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An Improved Block Pulse Function Method for Solving Integral Equations

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Abstract:
The article proposes a novel method for solving the first class of Volterra integral problems. It is based on a new modification of Block Pulse Functions (KBPFs). The application of (KBPFs) and their operational integration matrix facilitates the transformation of systems based on integral equations into linear systems of algebraic equations. We also examined errors created by the application of these modifications, and the numerical examples illustrate the significant efficiency and precision of the presented improvement.

Keywords: Volterra integral equations, improved block-pulse functions (KBPFs), Operational Matrix of Integration, Error Analysis.

1. Introduction

Several kinds of mathematical problems, such as harmonic continuation, the backward heat equation, numerical inversion of the Laplace transform, and numerical differentiation, can be formulated as first-kind Fredholm integral equations. Recently, numerous orthonormal basis functions, such as Fourier functions, wavelets ,sincfunctions , rationalized Haar functions , and block-pulse functions (BPFs) , have been utilized to estimate the solutions of these integral equations. But the most attractive solution for tackling linear and nonlinear integral equations may be the Block Pulse Functions (BPFs). The Block Pulse Functions (BPFs) are extensively employed; nonetheless, their convergence appears to be inadequate. Numerous published research studies have aimed to accelerate the convergence speed of Block-Pulse Functions (BPFs) by various methodologies, including hybrid BPFs and Improved Block-Pulse Functions (IBPFs) . This research study presents Modified Block Pulse Functions (KBPFs) as a superior alternative to the IBPFs modification suggested by Farshid Mirzaee , owing to its improved accuracy. When K equals 1, the alteration of the IBPFs is obtained as a particular case. In this article we provide a new category of Modified Block Pulse Functions (KBPFs) and clarify their characteristics, thereafter utilizing KBPFs to approximate solutions of Volterra Integral Equations. We juxtaposed the findings of article with the new data to demonstrate the improvements in accuracy and efficiency achieved by the suggested technique. We calculated the absolute error of the method and subsequently implemented it on two examples. The application of KBPFs, along with their operational integration matrix, facilitates the transformation of systems founded on integral equations into linear systems of algebraic equations. We evaluated the inaccuracies resulting from the execution of this adjustment, and the numerical examples demonstrate the considerable efficiency and accuracy of the proposed enhancements.

2. Definition

The k-IBPFs are defined piecewise over the interval $[0,1)$ such that:

$$\varphi_0(x)=\begin{cases}1 & x\in\left[0,\dfrac{h}{2k}\right),\\ 0 & other\ wise\end{cases}$$
$$\varphi_n(x)=\begin{cases}1 & x\in\left[(n-1)h+\frac{h}{2k},1\right),\\ 0 & other\ wise\end{cases} \tag{1}$$
$$\varphi_i(x)=\begin{cases}1 & x\in\left[(i-1)h+\dfrac{h}{2k},ih+\dfrac{h}{2k}\right),\\ 0 & other\ wise\end{cases}$$
$$for\quad i=1,2,\dots,n-1$$

where n,k is an arbitrary positive integers and $h=\frac{1}{n}$

2.1 Properties

- **Orthogonality**:

$$\int_0^1\varphi_i(x)\varphi_j(x)dx=\begin{cases}\dfrac{h}{2k}, i=j=0,\\ h, i=j\in\{1,2,\dots,n-1\},\\ \left(\dfrac{2k-1}{2k}\right)h, i=j=n.\\ 0, i\neq j.\end{cases}$$

- **Disjoint**:

$$\varphi_i(x)\varphi_j(x)=\begin{cases}\varphi_i(x), i=j,\\ 0, otherwise.\end{cases}$$

- **Completeness**: For every $f(x)\in L^2([0,1))$ when n go to infinity Parseval identity holds:

$$\int_0^1 f^2(x)dx=\sum_{i=0}^{\infty} f_i^2\,\|\varphi_i(x)\|^2.$$

Where

$$f_i = \begin{cases} 2kn\int_0^{\frac{h}{2k}} f(t)\,dt & \text{if } i=0, \quad t\in[0,\frac{h}{2k}), \\ n\int_{(i-1)h+\frac{h}{2k}}^{ih+\frac{h}{2k}} f(t)\,dt & \text{if } t\in[(i-1)h+\frac{h}{2k}, ih+\frac{h}{2k}), \\ \frac{2kn}{2k-1}\int_{(n-1)h+\frac{h}{2k}}^{1} f(t)\,dt & \text{if } i=n, \quad t\in[(n-1)h+\frac{h}{2k},1). \end{cases}$$

