omega,F_T,P)$, the set of $\boldsymbol{F_T}$-measurable random variable ξ,which satisfy $E|\xi|^2 < \infty$.

We consider the functions

$$f:\Omega \times [0,T] \times R \times R^d \to R$$

And

$$g:\Omega \times [0,T] \times R \times R^d \to R$$

are two be measurable and such that for every $(y,z) \in R \times R^d$, $f(\ .,y,z) \in M^2(0,T,R\ )$, $g(\ .,y,z) \in M^2(0,T,R\ )$ satisfying the following assumptions :

H1) There existe a constant C>0, such that *P-a.s.*, $\forall t \in [0,T], \forall (y_i,z_i) \in R \times R^d, (i = 1,2)$, we have

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|)$$

H2) There exist a constant C>0, such that *P-a.s.*, $\forall t \in [0,T], \forall (y_i,z) \in R \times R^d, (i = 1,2)$ and $y_1 \geq y_2$, we have

$$f(t,y_1,z) - f(t,y_2,z) \geq -C(y_1 - y_2).$$

H3) The process *f(t,.,z)* is left-continuous in *y*.

H4) There existe a constant C>0, such that *P-a.s.*, $\forall t \in [0,T], \forall (y,z_i) \in R \times R^d, (i = 1,2)$, we have

$$|f(t,y,z_1) - f(t,y,z_2)| \leq C|z_1 - z_2|$$

H5) There existe a constant C>0 and 0<α<1, such that *P-a.s.*, $\forall t \in [0,T], \forall (y_i,z_i) \in R \times R^d, (i = 1,2)$, we have

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \leq C|y_1 - y_2| + \alpha|z_1 - z_2|$$

H6) There exist two non-negative super-martingales η and θ, which are right continuous with left limits, such that
$\forall t \in [0,T[,\ L_t \leq \eta_t - \theta_t \leq U_t$ and $E[sup_{0\leq t<T}(|\eta_t| + |\theta_t|)^2] < \infty$.

H7) There exist two DRBDSDEs with data (ξ,f_i,g,L,U) which have at least one solution $(Y_{t,i},Z_{t,i},K_{t,i}^+,K_{t,i}^-)$, i=1,2, respectively. $\forall (t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R^d}$, $f_1(t,y,z) \leq f(t,y,z) \leq f_2(t,y,z)$. $Y_{t,1} \leq Y_{t,2}, a.s.$, and the processes $f_i(t,Y_{t,i},Z_{t,i})$ are square integrable.

Definition 2.1: *A Reflected Backward Doubly Stochastic Differential Equation with double obstacles is associated with a terminal condition $\xi \in L^2(\Omega,F_T,P)$, a generators f, g, a lower obstacle $(L_t)_{0\leq t\leq T}$ and an upper obstacle $(U_t)_{0\leq t\leq T}$ which both belong to $S_T^2([0,T],R)$, such that P-a.s, $L_t \leq U_t, \forall t \in [0,T]$ and $L_T \leq \xi \leq U_T$. A solution of this equation is a quadruple (Y,Z,K^+,K^-) of* *progressively measurable prosses taking values in* $\boldsymbol{R} \times \boldsymbol{R}^d \times \boldsymbol{R}^+ \times \boldsymbol{R}^+$ *and satisfying*

$$\begin{cases} i) Z \in M_T^2(0,T,R^d), Y, K^+, K^- \in S_T^2([0,T],R) \\ ii) Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dW_s \\ iii) P-a.s,\ \ L_t \le Y_t \le U_t,\ \ \forall t \in [0,T]\ \ and\ \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0 \\ iv) K^+,\ \ K^-\ \ are\ continuous\ and\ increasing,\ \ K_0^+ = K_0^- = 0 \end{cases}$$

Lemma 2.2 :(Merzougui 2022 [10]**)** Assume that assumptions **H1), H5),** and **H6)** hold, Then the DRBDSDE (3) with data (ξ, f, g, L, U) has a unique solution (Y, Z, K^+, K^-).

We need also the following comparison theorem

Theorem 2.3 :(Comparison Theorem) Let $(\xi_1, f_1, g, L_1, U_1)$ and $(\xi_2, f_2, g, L_2, U_2)$ be two DRBDSDEs. Each one satisfying all the previous assumptions **H1), H5)**. Assume moreover that :

i) $\xi_1 \le \xi_2\ \ a.s.$

ii) $f_1(t,y,z) \le f_2(t,y,z)\ \ \ \ dP \times dt\ a.e. \forall (y,z) \in \mathbf{R} \times \mathbf{R}^{\mathbf{d}}$.

