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Prime Spectrum Graph of C-Lattices

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Abstract:

Let £ be a C-lattice. Let σ(£) be the set of all prime elements of £ and M(£) be the collection of all maximal elements of £. For S ⊆M(£), we introduce the new graph called S-join graph on σ(£), denoted by $\Gamma_S(\sigma(£))$. We have studied properties like connectivity, diameter and domination number of the graph σ(£) . In this paper, we established that the topological space σ(£) is connected if and only if the graph $\Gamma_S(\sigma(£))$ is connected.

Keywords: Prime element; Maximal element; S-join graph.

1. INTRODUCTION

The study of commutative rings indeed enriched by using graph theory techniques. In 1988, I. Beck proved that how graph theory can be applied to the study of commutative rings. According to I. Beck, the zero-divisor graph of a commutative ring is a graph where the vertices represent the elements of the ring, and two vertices are connected by an edge if their product is zero (see [5]). Further, by employing these graph-theoretic approach, many researchers visually represented and analyzed various aspects of commutative rings, providing additional insights into their structure, properties, and relationships (see [1]-[4], [9]-[10], [12]).

The ideals of ring play a fundamental role in the study of ring structure. Therefore M. Behboodi et. al. introduced and studied annihilating-ideal graph whose vertices are annihilating ideals of a commutative ring with unity (see [6]-[7]).

The set of ideals of a ring is naturally endowed with a lattice structure. It is very much interesting that the set of ideals of a ring, denoted as Id(R), forms a multiplicative lattice.

Definition 1.1. A multiplicative lattice is denoted as (£,0,1,∗), where £ is a complete lattice with least element 0, greatest element 1 and ∗ is a binary operation defined on £ that satisfies the following properties for all a,b,c ∈ £:

1. $a * b \leq a \wedge b$.
2. $a * b = b * a$.

3. (a ∗ b) ∗ c = a ∗ (b ∗ c).

4. a ∗ ($\vee_{\alpha\in I}b_\alpha$) = $\vee_{\alpha\in I}$(a ∗ b_α), where b_α ∈ £ and I is an indexing set.

5. a ∗ 1 = a.

Henceforth, we write a ∗ b = ab for the sake of convenience only.

A member a ∈ £ is called compact if a ≤ $\vee_{\beta\in I}$ a_β implies a ≤ $\vee_{i=0}^{n} a_{\beta_i}$. Let $£_c$ the set of all compact elements of £. A multiplicative lattice £ is called compactly generated if each a ∈ £ is of the form $\vee b_i$ for b_i ∈ $£_c$. By C-lattice £, we mean a multiplicative lattice (£,0,1,•) which is generated under join by a multiplicatively closed set C of compact elements and the greatest element 1 is compact as well as multiplicative identity.

An element p ∈ £ is said to be proper if p <1. In a multiplicative lattice £, a proper element m is maximal, if m is not properly contained within any other element of £ under the partial order relation ≤. We denote M(£), the collection of all maximal elements of £. In a multiplicative lattice £, if greatest element 1 is compact then each a <1 lies below some m ∈ M(£). If M(£) = {m}, then £ is called as local. For a,b ∈ £, a proper element p ∈ M is said to be prime, if ab ≤ p, then a ≤ p or b ≤ p. Let σ(£) the collection of all prime elements of £. As each maximal element is prime, we have M(£) ⊆ σ(£). A multiplicative lattice £ is said to be domain if 0 ∈ σ(£).

In this paper, we used a non-empty subset S of M(£) and defined a new simple, undirected graph called the S-join graph Γ_S(σ(£)) with the vertex set σ(£), where σ(£) is the collection of all prime elements of £ and two distinct vertices a and b are adjacent i.e., a ∼ b if and only if a ∨ b ≤ m for some m ∈ S. Here, we study some basic properties like connectivity, girth and clique number of the graph Γ_S(σ(£)). Throughout this paper, multiplicative lattice £ assumed to be C-lattice £. By σ(£) and M(£), we mean the collection of all prime elements, and the collection of all maximal elements of £, respectively.