The set of KBPFs may be written as a $(n+1)$-vector $\Phi(x)$:

$$\Phi(x) = [\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)]^T,$$

Where $x \in [0,1)$. From the above representation and disjointness property, it follows:

$$\Phi(x)\Phi^T(x) = \begin{bmatrix} \varphi_0(x) & 0 & \cdots & 0 \\ 0 & \varphi_1(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_n(x) \end{bmatrix}$$

Now suppose that V be can $(n+1)$-vector. Hence by using Eq. [eq:7] we obtain

$$\Phi(x)\Phi^T(x)V = \tilde{V}\Phi(x).$$

where $\tilde{V} = diag(V)$.

Now suppose that $B(n+1)\times(n+1)$ matrix

$$\Phi^T(x)B\Phi(x) = \hat{B}^T\Phi(x).$$

where $\hat{B}$ is an $(n+1)$-vector with elements equal to the diagonal entries of matrix B.

2.2 Function Approximation

A continues function $f(x) \in L^2([0,1))$ may be expanded by the KBPFs as

$$f(x) \approx f_n(x) = \sum_{i=0}^{n} f_i\,\varphi_i(x) = F^T\Phi(x).$$

where F is an $(n+1)\times 1$ vector given by:

$$F = [f_0, f_1, f_2, f_3, \ldots f_n]^T.$$

and f_i is obtained as Eq. [eq:5].

Similarly a function of two variables, $k(x,y) \in L^2([0,1)\times[0,1))$ can be approximated by KBPFs as follows:

$$k(x,y) \approx \Phi^T(x)K\Phi(y).$$

where $\Phi(x)$ and $\Phi(y)$ are KBPFs vector of dimension $(n+1)$. Eq. [eq:6]

$$k_{ij} = \int_0^1\int_0^1 \varphi_i(x)k(x,y)\varphi_j(y)\,dx\,dy.$$

and $K=[k_{ij}]$ is the $(n+1)\times(n+1)$ KBPFs coefficients matrix of $k(x,y)$ is given by the following expression:

$$k_{ij}=\begin{cases}(2nk)^2\int_0^{\frac{1}{2nk}}\int_0^{\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & i=j=0,\\ 2kn^2\int_0^{\frac{1}{2nk}}\int_{(j-1)h+\frac{1}{2nk}}^{jh+\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & i=0,\ j=1,2,\dots,n-1,\\ 2kn^2\int_{(i-1)h+\frac{1}{2nk}}^{ih+\frac{1}{2nk}}\int_0^{\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & i=1,2,\dots,n-1,\ j=0,\\ n^2\int_{(i-1)h+\frac{1}{2nk}}^{ih+\frac{1}{2nk}}\int_{(j-1)h+\frac{1}{2nk}}^{jh+\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & i,j=1,2,\dots,n-1,\\ \frac{2n^2k}{2k-1}\int_{(j-1)h+\frac{1}{2nk}}^{jh+\frac{1}{2nk}}\int_{(n-1)h+\frac{1}{2nk}}^{1}k\,(x,y)\,dx\,dy, & i=n,\ j=1,\dots,n-1,\\ \frac{2n^2k}{2k-1}\int_{(n-1)h+\frac{1}{2nk}}^{1}\int_{(i-1)h+\frac{1}{2nk}}^{ih+\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & j=n,\ i=1,\dots,n-1,\\ \left(\frac{2nk}{2k-1}\right)2nk\int_0^{\frac{1}{2nk}}\int_{(n-1)h+\frac{1}{2nk}}^{1}k\,(x,y)\,dx\,dy, & i=n,\ j=0,\\ \left(\frac{2nk}{2k-1}\right)2nk\int_{(n-1)h+\frac{1}{2nk}}^{1}\int_0^{\frac{1}{2nk}}k\,(x,y)\,dx\,dy, & i=0,\ j=n,\\ \left(\frac{2nk}{2k-1}\right)^2\int_{(n-1)h+\frac{1}{2nk}}^{1}\int_{(n-1)h+\frac{1}{2nk}}^{1}k\,(x,y)\,dx\,dy, & i=j=n.\end{cases}$$