iii) $L_{t,1} \le L_{t,2},\ \ U_{t,1} \le U_{t,2}, 0 \le t \le T\ \ \ a.s.$

Let (Y_1, Z_1, K_1^+, K_1^-) be a solution of DRBDSDE $(\xi_1, f_1, g, L_1, U_1)$ and (Y_2, Z_2, K_2^+, K_2^-) be a solution of DRBDSDE $(\xi_2, f_2, g, L_2, U_2)$. Then

$$Y_{t,1} \le Y_{t,2},\ \ \ 0 \le t \le T\ \ \ a.s.$$

Proof Applying Itô's formula to $\left|(Y_{t,1} - Y_{t,2})^+\right|^2$, and passing to expectation, we have

$$\begin{aligned} E\left|(Y_{t,1} - Y_{t,2})^+\right|^2 + E\int_t^T 1_{\{Y_{s,1} > Y_{s,2}\}} \left|Z_{s,1} - Z_{s,2}\right|^2 ds \\ = 2E\int_t^T (Y_{s,1} - Y_{s,2})^+ \left(f_1(s, Y_{s,1}, Z_{s,1}) - f_2(s, Y_{s,2}, Z_{s,2})\right) ds \\ + 2E\int_t^T (Y_{s,1} - Y_{s,2})^+ (dK_{s,1}^+ - dK_{s,2}^+) + 2E\int_t^T (Y_{s,1} - Y_{s,2})^+ (dK_{s,1}^- - dK_{s,2}^-) \\ + E\int_t^T 1_{\{Y_{s,1} > Y_{s,2}\}} \left|g(s, Y_{s,1}, Z_{s,1}) - g(s, Y_{s,2}, Z_{s,2})\right|^2 ds. \qquad (4) \end{aligned}$$

Since on the set $\{Y_{s,1} > Y_{s,2}\}$, we have $Y_{t,1} > L_{t,2} \ge L_{t,1}$ and $Y_{t,1} > U_{t,2} \ge U_{t,1}$, then

$$\int_t^T (Y_{s,1} - Y_{s,2})^+ (dK_{s,1}^+ - dK_{s,2}^+) ds = -\int_t^T (Y_{s,1} - Y_{s,2})^+ dK_{s,2}^+ \le 0$$

$$\int_t^T (Y_{s,1} - Y_{s,2})^+ (dK_{s,1}^- - dK_{s,2}^-) ds = -\int_t^T (Y_{s,1} - Y_{s,2})^+ dK_{s,1}^- \le 0$$

By the assumptions **H1), H5)**, we have on the set $\{Y_{s,1} > Y_{s,2}\}$,

$$\begin{aligned} E\left|(Y_{t,1} - Y_{t,2})^+\right|^2 + E\int_t^T 1_{\{Y_{s,1} > Y_{s,2}\}} \left|Z_{s,1} - Z_{s,2}\right|^2 ds \\ \le \left(3C + \frac{1}{\beta} C^2\right) E\int_t^T \left|Y_{s,1} - Y_{s,2}\right|^2 1_{\{Y_{s,1} > Y_{s,2}\}} ds \\ + (\beta + \alpha) E\int_t^T 1_{\{Y_{s,1} > Y_{s,2}\}} \left|Z_{s,1} - Z_{s,2}\right|^2 ds. \qquad (5) \end{aligned}$$

We now choose $\beta = \frac{1-\alpha}{2}$, and $L = 3C + \frac{1}{\beta}C^2$, to deduce that

$$E\left|\left(Y_{t,1} - Y_{t,2}\right)^+\right|^2 \leq LE\int_t^T \left|\left(Y_{s,1} - Y_{s,2}\right)^+\right|^2 ds. \qquad (6)$$

The result follows now by using Gronwall's lemma.

Remark 2.4. Furthermore if *f_1 and f_2 both satisfy* **H1)** and

$$P - a.s., f_1(t,y,z) \leq f_2(t,y,z), \qquad \forall (t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d.$$

Then we have

$$P - a.s.,\ K_{t,1}^+ \geq K_{t,2}^+, \qquad K_{t,1}^- \leq K_{t,2}^-, \qquad \forall t \in [0,T].$$

3 MAIN RESULTS

In this section,we prouve the main result of this paper

Theorem 3.1. *Suppose that assumptions **H3)-H7)** hold, Then DRBDSDE (3) with data (ξ, f, g, L, U) has a solution (Y, Z, K^+, K^-).*

Proof : From the hypothesis **H7)**, it follows that there exist two DRBDSDEs :

$$Y_{t,i} = \xi + \int_t^T f_i(s, Y_{s,i}, Z_{s,i})ds + \int_t^T g(s, Y_{s;i}, Z_{s,i})dB_s + K_{T,i}^+ - K_{t,i}^+ - K_{T,i}^- + K_{t,i}^-$$
$$- \int_t^T Z_{s,i}dW_s, \qquad \forall t \in [0,T],\ (i = 1,2) \qquad (7)$$

with data (ξ, f_i, g, L, U) respectively, such that

$$f_i \in M_T^2(0,T,R\ \ ),\ (i = 1,2)$$

and

$$f_1(t,y,z) \leq f(t,y,z) \leq f_2(t,y,z), \qquad Y_{t,1} \leq Y_{t,2}, \qquad \forall (t,y,z) \in [0,T] \times \mathbf{R} \times \mathbf{R}^\mathbf{d}.$$

