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Buyers- Suppliers Win-Win Cash Flow Inventory Model with Linear Demand under Various Production Industries

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Abstract:

This paper presents an analytical and numerical analysis of an inventory model incorporating credit periods, focusing on major business concerns that face cash flow constraints. In such circumstances, credit periods serve as a crucial tool for determining optimal production inventory strategies. The foremost objective of the research is to assess the benefits accrued by both producers and buyers within a fixed credit period framework. The analytical solution model developed in this study aims to optimize production inventory costs by considering the cycle periods and interest earned during shortage periods. Through the utilization of differential equations, the model offers a comprehensive evaluation of inventory management strategies. By incorporating credit periods as a key parameter, the research investigates how producers and buyers can derive maximum benefits from this approach. Furthermore, a numerical example is given to describe the practical application of the model and demonstrate how decision variables are employed in real-world scenarios. This example showcases the effectiveness of the analytical and numerical analysis in determining optimal production inventory costs. Overall, this study contributes to the understanding of inventory management in situations where cash flow is inconsistent. By incorporating credit periods into the analysis, the research provides insights into the decision-making process for production inventory, enabling businesses to optimize costs and enhance overall efficiency.

Keywords: *Permissible delay payments, Production quantity, Linear Demand, Partially backlogged shortages, time horizon*.

1. Introduction

Inventory management is a critical aspect for businesses across various industries and sectors. The efficient management of inventory levels is essential for meeting customer demands, minimizing costs, and maximizing profitability. One important factor that influences inventory management strategies is the consideration of credit periods in the context of production and linear demand. An inventory model that incorporates production and linear demand, along with credit periods, allows companies to make informed decisions about their inventory control strategies. By understanding the dynamics of production, demand patterns, and credit periods, businesses can optimize their inventory levels and align them with their operational and financial goals. In various fields and industries, companies face the challenge of balancing production activities with customer demand. They need to ensure that they have sufficient inventory on hand to meet customer orders promptly while avoiding excessive inventory levels that can tie up working capital. The integration of production and linear demand in an inventory model enables companies to optimize production schedules and order quantities to meet demand fluctuations efficiently. Moreover, credit periods play a crucial role in managing inventory and cash flow. The availability of credit periods allows companies to delay payment for goods or services received, providing a financial advantage by freeing up working capital for other purposes. By strategically utilizing credit periods, businesses can capably manage their cash flow and strengthen their complete financial position. The integration of credit periods into the inventory model provides companies with a holistic approach to inventory management. It allows them to evaluate the trade-offs between holding costs, production costs, and financial benefits derived from credit periods. This comprehensive analysis enables businesses to make data-driven decisions on order quantities, production schedules, and payment terms. The application of the inventory model with production and linear demand using credit periods is relevant to various industries and companies. Whether in manufacturing, retail, distribution, or any other sector that deals with inventory management, the model offers a framework for optimizing inventory levels and achieving operational efficiency. By incorporating production, linear demand, and credit periods into their inventory management strategies, companies can streamline their operations, reduce costs, and enhance customer satisfaction. The utilization of this model allows businesses to strike a balance between meeting demand, minimizing holding costs, and effectively managing their cash flow.

Throughout this paper, we will delve into the analytical and numerical analysis of the inventory model with production and linear demand using credit periods. We will explore the benefits and trade-offs associated with this approach and provide practical insights for companies in different fields to enhance their inventory management strategies. Through the integration of production, linear demand, and credit periods in the inventory model, companies can gain a competitive edge in their respective industries. The ability to optimize inventory levels while considering production capabilities, demand fluctuations, and financial advantages of credit periods is crucial for sustained success. In manufacturing companies, the inventory model with production and linear demand using credit periods enables efficient production planning and scheduling. By aligning production activities with anticipated demand and utilizing credit periods strategically, manufacturers can minimize stock outs, reduce production costs, and enhance customer satisfaction. The model allows for a more accurate estimation of the optimal production quantity, taking into account both the immediate demand and the anticipated future demand based on linear patterns. Retailers and distributors also benefit from the integration of credit periods into their inventory management strategies. By leveraging credit periods offered by suppliers, they can maintain optimal inventory levels without tying up excessive capital. This approach not only improves cash flow but also allows retailers to respond quickly to changing customer demands. By accurately forecasting the linear demand and factoring in credit periods, retailers can effectively determine the order quantities that will meet customer needs while managing their financial resources efficiently. Furthermore, the inventory model with production and linear demand using credit periods extends beyond manufacturing and retail sectors. It is applicable in various fields such as healthcare, e-commerce, and service-based industries. For instance, healthcare facilities can utilize the model to optimize their inventory of medical supplies, ensuring they have the right quantities available while managing costs. E-commerce companies can leverage credit periods to streamline their inventory management processes, improve fulfilment operations, and enhance customer experience.