3. Operational Matrix of Integration

In this section of the article, we search for the matrix $P$where

$$\int_0^x \Phi\,(t)dt=P\Phi(x)$$

. We have

$$\int_0^x \varphi_0\,(t)dt=\begin{cases}x,x\in\left[0,\frac{h}{2k}\right),\\ \frac{h}{2k},otherwise.\end{cases}$$

we can approximate it

$$\int_0^x \varphi_0(t)dt = \left[\frac{h}{4k}, \frac{h}{2k}, \frac{h}{2k}, \dots, \frac{h}{2k}\right] \Phi(x).$$

$$\int_0^x \varphi_n(t)dt = \begin{cases} 0, x < (n-1)h + \dfrac{h}{2k}, \\ x - (n-1)h - \dfrac{h}{2k}, x \geq (n-1)h + \dfrac{h}{2k}. \end{cases}$$

we can approximate it

$$\int_0^x \varphi_n(t)dt = \left[0,0,0, \dots, \left(\frac{2k-1}{4k}\right)h\right] \Phi(x),$$

For $i = 1,2, \dots, n-1$.

$$\int_0^x \varphi_i(t)dt = \begin{cases} 0, & x < (i-1)h + \dfrac{h}{2k} \\ x - (n-1)h - \dfrac{h}{2k}, & x \in \left[(i-1)h + \dfrac{h}{2k}, ih + \dfrac{h}{2k}\right), \\ h, & x \geq ih + \dfrac{h}{2k}, \end{cases}$$

we can approximate it

$$\int_0^x \varphi_i(t)dt = \left[0,0, \dots, \frac{h}{2}, h, \dots, h\right] \Phi(x).$$

Therefore, the integration of the vector $\Phi(x)$ defined in Eq. [eq:6]

$$\int_0^x \Phi(t)dt = P\Phi(x)$$

where

$$P = \frac{h}{4} \begin{bmatrix} \frac{1}{k} & \frac{2}{k} & \frac{2}{k} & \cdots & \frac{2}{k} & \frac{2}{k} \\ 0 & 2 & 4 & \cdots & 4 & 4 \\ 0 & 0 & 2 & \cdots & 4 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 4 \\ 0 & 0 & 0 & \cdots & 0 & 2 - \frac{1}{k} \end{bmatrix}$$

where P called operational matrix of integration.

4. Method for Solving First-Kind Volterra Integral Equations

Consider Volterra integral equation of the first kind of the form:

$$\int_0^x k\,(x,y)u(y)dy = f(x), 0 \leq x \leq 1.$$

where f and k are known functions but u is not. Moreover, $k(t,s) \in L^2([0,1) \times [0,1))$ and $f(t) \in L^2([0,1))$.

Approximating functions f, u, and k with respect to KBPFs gives:

$$f(x) \simeq F^T\Phi(x) = \Phi^T(x)F,$$

$$u(x) \simeq U^T\Phi(x) = \Phi^T(x)U,$$

$$k(x,y) \simeq \Phi^T(x)K\Phi(y),$$

where the vectors F, U, and matrix K are KBPFs coefficients of $f(x), u(x)$, defined by Eq. [eq:5] and $k(t,s)$ defined by Eq. [eq:14] respectively.U is the unknown vector.
Substituting into the equation Eq. [eq:17], we obtain

$$F^T\Phi(x) = \int_0^x \Phi^T\,(x)K\Phi(y)\Phi^T(y)Udy.$$

$$F^T\Phi(x) = \Phi^T(x)K\int_0^x \Phi\,(y)\Phi^T(y)Udy.$$

Using Eq. [eq:8] follows:

$$F^T\Phi(x) = \Phi^T(x)K\int_0^x \widetilde{U}\,\Phi(y)dy.$$

$$F^T\Phi(x) = \Phi^T(x)K\widetilde{U}\int_0^x \Phi\,(y)dy.$$

Using operational matrix P, in Eq. [eq:16] gives.

$$F^T\Phi(x) = \Phi^T(x)K\widetilde{U}P\Phi(x).$$

in which $K\widetilde{U}P$ is an $(n+1) \times (n+1)$ matrix. Eq. [eq:9] follows:

$$\Phi^T(x)K\widetilde{U}P\Phi(x) = \widehat{U}\Phi(x).$$

where $\widehat{U}$ is an $(n+1)$-vector with components equal to the diagonal entries of matrix $K\widetilde{U}P$.