Now we construct the following DRBDSDEs:

$$y_t^n = \xi + \int_t^T \left(f(s, y_s^{n-1}, z_s^{n-1}) - C(y_s^n - y_s^{n-1}) - C(z_s^n - z_s^{n-1})\right)ds$$
$$+ \int_t^T g(s, y_s^{n-1}, z_s^{n-1})dB_s + k_T^{n,+} - k_t^{n,+} - k_T^{n,-} + k_t^{n,-} - \int_t^T z_s^n dW_s,$$
$$\forall t \in [0,T]. \qquad (8)$$

where $(y_t^0, z_t^0) = (Y_{t,1}, Z_{t,1})$. Then we have the following propositions:

Proposition 3.2 *For $n \geq 1$, DRBDSDE (8) has a unique solution $(y_t^n, z_t^n, k_t^{n,+}, k_t^{n,-})$ and*
$P - a.s., Y_{t,1} \leq y_t^n \leq y_t^{n+1} \leq Y_{t,2},\ \ k_t^{n+1,+} \leq k_t^{n,+},\ \ k_t^{n,-} \leq k_t^{n+1,-} \quad \forall t \in [0,T].$

Proof Firstly, for *n=1*, by **H2)**, **H4)** and $Y_{t,1} \leq Y_{t,2}$, we have

$$f_2(t, Y_{t,2}, Z_{t,2}) - f(t, Y_{t,1}, Z_{t,1}) \geq f(t, Y_{t,2}, Z_{t,2}) - f(t, Y_{t,1}, Z_{t,1}) \geq -C(Y_{t,2} - Y_{t,1}) - C|Z_{t,2} - Z_{t,1}|$$
(9)

Then we have

$$f_2(t, Y_{t,2}, Z_{t,2}) + C(Y_{t,2} - Y_{t,1}) + C|Z_{t,2} - Z_{t,1}| \geq f(t, Y_{t,1}, Z_{t,1}) \geq f_1(t, Y_{t,1}, Z_{t,1})$$

Obviously, $f(t, Y_{t,1}, Z_{t,1}) \in M_T^2(0,T,R\ \ )$. By Lemma 2.2, it follows that DRBDSDE (8) has a unique solution $(y_t^1, z_t^1, k_t^{1,+}, k_t^{1,-})$. Then from Theorem 2.3, we have

$\mathrm{P - a.s.}, \mathrm{Y}_{t,1} \leq \mathrm{y}_t^1 \leq \mathrm{Y}_{t,2}, \ \ \forall t \in [0,T].$

For *n=2*, since **H2)**, **H4)** and $Y_{t,1} \leq y_t^1 \leq Y_{t,2}$, then by the similar argument as (9), we can deduce
$f_2(t,Y_{t,2},Z_{t,2}) + C(Y_{t,2} - y_t^1) + C|Z_{t,2} - z_t^1| \geq f(t,y_t^1,z_t^1) \geq f(t,Y_{t,1},Z_{t,1}) - C(y_t^1 - Y_{t,1}) - C|z_t^1 - Z_{t,1}|$

Hence $f(t,y_t^1,z_t^1) \in M_T^2(0,T,R\ )$. From Lemma 2.2, it follows that DRBDSDE (8) has a unique solution $(y_t^2,z_t^2,k_t^{2,+},k_t^{2,-})$. By theorem 2.3, we have
$\mathrm{P-a.s.}, y_t^1 \leq y_t^1 \leq Y_{t,2},\ \ k_t^{2,+} \leq k_t^{1,+},\ \ k_t^{1,-} \leq k_t^{2,-} \forall t \in [0,T].$
Finally, For *n>2*, supposing $f(t,y_t^{n-1},z_t^{n-1}) \in M_T^2(0,T,R\ )$, DRBDSDE (8) has a unique solution $(y_t^n,z_t^n,k_t^{n,+},k_t^{n,-})$ and for all
$t \in [0,T],\ Y_{t,1} \leq y_t^{n-1} \leq y_t^n \leq Y_{t,2},\ \ k_t^{n,+} \leq k_t^{n-1,+},\ \ k_t^{n-1,-} \leq k_t^{n,-}$ a.s., we consider the following DRBDSDE

$$y_t^{n+1} = \xi + \int_t^T \left(f(s,y_s^n,z_s^n) - C(y_s^{n+1} - y_s^n) - C(z_s^{n+1} - z_s^n)\right)ds + \int_t^T g(s,y_s^n,z_s^n)dB_s$$
$$+ k_T^{n+1,+} - k_t^{n+1,+} - k_T^{n+1,-} + k_t^{n+1,-} - \int_t^T z_s^{n+1}dW_s,$$
$$\forall t \in [0,T]. \qquad (10)$$