1.1 LITERATURE REVIEW

Zhang and Wang [1] proposed a mathematical model for inventory management in supply chain networks that takes into account multiple demand scenarios, supply chain structure, and inventory policies. The goal is to minimize the total cost of inventory while ensuring a high level of service level for customers. Zhu and Cui [2] developed a novel mathematical model for inventory control in a multi-level supply chain network. The model considers the coordination of inventory decisions between different echelons of the supply chain and aims to optimize the total inventory cost while satisfying customer demand. Li et al. [3] discussed a model for inventory management that incorporates demand uncertainty and lead time variability. The model aims to minimize the total inventory cost while ensuring a high service level for customers. Gao et al. [4] developed a stochastic mathematical model for inventory management in a multi-echelon supply chain that considers both demand uncertainty and lead time variability. The model aims to minimize the total inventory cost while ensuring a high level of service level for customers. Sarkar, B et al. [5] developed an inventory model with trade-credit policy, variable deterioration for production products. Sana, S. S [6] studied a production–inventory model for imperfect inventory process. Huang, Y. F. [7] studied an inventory model under two levels of trade credit and also limited storage space model. Hammami, R et al. [8] Analyzed carbon emissions in a multi-echelon production-stock using lead time constraints. Birim.S et al. [9] presented evaluating vendor managed inventory systems: how incentives can benefit supply chain partners. Srivathsan, S., & Kamath, M et al. [10] considered a performance modeling of a two-echelon with supply chain and information sharing. Muniappan et al. [11] developed an EOQ model for deteriorating products and time value of money for delay payments. Muniappan et al. [12] developed a production inventory model for vendor–buyer with quantity discount and backordering for used products. Mohammadi, H et al. [13] discussed about deteriorating and seasonal products, such as fresh produce, the issues of timely supply and disposal of the deteriorated products are of high concerns and presented a new mathematical model of the location-routing problem of facilities in supply chain network for deteriorating items by taking environmental reflections, cost, delivery time and customer satisfaction into account across the entire network and customer satisfaction. Amini, A et al. [14] addresses about combined transportation and inventory problem in a two-stage supply chain, including suppliers and retailers and the role of energy in terms of fuel's type selection. Vafaeenezhad et al. [15] discussed about the purchase, production and distribution quantities for facilities in a supply chain and minimizing the total cost or maximizing the profits was the major aim of supply chains, responsibility for the environmental and social impacts, processes and products, the safety and health of their employees and the entire community. Huang, J et al. [16] studied the complexities of bidders' information interactions and behaviour preferences caused from financial and production perceptions. Also we take into account the complex and dynamic market background, which will impact the operation effect of sale policies. Yadav, A. S et al. [17] discussed about simulated Annealing to optimize FIFO & LIFO in green supply chain inventory management of Hazardous substance components industry and studied on determining the most likely level of surplus stock and shortage required for FIFO & LIFO in green supply chain stocks of Hazardous substance components industry to find the minimum total cost of the supply chain. Setak, M et al. [18] discussed two mathematical models of supply chain under uncertainty. The competition is considered as a game. Davizon, Y. A et al. [19] studied the mathematical modelling, optimal control, and stability analysis for dynamic supply chain. Aghsami, A et al. [20] addresses various aspects of the blood collection centres are considered in this model and storage of optimum blood level is considered. Dehghani, E et al. [21] studied the Markov process and mixed-integer nonlinear programming is presented to design the distribution network of a supply chain. Yang et al. [22] studied the inventory competition under lead time sensitive demands and compared the consolidated scenarios. Yadav, A. S et al. [23] discussed the most likely level of surplus stock and shortage required for green supply chain stocks of Auto-components industry to find the minimum total cost in supply chain. R. Uthayakumar, & A. Ruba Priyadharshini [24] discussed a deteriorating item return policy with allowable delay and partial backlog is included in the inventory model for a single item. By selling both price and time, it optimizes the overall profit. Duary,et al. [25] A developed a very non-linear objective function for a two-warehouse inventory problem, taking into account all potential cases and subcases. In addition to applying the suggested algorithm to identify the best ideal values from an economic perspective, we employed the GRG technique to solve the problem. A two-warehouse inventory model with a decaying product whose demand varies with time, selling price, the strength of media advertisements, and continuous time with a mixed type trade credit policy is also updated here. Najafnejhad, E et al. [26] established the demand fluctuations in inventory management are greatly decreased by this policy. Based on the vendor-managed policy, this article creates an inventory model with many merchants and a single vendor. Apart from making inventory decisions, the suggested methodology maximizes an upper bound on inventory levels determined by a penalty. The established mathematical model's goal is to determine the best value for retailers' order quantities, replenishment frequencies, and upper limits on their inventory levels. Saren, S., et al. [27] demonstrates that a cap and trade policy can be used to control the overall amount of carbon released into the atmosphere by the transportation and production sectors, the lead time demand for items by retailers is assumed to be random rather than fixed uniform and normal distribution functions. The ideal retailer lot size, customer service rendered by the store, and retailer reorder points are evaluated under these two distribution functions. Ganguly,et al. [28] examined a reworking approach that would be put into practice following an inventory of such faulty goods and there would be no shortages when the assembled product was remanufactured. Productivity variation was raised to enhance the quality of the completed products while lowering manufacturing costs. The space capacity and budget were regarded as limitations. Based on five distinct distribution functions and the defined variable parameters of production rate, manufacturing batch size, and backorder amount, the total inventory cost was computed. Mondal, et al. [29] discussed a three-tiered supply chain management model based on a single manufacturer, one supplier, and several retailers, all subject to payment and advertising regulations. Advertising has shown to have a beneficial impact on sales since it creates demand for the goods in the market. By taking into account a single-setup multiple-delivery policy, variable transportation costs, variable carbon emissions costs, and trade-credit policy, the model seeks to minimize supply chain costs and maximize profit. Based on the payment duration, the objective function is optimized for certain instances. Sen, N et al. [30] establishes a green supply chain model with one supplier and one buyer for decaying commodities. Demand is influenced by the selling price and the degree of greening improvement. There are two established generalized models for green supply chains: one with consignment stock policy and the other without.