2. GRAPH THEORETIC DEFINITIONS

Let the undirected graph G = (V,E), where V = V (G) is the set of vertices of G and E = E(G) is the set of edges of G. A graph with empty vertex set is called an empty graph. Let b ∈ V , the number of edges incident on b is called degree of a vertex b and it is denoted by deg(b). In a graph G, d(a,c) represents the length of shortest path between a and c. Note that, d(a,c) = ∞, if there is no path between a and c. The diameter of a graph G is defined as diam(G) = sup{d(a,c)|a,c ∈ V (G)}. The length of shortest cycle in G is called the girth of G, denoted by gr(G). A clique of graph is its maximal complete subgraph. For a graph G, a subset S ⊆ V (G) is supposed to be independent, if no two vertices in S are adjacent. The independence number α(G) is the maximum size of an independent set in G. Let ∅ ≠ S ⊆ V. If each vertex in V−S is adjacent to some vertex in S, then S is called a dominating set. Number of vertices in smallest dominating set is called domination number and it is denoted by γ(G).

For more information on graph theory, the reader may refer ([11], [14]).

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3. S-JOIN GRAPH $\Gamma_S((\sigma(£))$

We start this section with the following definition.

Definition 3.1. Let M(£) be the set of all maximal elements of C-lattice £ and $\emptyset \neq S \subseteq M(£)$. The S-join graph $\Gamma_S(\sigma(£))$ is simple, undirected graph with vertex set σ(£) and two distinct vertices a and b are adjacent if and only if $a \vee b \leq m$ for some $m \in S$.

Example 3.2. The lattice in Figure (1) is a multiplicative lattice £ and Figure (2) represents the S-join graph $\Gamma_S(\sigma(£))$ with the vertex set σ(L) = {a,c,d} and S = {c,d}.

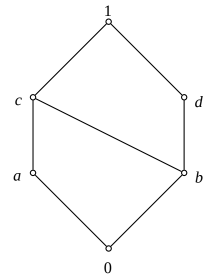

Figure (1) Multiplicative Lattice £

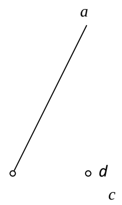

Figure (2) $\Gamma_S(\sigma(£))$

Definition 3.3. A non-empty subset S of M(£) is said to be small with respect to ∧ in short ∧-min set, if for any $a,b \in S$, there is no $0 \neq p \in \sigma(£)$ such that $p \leq a \wedge b$.

Proposition 3.4. Let £ be a C-lattice and S be a ∧-min set. If the S-join graph $\Gamma_S(\sigma(£))$ is connected, then £ is a local or a domain.

Proof. Suppose that zero element of £ is not a prime and $m_1,m_2 \in M(£)$. It is given that the S-join graph $\Gamma_S(\sigma(£))$ is connected, we have a path $m_1 \sim p_1 \sim \cdots \sim p_n \sim m_2$ between m_1 and m_2. Clearly, all p_i's are non-zero members of σ(£). Since m_1 is adjacent to p_1 and p_1 is adjacent to p_2, we have $p_1 \leq m_1$ and there exists $m \in S$ with $p_1 \vee p_2 \leq m$. This implies that, $p_1 \leq m_1 \wedge m$. Since S is ∧-min set and $p_1 \neq 0$ implies that $m_1 = m$. By the similar arguments, we have $m_1 = m_2$. Hence, £ is a local.

Proposition 3.5. If £ is a C-lattice which is either a local or a domain S is non-empty subset of M(£), then the S-join graph $\Gamma_S(\sigma(£))$ is connected.

Proof. Suppose that £ is a C-lattice which is a domain, then $0 \in \sigma(£)$. Therefore for any prime elements p and q other than 0, we have, a path $p \sim 0 \sim q$. Hence, $\Gamma_S(\sigma(£))$ is connected. Now, if £ is a C-lattice which is a local with M(£) = {m}. Then for any $p_1,p_2 \in \sigma(£)$, we have $p_1 \vee p_2 \leq m$. Therefore, $p_1 \sim p_2$. Consequently, $\Gamma_S(\sigma(£))$ is connected.

From the Proposition 3.5, we have following corollary.

Corollary 3.6. Let £ be a C-lattice. If £ is a local or a domain, then diam(Γ_S(σ(£))) ≤2.