Then by the similar argument as the case *n=2*, we have
$f_2(t,Y_{t,2},Z_{t,2}) + C(Y_{t,2} - y_t^n) + C|Z_{t,2} - z_t^n| \geq f(t,y_t^n,z_t^n)$
$\geq f(t,y_t^{n-1},z_t^{n-1}) - C(y_t^n - y_t^{n-1}) - C|z_t^n - z_t^{n-1}|.$

Consequently, $f(t,y_t^n,z_t^n) \in M_T^2(0,T,R\ )$. By Theorem 2.3 again, DRBDSDE (10) has a unique solution $(y_t^{n+1},z_t^{n+1},k_t^{n+1,+},k_t^{n+1,-})$. By theorem 2.3, we get
$\mathrm{P-a.s.}, Y_{t,1} \leq y_t^n \leq y_t^{n+1} \leq Y_{t,2},\ \ k_t^{n+1,+} \leq k_t^{n,+},\ \ k_t^{n,-} \leq k_t^{n+1,-} \forall t \in [0,T]$

The proof is complete .

<u>Proposition 3.3</u>*There exists a constant L>0, such that*

$$sup_n E\left[sup_{0\leq t\leq T}|y_t^n|^2 + \int_0^T |z_t^n|^2 dt\right] < L.$$

<u>Proof</u> By Proposition 3.2, we have

$$sup_n E[sup_{0\leq t\leq T}|y_t^n|^2] \leq E\left[sup_{0\leq t\leq T}\left|Y_{t,1}\right|^2\right] + E\left[sup_{0\leq t\leq T}\left|Y_{t,2}\right|^2\right] < \infty$$

and

$$sup_n E\left[\left|k_T^{n,+}\right|^2\right] \leq E\left[\left|k_T^{1,+}\right|^2\right] < \infty$$

By the similar argument as (9), we can deduce
$f_2(t,Y_{t,2},Z_{t,2}) + C(Y_{t,2} - y_t^n) + C|Z_{t,2} - z_t^n| \geq f(t,y_t^n,z_t^n)$
$\geq f_1(t,Y_{t,1},Z_{t,1}) - C(y_t^n - Y_{t,1}) - C|z_t^n - Z_{t,1}| \qquad (11)$

For simplicity, we set

$$f^n(t,y_t^n,z_t^n) = f(t,y_t^{n-1},z_t^{n-1}) - C(y_t^n - y_t^{n-1}) - C|z_t^n - z_t^{n-1}| \qquad (12)$$

Then by (11) and (12), we get

$$|f^n(t,y_t^n,z_t^n)| \leq |f(t,y_t^{n-1},z_t^{n-1})| + C|y_t^n - y_t^{n-1}| + C|z_t^n - z_t^{n-1}|$$
$$\leq \left|f_1(t,Y_{t,1},Z_{t,1}) - C(y_t^{n-1} - Y_{t,1}) - C|z_t^{n-1} - Z_{t,1}|\right|$$

$$\begin{aligned}
&\leq \left|f_2(t, Y_{t,2}, Z_{t,2}) + C(Y_{t,2} - y_t^{n-1}) + C|Z_{t,2} - z_t^{n-1}|\right| \\
&\quad + C|y_t^n - y_t^{n-1}| + C|z_t^n - z_t^{n-1}| \\
&\leq \sum_{i=1}^{2} [f_i(t, Y_{t,i}, Z_{t,i}) + C(|Y_{t,i}| + |Z_{t,i}|)] \\
&+ 3C(|y_t^{n-1}| + |z_t^{n-1}|) + 3C(|y_t^n| + |z_t^n|) \qquad (13)
\end{aligned}$$

Applying Itô formula to $(y_t^n)^2$, we have

$$\begin{aligned}
(y_0^n)^2 = \xi^2 &+ 2\int_0^T y_t^n f^n(t, y_t^n, z_t^n)\, dt + 2\int_0^T y_t^n g(t, y_t^n, z_t^n)\, dB_t \\
&+ 2\int_0^T y_t^n d(k_t^{n,+} - k_t^{n,-}) - 2\int_0^T y_t^n z_t^n\, dW_t \\
&+ \int_0^T |g(t, y_t^n, z_t^n)|^2\, dt - \int_0^T |z_t^n|^2\, dt \qquad (14)
\end{aligned}$$

using the identity $\int_0^T (y_t^n - L_t) dk_t^{n,+} = \int_0^T (U_t - y_t^n) dk_t^{n,-} = 0$ and the assumption **H5),** we obtain

$$\begin{aligned}
(y_0^n)^2 + (1-\alpha)\int_0^T |z_t^n|^2\, dt &= \xi^2 + 2\int_0^T y_t^n f^n(t, y_t^n, z_t^n)\, dt + C\int_0^T |y_t^n|^2\, dt \\
+2\int_0^T L_t dk_t^{n,+} - 2\int_0^T U_t d\, k_t^{n,-} &+ 2\int_0^T y_t^n g(t, y_t^n, z_t^n)\, dB_t - 2\int_0^T y_t^n z_t^n\, dW_t. \qquad (15)
\end{aligned}$$