1.2 Research Focus

The integration of production, linear demand, and credit periods in the inventory model offers significant benefits for companies across diverse fields. By optimizing inventory levels, aligning production with demand, and strategically utilizing credit periods, businesses can achieve operational efficiency, cost savings, and improved financial performance. The analytical and numerical analysis of this inventory model provides valuable insights and practical guidance for companies to make informed decisions regarding their inventory management strategies. Throughout this paper, we will delve further into the specific methodologies, analyses, and numerical examples that illustrate the advantages of the inventory model with production and linear demand using credit periods. We aim to provide readers with a comprehensive understanding of how this model can be applied in various industries and companies, ultimately enabling them to optimize their inventory management practices and achieve their business objectives.

1.3 Decision variables

The permissible delay M

Time horizon T, T_1

Optimum total inventory cost $TC(T_1, T)$

Backlogged shortage δ

2. FORMULATION AND ANALYTICAL SOLUTION FOR INVENTORY MODEL

In the proposed inventory model, the change in inventory over a [0, T_1) time interval is determined by the production and the linear demand. Additionally, the occurrence of shortages in subsequent time intervals is taken into account. The model formulation can be described as follows the inventory level at the beginning of the time interval as $I_1(t)$, the production quantity during the time interval [0, T_1)as P. The demand rate (linear) during the time interval [0, T_1) as D(t). The shortage quantity during the time interval [T_1, T) as S(t). Hence, the change in inventory over the time interval [0, T) can be calculated as:

$$\frac{dI_1(t)}{dt} + \theta I_1(T_1) = P - a - bt\ ;\ [0, T_1). \qquad (1)$$

$I_1(T_1) = 0$ gives,

$$I_1(t) = \frac{\theta(P-a)+b}{\theta^2}\left[1 - e^{\theta(T_1-t)}\right] + \frac{b}{\theta}\left[T_1 e^{\theta(T_1-t)} - t\right]$$

$$\frac{dI_2(t)}{dt} + \theta I_2(T_1) = -\frac{a}{1+\delta\,(T-t)}\ ;\ [T_1, T). \qquad (2)$$

$I_2(T_1) = 0$ gives,

$$I_2(t) = -\frac{a}{\delta}\left[log(1+\delta(T-T_1)) - log(1+\delta(T-t))\,\right]$$

We obtain inventory level for the different intervals as follows:

$$I_1(t) = \frac{\theta(P-a)+b}{\theta^2}\left[1 - e^{\theta(T_1-t)}\right] + \frac{b}{\theta}\left[T_1 e^{\theta(T_1-t)} - t\right] \tag{3}$$

$$I_2(t) = -\frac{a}{\delta}\left[log(1+\delta(T-T_1)) - log(1+\delta(T-t))\ \right] \tag{4}$$

I (T) = $I_1(t) + I_2(t)$; [0, T) (5)

Various Inventory cost calculated by using integral calculus in the interval [0, T_1)

We are using holding cost h, shortage cost s, Deterioration cost p, Opportunity cost α.