$$F^T\Phi(x) = \widehat{U}^T\Phi(x).$$

$$\widehat{U} = F.$$

$$\widetilde{U}=\frac{h}{4}\begin{bmatrix} \frac{1}{k}u_0k_{00} \\ \frac{2}{k}u_0k_{10}+2u_1k_{11} \\ \frac{2}{k}u_0k_{20}+4u_1k_{21}+2u_2k_{22} \\ \vdots \\ \frac{2}{k}u_0k_{(n-1)0}+4u_1k_{(n-1)2}+\cdots+2u_{n-1}k_{(n-1)(n-1)} \\ \frac{2}{k}u_0k_{n0}+4u_1k_{n1}+\cdots+\left(\frac{2k-1}{k}\right)u_nk_{nn} \end{bmatrix}.$$

which is a linear system

$$\frac{h}{4}\begin{bmatrix} \frac{1}{k}k_{00} & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{k}k_{10} & 2k_{11} & 0 & \cdots & 0 & 0 \\ \frac{2}{k}k_{20} & 4k_{21} & 2k_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2}{k}k_{(n-1)0} & 4k_{(n-1)1} & 4k_{(n-1)2} & \cdots & 2k_{(n-1)(n-1)} & 0 \\ \frac{2}{k}k_{n0} & 4k_{n1} & 4k_{n2} & \cdots & 4k_{n(n-1)} & \frac{2k-1}{k}k_{nn} \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}=\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}.$$

5. Numerical examples

5.1 Example 1

Consider the following integral equation :

$$\int_0^x e^{x+s}u(s)ds=xe^x.$$

with exact solution $u(x)=e^{-x}$ for $0<x<1$,
Table 1 shows the numerical results.

5.2 Example 2

Consider the following integral equation :

$$\int_0^x \cos(x-s)u(s)ds=x\sin(x).$$

with exact solution $u(x)=2\sin(x)$,
Table 3 shows the numerical results for $0\leq x\leq 1$
We will plot the absolute error of the method’s approximations for $k=1, k=2, k=3, k=5$, and $k=8$. We are going to determine the mean absolute error and juxtapose it with the precise solution.

Table 1:Comparison between exact solution and approximate solutions for different K values for Example *1*

x	Exact solution	K=1	K=2	K=3	K=5	K=8	K=20
0.0	1.000000	0.997396	0.998698	0.999132	0.999479	0.999674	0.999870
0.1	0.904837	0.909316	0.901944	0.903282	0.904307	0.904866	0.905411
0.2	0.818731	0.817231	0.821229	0.822448	0.823381	0.823890	0.824387
0.3	0.740818	0.744097	0.747738	0.733620	0.734316	0.734723	0.735141
0.4	0.670320	0.665269	0.667220	0.667968	0.668603	0.668973	0.669354
0.5	0.606531	0.605735	0.607511	0.608192	0.608770	0.609107	0.609454
0.6	0.548812	0.551528	0.547057	0.547868	0.548490	0.548829	0.549160
0.7	0.496585	0.495675	0.498101	0.498840	0.499406	0.499714	0.500016
0.8	0.449329	0.451318	0.453526	0.454199	0.445385	0.445632	0.445886
0.9	0.406570	0.403506	0.404690	0.405143	0.405528	0.405753	0.405984

Table 2:Comparison of approximation methods by mean absolute error for Example 1

Method	$k = 1$	$k = 2$	$k = 3$	$k = 5$	$k = 8$	$k = 20$
Mean Absolute Error	0.008218	0.003877	0.002549	0.001592	0.001015	**0.000255**

Table 3:Comparison between exact solution and approximate solutions for different K values for Example *2*

x	Exact solution	K=1	K=2	K=3	K=5	K=8	K=10
0.0	0.000000	0.005208	0.002604	0.001736	0.001042	0.000651	0.000521
0.1	0.199667	0.189831	0.205991	0.203040	0.200783	0.199553	0.199150
0.2	0.397339	0.400862	0.391254	0.388339	0.386111	0.384898	0.384501
0.3	0.591040	0.582469	0.573040	0.609882	0.608081	0.607027	0.606669
0.4	0.778837	0.792947	0.787717	0.785682	0.783949	0.782933	0.782588
0.5	0.958851	0.961465	0.956555	0.954625	0.952976	0.952008	0.951678
0.6	1.129285	1.121562	1.133749	1.131245	1.129345	1.128316	1.127980
0.7	1.288435	1.290618	1.282696	1.280340	1.278558	1.277595	1.277281
0.8	1.434712	1.427761	1.420337	1.418146	1.448512	1.447764	1.447508
0.9	1.566654	1.577013	1.574137	1.572883	1.571774	1.571109	1.570880

Table 4:Absolute Errors for Different K Values (Example 2)

	K=1	K=2	K=3	K=5	K=8	K=10
Absolute Error	0.007068	0.007625	0.007697	0.006727	0.006947	0.006590