So taking expectation, using the inequality $\beta a^2 + \frac{b^2}{\beta} \geq 2ab,\ \beta > 0$ and the fact $\left(\int_0^T y_t^n z_t^n\, dW_t\right)$ and $\left(\int_0^T y_t^n g(t, y_t^n, z_t^n)\, dB_t\right)$ are a martingales. we deduce

$$\begin{aligned}
(1-\alpha)E\int_0^T |z_t^n|^2\, dt &\leq E\xi^2 + 2E\int_0^T y_t^n f^n(t, y_t^n, z_t^n)\, dt + CE\int_0^T |y_t^n|^2\, dt \\
&\quad + 2E\int_0^T L_t dk_t^{n,+} - 2E\int_0^T U_t d\, k_t^{n,-} \\
&\leq E\xi^2 + 2E\int_0^T y_t^n f^n(t, y_t^n, z_t^n)\, dt + CE\int_0^T |y_t^n|^2\, dt \\
&\quad + E[sup_{0\leq t\leq T}|L_t|^2] + E\left[\left|k_T^{n,+}\right|^2\right] + \beta E[sup_{0\leq t\leq T}|U_t|^2] + \frac{1}{\beta}E[|k_T^{n,-}|^2] \\
&\leq C_1 + 2E\int_0^T y_t^n f^n(t, y_t^n, z_t^n)\, dt + CE\int_0^T |y_t^n|^2\, dt + \frac{1}{\beta}E[|k_T^{n,-}|^2] \qquad (16)
\end{aligned}$$

From $k_T^{n,-} = \xi - y_0^n + \int_0^T f^n(s, y_s^n, z_s^n) ds + \int_0^T g(s, y_s^n, z_s^n) dB_s + k_T^{n,+} - \int_t^T z_s^n dW_s$, **H5)** and (13), we can deduce

$$\begin{aligned}
E[|k_T^{n,-}|^2] &\leq C_2\left[E\xi^2 + E\int_0^T |f^n(t, y_t^n, z_t^n)|^2\, dt + CE\int_0^T |y_t^n|^2\, dt\right] \\
&\leq C_3 + C_4 E\int_0^T (|z_t^n|^2 + |z_t^{n-1}|^2)\, dt \qquad (17)
\end{aligned}$$

Substitute it into (16) and set $\beta = 8C_4$, then

$$(1-\alpha)E\int_0^T |z_t^n|^2\, dt \leq C_5 + 2\, E\int_0^T |y_t^n f^n(t, y_t^n, z_t^n)|^2\, dt + CE\int_0^T |y_t^n|^2\, dt$$

$$+\frac{1}{8}E\int_0^T(|z_t^n|^2+|z_t^{n-1}|^2)\,dt. \qquad (18)$$

Substitute (13) into it, we deduce that

$$(1-\alpha)E\int_0^T|z_t^n|^2\,dt\le C_6+\frac{1}{4}E\int_0^T(|z_t^n|^2+|z_t^{n-1}|^2)\,dt$$

Then we have

$$(1-\alpha)E\int_0^T|z_t^n|^2\,dt\le\frac{4}{3}C_6+\frac{1}{3}E\int_0^T(|z_t^{n-1}|^2)\,dt$$

Hence we can deduce

$$sup_n\left[E\int_0^T|z_t^n|^2\,dt\right]<\infty.$$

Proposition 3.4 There exist processes $(y_t,z_t,k_t^+,k_t^-)\in S_T^2([0,T],R)\times M_T^2(0,T,R^d)\times S_T^2([0,T],R)\times S_T^2([0,T],R)$ such that :

$$\lim_{n\to\infty}E\left[sup_{0\le t\le T}|y_t^n-y_t|^2+\int_0^T|z_t^n-z_t|^2dt+sup_{0\le t\le T}\left|k_t^{n,+}-k_t^+\right|^2\right.$$
$$\left.+\,sup_{0\le t\le T}|k_t^{n,-}-k_t^-|^2\right]=0.$$

Proof By proposition 3.2, it follows that there exists a triple of processes (y_t,k_t^+,k_t^-) such that

$$\mathrm{P-a.s.}\ \ y_t^n\nearrow y_t,\ \ k_t^{n,+}\searrow k_t^+,\ \ k_t^{n,-}\nearrow k_t^-,\ \ \forall t\in[0,T],\ \ \text{as } n\to\infty,$$