In the following Theorem, we have obtained a characterization of a local C-lattice £.

Theorem 3.7. Let £ be a C-lattice and let Γ_S(σ(£)) be the S-join graph of £. Then the graph Γ_S(σ(£)) is complete if and only if £ is a local.

Proof. Suppose that, Γ_S(σ(£)) is a complete graph of a C-lattice £ with m_1,m_2 ∈ M(£). Since the graph Γ_S(σ(£)) is complete and every maximal element of £ is prime, we have $m_1 \sim m_2$, implies that $m_1 \vee m_2 \leq m$ for some m ∈ M(£) such that $m \neq 1$, a contradiction to the fact that $m_1 \vee m_2 = 1$ so that $m_1 = m_2$. Hence, £ is a local C-lattice. Conversely, suppose that £ is a local C-lattice with the maximal element m. Then for any p_1,p_2 ∈ σ(£), we have $p_1 \vee p_2 \leq m$ and hence p_1 and p_2 are adjacent. Consequently, the S-join graph Γ_S(σ(£)) is complete.

Proposition 3.8. If a C-lattice £ is such that M(£) = σ(£) − {0}, with unique atom, then Γ_S(σ(£)) is a star graph.

Proof. Suppose that a C-lattice £ is a domain. so that, 0 ∈ σ(£). Also, any two non-zero elements a,b ∈ £ are co-maximal. Therefore, for any $0 \neq a$ ∈ £, we have a path $0 \sim a$. Also, there does not exist edge between any two non-zero elements of £. Hence, Γ_S(σ(£)) is a star graph.

The following definition of dimension of a C-lattice £ is used in results that follows.

Definition 3.9. Let £ be a C-lattice. The dimension of £, denoted as dim(£), is the supremum of the lengths of chains of members of σ(£).

Theorem 3.10. Let £ be a C-lattice which is not a local and let S be a ∧-min set of £. If a S-join graph Γ_S(σ(£)) is a star graph, then £ is a domain with dim(£) ≤ 1.

Proof. Suppose that Γ_S(σ(£)) is a star graph of £. Hence, there is a vertex p in the graph Γ_S(σ(£)) such that $p \sim q$, for all q ∈ σ(£). Therefore, for m_1,m_2 ∈ M(£), we have $p \sim m_1$ and $p \sim m_2$. Since m_1,m_2 ∈ M(£), $p \vee m_1 = m_1$ and $p \vee m_2 = m_2$. This implies that $p \leq m_1 \wedge m_2$. But it is given that S is a ∧-min set, therefore p = 0. Hence, 0 is a prime element of £, consequently, £ is a domain. Now, suppose that dim(£) ≥ 2, then there exists a chain of prime elements having length at least two, say, $p_1 \leq p_2 \leq p_3$. It is clear that $p_1 = 0$. Since Γ_S(σ(£)) is a star graph, we have $p_1 \sim p_2$, by definition there exists m ∈ M(£) such that $p_1 \vee p_2 \leq m$, i.e., $p_1 \vee p_2 = p_2 \leq m$. If $p_2 \neq m$, then we have a cycle $p_1 \sim p_2 \sim m \sim \cdots \sim p_1$, a contradiction to the fact that the graph Γ_S(σ(£)) is a star graph. If $p_2 = m$, then we have $p_1 \leq p_2 = m \leq p_3$ which not possible because m is maximal. Consequently, dim(£) ≤ 1.

Proposition 3.11. Let £ be a C-lattice and Γ_S(σ(£)) be a S-join graph of £. If Γ_S(σ(£)) is a star graph, then S = M(£).

Proof. Suppose that the S-join graph Γ_S(σ(£)) of £ is a star graph. Then, we have a fixed vertex, say a, in the graph Γ_S(σ(£)) such that $a \sim b$ for each vertex b in Γ_S(σ(£)). Therefore, for any m ∈ M(£), we have a ∼ m. This implies that, a ∨ m ≤ m_1, for some m_1 ∈ S. As, m ∈ M(£), implies that m = m_1. Hence, S = M(£).

We need the following proposition to establish the some important theorem ahead.