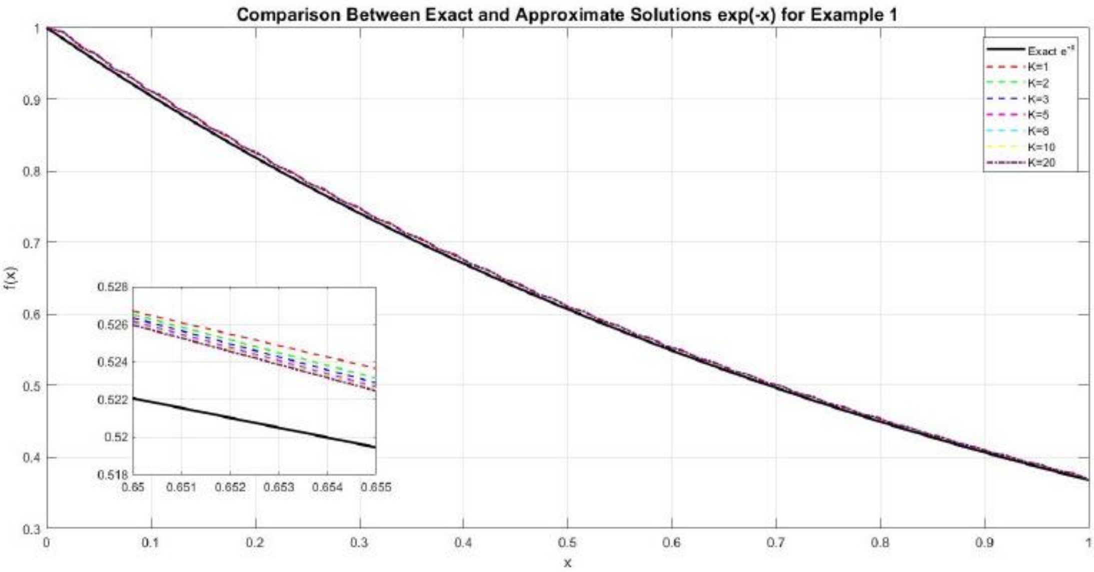

Figure 1: Absolute Error (Log Scale) Between Exact and Approximate Solutions (Example1)

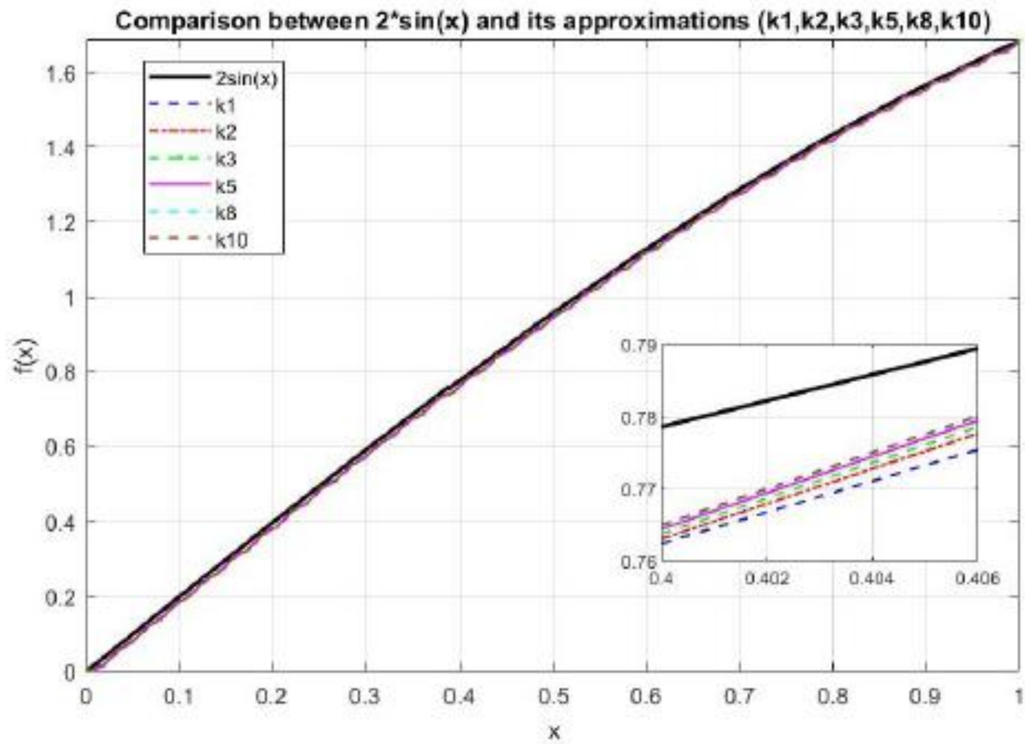

Figure 2: Absolute Error (Log Scale) Between Exact and Approximate Solutions (Example2).