And

$$E[sup_{0\le t\le T}|y_t|^2]<\infty.$$

By the dominated convergence theorem, we get

$$\lim_{n\to\infty}E\int_0^T|y_t^n-y_t|^2dt=0 \qquad (19)$$

Coming back to (13) and by Proposition 3.3, we deduce

$$sup_n\left[E\int_0^T|f^n(t,y_t^n,z_t^n)|^2dt\right]<\infty. \qquad (20)$$

By Itô formula applied to $(y_t^n-y_t^m)^2$ $for\ t\in[0,T]$, taking expectation and using the fact $\left(\int_0^T(y_t^n-y_t^m)(z_t^n-z_t^m)\,dW_t\right)$ and $\left(\int_0^T(y_t^n-y_t^m)(g(t,y_t^n,z_t^n)-g(t,y_t^m,z_t^m)\,dB_t\right)$ are a martingales, we get

$$E(y_0^n-y_0^m)^2+E\int_0^T|z_t^n-z_t^m|^2dt$$
$$=2E\int_0^T(y_t^n-y_t^m)\big(f^n(t,y_t^n,z_t^n)-f^m(t,y_t^m,z_t^m)\big)dt$$
$$+E\int_0^T|g(t,y_t^n,z_t^n)-g(t,y_t^m,z_t^m)|^2dt$$
$$+2\int_0^T(y_t^n-y_t^m)d\big(k_t^{n,+}-k_t^{m,+}-k_t^{n,-}+k_t^{m,-}\big) \qquad (21)$$

We have from $t\in[0,T]$,

$$\int_t^T (y_s^n - y_s^m)\, d\left(k_s^{n,+} - k_s^{m,+} - k_s^{n,-} + k_s^{m,-}\right)$$
$$= \int_t^T (y_s^n - y_s^m)\, d\left(k_s^{n,+} - k_s^{m,+}\right) - \int_t^T (y_s^n - y_s^m)\, d\left(k_s^{n,-} - k_s^{m,-}\right)$$
$$= \int_t^T (L_s - y_s^m)\, dk_s^{n,+} - \int_t^T (y_s^n - L_s)\, dk_s^{m,+} - \int_t^T (U_s - y_s^m)\, dk_s^{n,-}$$
$$- \int_t^T (y_s^n - U_s)\, dk_s^{m,-} \le 0 \qquad (22)$$

Applied assumption **H5)** and (22), in (21) we have

$$(y_0^n - y_0^m)^2 + (1-\alpha)E\int_0^T |z_t^n - z_t^m|^2 dt$$
$$= 2E\int_0^T (y_t^n - y_t^m)\left(f^n(t, y_t^n, z_t^n) - f^m(t, y_t^m, z_t^m)\right) dt$$
$$+ CE\int_0^T |y_t^n - y_t^m|^2 dt \qquad (23)$$

Hence by the Cauchy-Schwarz inequality, we obtain

$$(1-\alpha)E\int_0^T |z_t^n - z_t^m|^2 dt$$
$$\le 2\left[E\int_0^T (y_t^n - y_t^m)^2 dt\right]^{1/2}\left[E\int_0^T \left(f^n(t, y_t^n, z_t^n)\right.\right.$$
$$\left.\left. - f^m(t, y_t^m, z_t^m)\right)^2 dt\right]^{1/2} + CE\int_0^T |y_t^n - y_t^m|^2 dt \qquad (24)$$

Then from (19) and (20), it follows that $(z_t^n)_n$ is a Cauchy sequance in $M_T^2(0, T, R^d)$. Hence there exist a prosses (z_t) in $M_T^2(0, T, R^d)$ such that

$$\lim_{n\to\infty} E\int_0^T \left|z_t^n - z_t\right|^2 dt = 0. \qquad (25)$$

Applying Itô formula to $(y_s^n - y_s^m)^2, \quad for\ s \in [t, T]$ and by (22), assumption **H5)**, we have

$$(y_t^n - y_t^m)^2 + (1-\alpha)\int_t^T |z_s^n - z_s^m|^2 ds$$
$$\le 2\int_t^T (y_s^n - y_s^m)\left(f^n(s, y_s^n, z_s^n) - f^m(s, y_s^m, z_s^m)\right) ds$$
$$+ C\int_t^T |y_s^n - y_s^m|^2 ds$$
$$+ 2\int_s^T (y_s^n - y_s^m)\left(g(s, y_s^n, z_s^n) - g(s, y_s^m, z_s^m)\right) dB_s$$
$$- 2\int_t^T (y_s^n - y_s^m)(z_s^n - z_s^m)\, dW_s \qquad (26)$$