Proposition 3.12. [13] Let £ be a multiplicative lattice with greatest element 1 is compact. Then for any a ∈ £ such that a ≠ 1, there exists m ∈ M(£) such that a ≤ m.

Theorem 3.13. Let £ be a C-lattice and $\Gamma_S(\sigma(£))$ be a S-join graph of £. If dim(£) ≥ 2, then gr($\Gamma_S(\sigma(£))$) = 3.

Proof. Suppose that $a_0 \leq a_1$ be a chain of members of σ(£). Since £ is a C-lattice, we have greatest element 1 is compact. Therefore, by Proposition 3.12, there exists a maximal element m such that $a_1 \leq m$. Now, we have $a_0 \sim a_1 \sim m \sim a_0$, since $a_0 \vee a_1 \leq m$ and $a_0 \vee m = m$.

Theorem 3.14. Let £ be a C-lattice with dim(£) = 1. If a S-join graph $\Gamma_S(\sigma(£))$ of £ contains a cycle, then £ is not a domain.

Proof. Suppose on the contrary that 0 ∈ σ(£). Since dim(£) = 1 and graph $\Gamma_S(\sigma(£))$ of £ contains a cycle, by Theorem 3.13 we have a cycle $a_0 \sim a_1 \sim a_2 \sim \cdots \sim a_0$ in $\Gamma_S(\sigma(£))$. As $a_0 \sim a_1$, there exist m ∈ M(£), such that $a_0 \vee a_1 \leq m$. Here either a_0 or a_1 are non-zero member of σ(£). Suppose that a0≠ 0. Since 0 ∈ σ(£) and dim(£) = 1, we have a_1 is maximal element. In fact, a_1 = m. Similarly, we can prove that a_2 is also maximal element. Hence a_1 and a_2 are co-maximal elements of £, which is contradiction to the fact that $a_1 \sim a_2$. Consequently, 0 ∈ σ(£).

Note, $\sigma_{min}(£)$ denotes the set of all minimal prime elements of £.

Theorem 3.15. Let £ be a C-lattice with $|\sigma_{min}(£)| \leq \infty$. Then $\gamma[\Gamma_S(\sigma(£))] \leq |\sigma_{min}(£)|$.

Proof. Suppose on the contrary that κ = {p_1,p_2,•••,p_n} be a dominating set in the S-join graph $\Gamma_S(\sigma(£))$ of £. By definition, for any q ∈ σ(£) − κ there is p_i ∈ κ such that q is adjacent to pi for some 1 ≤ i ≤ n. Let q_1,q_2,••• ,q_r be the distinct members of $\sigma_{min}(£)$. For each p_i ∈ κ there is at least one q_j (1 ≤ j ≤ r) with $q_j \leq p_i$. This implies that r ≤ n. Now we prove that τ = {q_1,q_2,••• ,q_r} is also a dominating set. Suppose that p ∈ σ(£) such that p ∉ τ. If p ∉ κ. Since κ is a dominating set, we have p_i ∈ κ such that p ∼ p_1. By definition there exists m′ ∈ M(£) such that p ∨ p_i ≤ m′. According to the construction of the set τ = {q_1,q_2,••• ,q_r}, there are some q_j ∈ τ with $q_j \leq p_i$, therefore p ∨ q_j ≤ m′ and hence p ∼ q_j for some q_j ∈ τ. Now, suppose p ∈ κ. By the definition of τ, there exists q_i ∈ τ such that q_i ≤ p. Since the greatest element 1 is compact, by Proposition 3.12 we have m ∈ M(£) such that p ≤ m, therefore q_i ∨ p ≤ m. This implies that q_i ∼ p, consequently, τ is a dominating set.

In [8], F. Callialp et. al. established some results on the Zariski topology over σ(£). For a ∈ £, define ϑ(a) = {p ∈ σ(£)|a ≤ p}. F. Callialp et. al. introduced a topology on σ(£) with the collection of all closed set {ϑ(a)|a ∈ £} using the following Proposition 3.16 (see [8]).