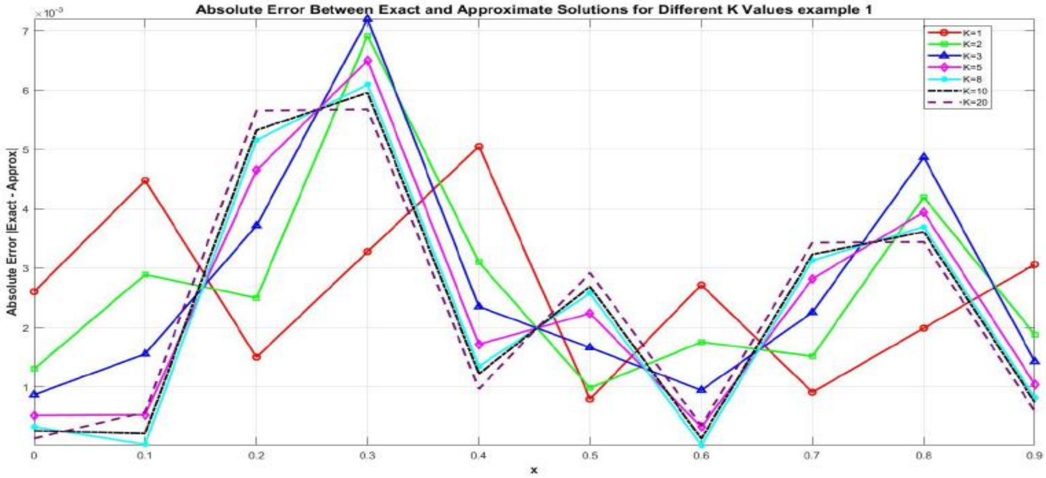

Figure 3: Absolute Error (Log Scale) Between Exact and Approximate Solutions (Example1).

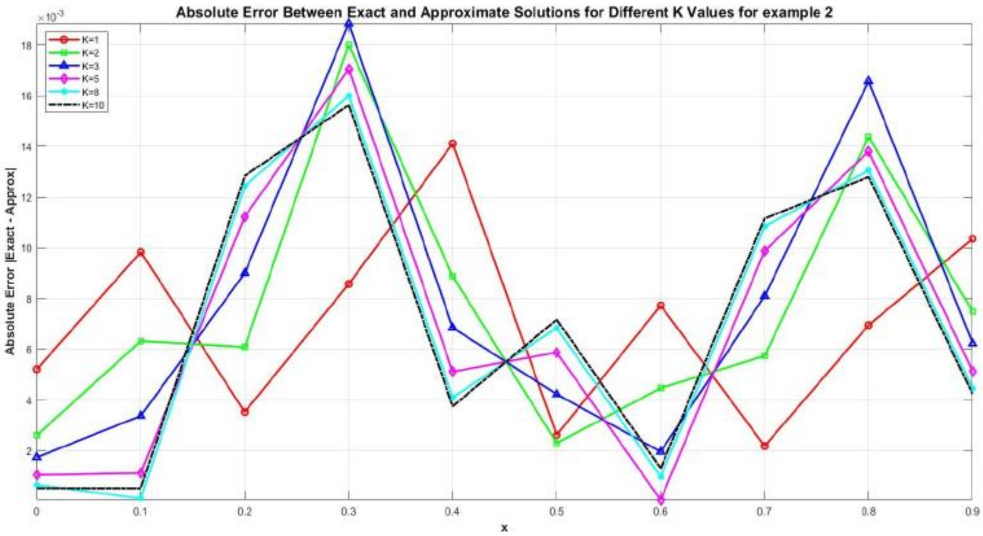

Figure 4: Absolute Error (Log Scale) Between Exact and Approximate Solutions (Example2).

6. Error Analysis

We define

$$\|f(x)\| = \sqrt{\int_0^1 |f(x)|^2\, dx}.$$

and

$$\|g(x)\| = \sqrt{\sum_{i=0}^{n} \int_0^1 \|g_i(x)\|^2\, dx}.$$

where $g(x) = [g_0(x), g_1(x), g_2(x), \ldots, g_n(x)]$.