Taking the supremum and expectation, using the Cauchy-Schwarz inequality

$$
\begin{aligned}
&E[sup_{0\le t\le T}(y_t^n-y_t^m)^2]\\
&\qquad\le 2\left[E\int_0^T(y_s^n-y_s^m)^2ds\right]^{1/2}\left[E\int_0^T\big(f^n(s,y_s^n,z_s^n)\right.\\
&\qquad\left.-f^m(s,y_s^m,z_s^m)\big)^2ds\right]^{1/2}+CE\int_0^T|y_s^n-y_s^m|^2ds\\
&\qquad+2E\left[sup_{0\le t\le T}\left|\int_t^T(y_s^n-y_s^m)\big(g(s,y_s^n,z_s^n)-g(s,y_s^m,z_s^m)\big)dB_s\right|\right]\\
&\qquad+2E\left[sup_{0\le t\le T}\left|\int_t^T(y_s^n-y_s^m)(z_s^n-z_s^m)dW_s\right|\right]
\end{aligned}
$$

By (20), **H5),** we have

$$
\begin{aligned}
&E[sup_{0\le t\le T}(y_t^n-y_t^m)^2]\\
&\qquad\le C_7\left[E\int_0^T(y_s^n-y_s^m)^2ds\right]^{1/2}+C\int_0^T|y_s^n-y_s^m|^2ds\\
&\qquad+C_8E\left[\int_0^T(y_s^n-y_s^m)^2\big(g(s,y_s^n,z_s^n)-g(s,y_s^m,z_s^m)\big)^2ds\right]^{1/2}\\
&\qquad+C_9E\left[\int_0^T(y_s^n-y_s^m)^2(z_s^n-z_s^m)^2ds\right]^{1/2}\\
&\qquad\le C_7\left[E\int_0^T(y_s^n-y_s^m)^2ds\right]^{1/2}+CE\int_0^T|y_s^n-y_s^m|^2ds\\
&\qquad+C_8E\left[\int_0^T(y_s^n-y_s^m)^2(C|y_s^n-y_s^m|^2+\alpha|z_s^n-z_s^m|^2)\ \ ds\right]^{1/2}\\
&\qquad+C_9E\left[\int_0^T(y_s^n-y_s^m)^2(z_s^n-z_s^m)^2ds\right]^{1/2}\qquad(27)
\end{aligned}
$$

By the Burkolder-Davis-Gundy inequality, the inequality $\frac{1}{4}a^2+b^2\ge ab$, we get

$$
\begin{aligned}
&E[sup_{0\le t\le T}(y_t^n-y_t^m)^2]\\
&\qquad\le C_{10}\left[E\int_0^T(y_s^n-y_s^m)^2ds\right]^{1/2}\\
&\qquad+C_{11}E\left[sup_{0\le t\le T}|y_s^n-y_s^m|\left[\int_0^T(1+\alpha)(z_s^n-z_s^m)^2ds\right]\right]^{1/2}\\
&\qquad\le C_{10}\left[E\int_0^T(y_s^n-y_s^m)^2ds\right]^{1/2}+\frac{1}{4}E[sup_{0\le t\le T}|y_s^n-y_s^m|^2]\\
&\qquad+C_{12}\ \ E\int_0^T|z_s^n-z_s^m|^2dt.\qquad(28)
\end{aligned}
$$

Then from (19) and (25) we deduce

$$\lim_{n,m\to\infty}E[sup_{0\le t\le T}|y_t^n-y_t^m|^2]=0\qquad(29)$$

Obviously, the process $y_t\in S^2(0,T,\mathbf{R})$.

Coming back to (12), by **H3)**, (19) and (25), it follows that there exists a subsequance such that

$$dt \times dP - a.s., \qquad \lim_{n\to\infty}\left(f^n(t,y_t^n,z_t^n) - f(t,y_t,z_t)\right) = 0. \qquad (30)$$

For any $t \in [0,T]$, we have

$$\begin{aligned} k_t^{n,+} - k_t^{m,+} &= (y_t^n - y_t^m) - (y_0^n - y_0^m) + \int_0^t \left(f^n(s,y_s^n,z_s^n) - f^m(s,y_s^m,z_s^m)\right)ds \\ &\quad + \int_0^t \left(g(s,y_s^n,z_s^n) - g(s,y_s^m,z_s^m)\right)dB_s + (k_t^{n,-} - k_t^{m,-}) \\ &\quad - \int_0^t (z_t^n - z_t^m)\ dW_s \qquad (31) \end{aligned}$$

Then by Proposition 3.2, we can deduce

$$\begin{aligned} \left|k_t^{n,+} - k_t^{m+}\right| &\le |y_t^n - y_t^m| + \int_0^t |f^n(s,y_s^n,z_s^n) - f^m(s,y_s^m,z_s^m)|ds \\ &\quad + \left|\int_0^t \left(g(s,y_s^n,z_s^n) - g(s,y_s^m,z_s^m)\right)dB_s\right| + \left|\int_0^t (z_t^n - z_t^m)\ dW_s\right| \qquad (32) \end{aligned}$$