Proposition 3.16. [8] Let £ be a C-lattice and for a ∈ £, let ϑ(a) = {p ∈ σ(£)|a ≤ p}. Then the following axioms hold:

1. $\vartheta(0) = \sigma(£)$ and $\vartheta(1) = \emptyset$.
2. $\cap_{\alpha\in\triangle}\vartheta(a_\alpha) = \vartheta(\vee_{\alpha\in\triangle}a_\alpha)$ for any index set $\triangle$.
3. $\vartheta(p) \cup \vartheta(q) = \vartheta(p \wedge q) = \vartheta(pq)$ for $p,q \in £$.

In the next Theorem 3.17, we studied the connected topological space $\sigma(£)$.

Theorem 3.17. Let £ be a C-lattice and $|\sigma_{min}(£)| \leq \infty$. Then $\sigma(£)$ is connected if and only if the S-join graph $\Gamma_S(\sigma(£))$ of £ is connected and $diam(\Gamma_S(\sigma(£))) \leq 2|\sigma_{min}(£)|$.

Proof. Suppose that $\sigma(£)$ is not connected. If $\Gamma_S(\sigma(£))$ is disconnected, then nothing to prove. Suppose that $\Gamma_S(\sigma(£))$ is connected, then for some $a,b \in £$, $\sigma(£) = \vartheta(a) \cup \vartheta(b)$ and $\vartheta(a) \cap \vartheta(b) = \emptyset$. Let $p,q \in \sigma(£)$ such that $p \in \vartheta(a)$ and $q \in \vartheta(b)$. Since $\Gamma_S(\sigma(£))$ is the connected graph, we have a path $p \sim p_1 \sim p_2 \sim p_3 \sim \cdots \sim p_n \sim q$ between p and q. As $p \sim p_1$, by definition there exists $m_1 \in M(£)$ such that $p \vee p_1 \leq m_1$. This implies that $p_1 \in \vartheta(a)$ since $p,m_1 \in \vartheta(a)$. Using similar arguments, we can prove that $p_n \in \vartheta(a)$. Also, note that $p_n \sim q$. Therefore by definition, there exists $m_n \in M(£)$ such that $p_n \vee q \leq m_n$. This implies that $m_n \in \vartheta(a)$ because $p_n \leq m_n$. Also, note that $q \leq m_n$, therefore we have $m_n \in \vartheta(b)$, a contradiction to the fact that $\vartheta(a) \cap \vartheta(b) = \emptyset$. Consequently, $\sigma(£)$ is connected. Conversely, suppose that the space $\sigma(£)$ is connected and $\sigma_{min}(£) = \{a_1,a_2,\cdots ,a_r\}$. Case I) Suppose $r = 2$. For $p,q \in \sigma(£)$, there exist $a_1,a_2 \in \sigma_{min}(£)$ such that $a_1 \leq p$ and $a_2 \leq q$. Since $\sigma(£)$ is connected, we have $a_3 \in \vartheta(a_1)\cap\vartheta(a_2)$. Also, since the greatest element 1 is compact, by Proposition 3.12 we have a path $p \sim a_1 \sim a_3 \sim a_2 \sim q$ of length $4 = 2|\sigma_{min}(£)|$ between p and p. Case II) Suppose that $r >2$. Let $p, q \in \sigma(£)$ such that for $1 \leq i \leq l_1$, $p \in \vartheta(a_i)$, for $l_1 \leq i \leq l_2$, $q \in \vartheta(a_i)$ and for $l_2 \leq i \leq r$, $p, q \notin \vartheta(a_i)$. But $\sigma(£)$ is connected, therefore there exist $1 \leq i_1 \leq \cdots \leq i_k \leq r$ with $\vartheta(a_{i_l}) \cap \vartheta(a_{i_{l+1}}) \neq \emptyset$ for $1 \leq l \leq k$ and $\vartheta(\vee_{i=1}^{l_1} a_i) \cap \vartheta(a_{i_1}) \neq \emptyset$, $\vartheta(\vee_{i=l_1+1}^{l_2} a_i) \cap \vartheta(a_{i_k}) \neq \emptyset$. This implies that there is a path between p and q with length at most $2|\sigma_{min}(£)|$. Consequently, the graph $\Gamma_S(\sigma(£))$ of £ connected and $diam(\Gamma_S(\sigma(£))) \leq 2|\sigma_{min}(£)|$.

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