Theorem 1: Let $f(x) \in L^2([0,1))$ and $f_n(x)$ be the KBPFs expansion of $f(x)$ that is defined as

$$f_n(x) = \sum_{i=0}^{n} f_i\, \varphi_i(x).$$

where $f_i; i = 0,1,\ldots,n$ are defined in Eq. [eq:5]. Then the criterion of this approximation is that the mean square error between $f(x)$ and $f_n(x)$ in the interval $x \in [0,1)$

$$\int_0^1 \left(f(x) - f_n(x)\right)^2 dx,$$

achieves its minimum value and also we have

$$\int_0^1 f^2\,(x)dx = \sum_{i=0}^{\infty} f_i^2\, \|\varphi_i(x)\|^2.$$

Proof. It is an immediate consequence of theorem which is proved in .

Theorem 2: Suppose $f(x)$ is continuous on $[0,1)$, differentiable on $[0,1)$ and there exists a positive scalar M such that $|f'(x)| < M$, for every $x \in [0,1)$ Then

$$|f(b) - f(a)| < M|b - a|.$$

Proof. See .

Theorem 3: Suppose $f_n(x)$ is the k-IBPFs expansions of $f(x)$ that is defined as Eq. [eq:5] and $f(x)$ is diferentiable on $[0,1)$ such that. Also, assume that $e_n(x) = f(x) - f_n(x)$, then

$$\|e_n(x)\| = O(h).$$

Proof. Suppose $x_0 = 0; x_i = (i-1)h + \frac{h}{2k}; i = 1,2,\ldots,n$ and $x_{n+1} = 1$. We define the error between $f(x)$ and its KBPFs expansion over every subinterval $I_i = [x_i, x_{i+1})$ as follows:

$$e_i(x) = f(x) - f_i(x) x \in I_i$$

where $i = 0,1,2,\ldots,n$, By using mean value theorem for integral, we have

$$\|e_0(x)\|^2 = \int_0^{\frac{h}{2k}} e_0^2\,(x)dx.$$

$$\Rightarrow \|e_0(x)\|^2 = \int_0^{\frac{h}{2k}} (f(x) - f_0)^2\, dx = \frac{h}{2k}(f(\zeta_0) - f_0)^2.$$

where $\zeta_0 \in \left[0, \frac{h}{2k}\right)$. Also, for $i = 0,1,2,\ldots,n-1$. we have

$$\|e_i(x)\|^2 = \int_{(i-1)h+\frac{h}{2k}}^{ih+\frac{h}{2k}} e_i^2\,(x)dx$$

$$\Rightarrow \|e_i(x)\|^2 = \int_{(i-1)h+\frac{h}{2k}}^{ih+\frac{h}{2k}} (f(x) - f_i)^2\, dx$$

$$\Rightarrow \|e_i(x)\|^2 = h(f(\zeta_i) - f_0)^2$$

where $\zeta_i \in \left[(i-1)h + \frac{h}{2k}, ih + \frac{h}{2k}\right)$. Ultimately, for n, we get

$$\|e_n(x)\|^2 = \int_{(n-1)h+\frac{h}{2k}}^{1} e_n^2\,(x)dx$$

$$\Rightarrow \|e_n(x)\|^2 = \int_{(n-1)h+\frac{h}{2k}}^{1} (f(x) - f_n)^2\, dx.$$

$$\Rightarrow \|e_n(x)\|^2 = \frac{2k-1}{2k} h(f(\zeta_n) - f_0)^2.$$

Using Eq. [eq:5] and the mean value theorem, we have

$$f_i = \begin{cases} 2kn\int_0^{\frac{h}{2k}} f\,(x)\,dx = 2kn\left(\frac{h}{2k}\right) = f(\eta_0) & \text{if } i = 0, \quad x \in [0, \frac{h}{2k}), \\ n\int_{(i-1)h+\frac{h}{2k}}^{ih+\frac{h}{2k}} f\,(x)\,dx = nhf(\eta_i) & \text{if } x \in [(i-1)h + \frac{h}{2k}, ih + \frac{h}{2k}), \\ \frac{2kn}{2k-1}\int_{(n-1)h+\frac{h}{2k}}^{1} f\,(x)\,dx = f(\eta_n) & \text{if } i = n, \quad x \in [(n-1)h + \frac{h}{2k}, 1). \end{cases}$$