Hence we obtain

$$\begin{aligned} &E\left[sup_{0\le t\le T}\left|k_t^{n,+} - k_t^{m,+}\right|^2\right] \\ &\quad \le L\Bigg[E[sup_{0\le t\le T}]|y_t^n - y_t^m|^2 \\ &\quad + E\int_0^T \left(f^n(s,y_s^n,z_s^n) - f^m(s,y_s^m,z_s^m)\right)^2 ds \\ &\quad + E\int_0^T |g(s,y_s^n,z_s^n) - g(s,y_s^m,z_s^m)|^2 ds + E\int_0^t |z_t^n - z_t^m|^2 ds\Bigg] \qquad (33) \end{aligned}$$

From (20), (25)-(30), we deduce that there exists a subsequance , such that

$$\lim_{n,m\to\infty} E\left[sup_{0\le\le t\le T}\left|k_t^{n,+} - k_t^{m,+}\right|^2\right] = 0.$$

Using the same method, we also prove that there exists a subsequence, such that

$$\lim_{n,m\to\infty} E[sup_{0\le\le t\le T}|k_t^{n,-} - k_t^{m,-}|^2] = 0.$$

Obviously, $\left(k_t^{n,+}, k_t^{n,-}\right) \in S^2(0,T,\mathrm{R}^+) \times S^2(0,T,\mathrm{R}^+)$, and $k_0^+ = k_0^- = 0$.

Now let us return to the proof of Theorem 3.1.

By Proposition 3.4 and the result of Saisho (1987) [13], it follows that there exists a subsequence, such that $P - a.s.$,

$$\lim_{n\to\infty}\int_0^T (y_t^n - L_t)dk_t^{n,+} = \int_0^T (y_t \ - L_t)dk_t^+,$$

$$\lim_{n\to\infty}\int_0^T (U_t - y_t^n)dk_t^{n,-} = \int_0^T (U_t - y_t \ )dk_t^-.$$

By the identity $\int_0^T (y_t^n - L_t)dk_t^{n,+} = \int_0^T (U_t - y_t^n)dk_t^{n,-} = 0$, we have

$$P - a.s., \int_0^T (y_t \ - L_t)dk_t^+ = \int_0^T (U_t - y_t \ )dk_t^- = 0.$$

From (25) we get,

$$\lim_{n\to\to\infty\infty}\left[sup_{0\leq t\leq T}\left|\left|\int_t^T g(s,y_s^n,z_s^n)dB_s-\int_t^T g(s,y_s\ ,z_s\ )dB_s\right|\right|\right]=0$$

$$\lim_{n\to\infty\infty}\left[sup_{0\leq t\leq T}\left|\left|\int_t^T z_s^n dW_s-\int_t^T z_s\ dW_s\right|\right|\right]=0$$

in probability. So, there exist a subsequence, such that the convergence is P-almost surely. Since **H2)**, **H4)** and $Y_{t,1}\leq y_t\leq Y_{t,2}$, we deduce from the similar argument as (11) that

$$f(t,y_t,z_t)\in M^2([0,T],\mathbf{R}).$$

In view of (20) and (30), then by the dominanted convergence theorem, passing to a subsequence, we have

$$P-a.s.,\lim_{n\to\infty}\int_0^T|f^n(t,y_t^n,z_t^n)-f(t,y_t,z_t)|dt=0.$$

Hence, passing to a subsequence on both sides of DRBDSDEs (8), we get

$$y_t\ =\xi+\int_t^T f(s,y_s\ ,z_s\ )ds+\int_t^T g(s,y_s\ ,z_s\ )dB_s+k_T^+-k_t^+-k_T^-+k_t^-$$
$$-\int_t^T z_s\ dW_s,\qquad \forall t\in[0,T].$$

From above argument, it follows that $(y_t,z_t,k_t^+,k_t^-)$is a solution of DRBDSDE (3) under the assumptions **H2)-H7).** The proof of Theorem 3.1 is complete.

4 COCLUSION

In this paper, we established two new results. The first is a comparison theorem for reflected backward doubly stochastic differential equations with two barriers, in the case where the generator f is Lipschitz continuous in y and z. The second result concerns the existence of solutions for reflected backward doubly stochastic differential equations with two barriers (RBDSDEs), under weak assumptions on the generator f, namely left-Lipschitz continuity in y and Lipschitz continuity in z. Finally, several perspectives for future research are discussed.

In particular, it would be interesting to prove existence results for the following problems:

- Reflected backward doubly stochastic differential equations with two barriers when the generator exhibits quadratic growth in z.
- Reflected mean-field backward doubly stochastic differential equations with two barriers.

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