Where$\eta_i \in I_i$

$$I_i = \begin{cases} I_0 = \left[0, \frac{h}{2k}\right) & i = 0, \\ I_i = \left[(i-1)h + \frac{h}{2k}, ih + \frac{h}{2k}\right) & i = 1,2,\ldots,n-1, \\ I_n = \left[(n-1)h + \frac{h}{2k}, 1\right) & i = n, \end{cases}$$

From the aboveequations and Theorem 2, wegetWhere$\zeta_n \in I_i$

$$\|e_i(x)\|^2 = \begin{cases} \frac{h}{2k}\left(f(\xi_0) - f(\eta_0)\right)^2 \le \frac{h}{2k}M^2|\xi_0 - \eta_0|^2 \le \frac{h}{2k}M^2\frac{h^2}{4k^2} \le \frac{M^2}{8}\left(\frac{h}{k}\right)^3 & i = 0 \\ h(f(\xi_i) - f_i)^2 = h\left(f(\xi_i) - f(\eta_i)\right)^2 \le hM^2[\xi_i - \eta_i]^2 \le M^2h^3 & i = 1,2\ldots,n-1 \\ \frac{2k-1}{2nk}(f(\xi_n) - f_n)^2 = \frac{2k-1}{2nk}\left(f(\xi_n) - f(\eta_n)\right)^2 \le \frac{2k-1}{2nk}M^2[\xi_n - \eta_n]^2 \le \left(\frac{2k-1}{2k}\right)^3 M^2h^3 & i = n \end{cases}$$

We have

$$\|e(x)\|^2 = \int_0^1 e^2(x)dx = \int_0^1\left(\sum_{i=0}^{i=n} e_i(x)\right)^2 dx = \int_0^1\left(\sum_{i=0}^{i=n} e_i^2(x)\right)dx + 2\sum_{i<j}\int_0^1 e_i(x)e_j(x)dx$$

Since for $i \neq j I_i \cap I_j = \emptyset, \int_0^1 e_i(x)e_j(x)dx = 0$ then

$$\|e(x)\|^2 = \int_0^1 \left(\sum_{i=0}^{i=n} e_i^2(x)\right)dx = \sum_{i=0}^{i=n} \|e_i(x)\|^2$$

$$\|e(x)\|^2 \leq \frac{M^2}{8}\left(\frac{h}{k}\right)^3 + (n-1)M^2h^3 + \left(\frac{2k-1}{2k}\right)^3 M^2h^3$$

$$\|e(x)\|^2 \leq \left(\frac{1}{8k^3} + (n-1) + \left(\frac{2k-1}{2k}\right)^3\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(n + \frac{1}{8k^3} - 1 + \left(\frac{2k-1}{2k}\right)^3\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(n + \frac{1}{8k^3} - 1 + \frac{(2k-1)^3}{8k^3}\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(n + \frac{1}{8k^3} - 1 + \frac{8k^3 - 12k^2 + 6k - 1}{8k^3}\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(n + \frac{-12k^2 + 6k}{8k^3}\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(n + \frac{-6k+3}{4k^2}\right)M^2h^3 \leq \left(n - \frac{3}{2k} + \frac{3}{4k^2}\right)M^2h^3$$

$$\|e(x)\|^2 \leq \left(\frac{1}{h} - \frac{3}{2k} + \frac{3}{4k^2}\right)M^2h^3$$

$$\|e(x)\|^2 \leq M^2h^2 + \frac{3M^2h^3}{4k^2} - \frac{3M^2h^3}{2k}$$

7. Conclusion

We propose a method to enhance block-pulse functions(KBPFs), which, alongside their corresponding operational integration matrix P, aids in solving the Fredholm integral equation of the first kind. The previously mentioned equation is transformed into a system of algebraic equations using the suggested method. The lower triangular structure of matrix P makes the method computationally efficient due to its simplicity. Additionally, we have demonstrated the intrinsic error in the method, and to improve our comprehension of the approximation's efficacy, we charted the absolute error. Although the improved block-pulse functions (IBPFs) outlined in reference are deemed effective, our proposed method exhibits superior effectiveness and enhanced accuracy. Indeed, we can derive the (IBPFs) by establishing $k = 1$. The analysis of the results from the examined examples demonstrates the accuracy and efficacy of the proposed method. This was accomplished by computing the mean absolute error between the approximations and the precise solution. In Table 2, we find that the approximation at $k = 20$ is closer to the exact solution than the approximation proposed in article .

Furthermore, Table 4 shows that the approximation at $k = 10$ is more accurate than the approximation proposed in the same article.

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