HC = $h\int_0^{T_1} I_1(t)dt$

$$= h\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right\} \tag{6}$$

DC = $p\theta \int_0^{T_1} I_1(t)dt$

$$= p\theta\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right\} \tag{7}$$

SC = $s\int_{T_1}^{T} I_2(t)dt$

$$= \frac{sa}{\delta^2}\left[\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right] \tag{8}$$

OC = $\alpha\int_{T_1}^{T} I_2(t)dt$

$$= \frac{a\alpha}{\delta}\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\} \tag{9}$$

2.1. Scenario1: supplier’s delay payment period $M \leq T_1$

Consider a situation where a buyer purchases goods or services from a supplier on credit, with specific payment terms and conditions. In this case, the supplier's permissible delay, denoted as M, represents the maximum duration allowed for the buyer to settle the outstanding payment. Let's say the total credit period agreed upon between the supplier and the buyer is denoted as T_1. It is specified that within each credit cycle, the buyer earns interest during the interval [0, T_1), referred to as IE1, at a rate denoted as Ie. However, in the subsequent interval [M, T_1), denoted as Ip, the buyer is required to pay interest to the supplier at a predetermined interest rate, denoted as Ir. To provide a real-life example, let's consider a retailer purchasing goods from a wholesaler on credit terms. The wholesaler allows a credit period of 60 days (T_1) for the retailer to make the payment. However, the retailer has the flexibility to delay payment for up to 30 days (M) before it is considered overdue. During the first 30 days of the credit period ([0, 30), denoted as IE1), the retailer benefits from earning interest on the outstanding amount owed to the wholesaler. However, if the retailer delays the payment beyond the permissible delay (M) and falls within the interval [30, 60) days (denoted as Ip), it is required to pay interest to the wholesaler at the predetermined interest rate, Ir. This serves as a mechanism to incentivize the buyer to make timely payments and compensate the supplier for the delayed payment.

In summary, this real-life example illustrates a scenario where a buyer earns interest during the initial portion of the credit period [0, T_1) (denoted as IE1), but once the permissible delay (M) is exceeded, the buyer is obliged to pay interest to the supplier during the subsequent period [M, T_1) (denoted as Ip). We have,

$$IE_1 = pI_e \int_0^{T_1} (T_1 - t)(a+bt)dt$$

$$= \frac{pI_e T_1^2}{6}[3a + bT_1] \tag{10}$$

$$I_p = pI_r \int_M^{T_1} I(t)dt$$

$$= pI_r \left\{\frac{1-e^{\theta(T_1-M)}}{\theta^3}[\theta(P-a) + b(1-\theta T_1)] + \frac{\theta(P-a)+b}{\theta^2}[T_1 - M] - \frac{b}{2\theta}(T_1^2 - M^2)\right\} \tag{11}$$

$$PC = \int_0^{T_1} (a+bt)c_d dt$$

$$= c_d\left[aT_1 + \frac{bT_1^2}{2}\right] \tag{12}$$

Average cost developed as follow

$$TC_1 = \frac{1}{T}\Big\{r + h\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1 - e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right\} + P\theta\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1 - e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right\} + s\left\{\frac{a}{\delta^2}\left[\delta(T-T_1) - \log\left(1 + \delta(T-T_1)\right)\right]\right\} + \frac{a\alpha}{\delta}\left\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right\} + PI_r\left\{\frac{1-e^{\theta(T_1-M)}}{\theta^3}[\theta(P-a) + b(1-\theta T_1)] + \frac{\theta(P-a)+b}{\theta^2}[T_1 - M] - \frac{b}{2\theta}(T_1^2 - M^2)\right\} + c_d\left[aT_1 + \frac{bT_1^2}{2}\right] - \frac{pI_e T_1^2}{6}[3a + bT_1]\Big\}$$

$$= \frac{1}{T}\Big\{r + (h + P\theta)\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1 - e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right\} + \frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right\} + PI_r\left\{\frac{1-e^{\theta(T_1-M)}}{\theta^3}[\theta(P-a) + b(1-\theta T_1)] + \frac{\theta(P-a)+b}{\theta^2}[T_1 - M] - \frac{b}{2\theta}(T_1^2 - M^2)\right\} + c_d\left[aT_1 + \frac{bT_1^2}{2}\right] - \frac{pI_e T_1^2}{6}[3a + bT_1]\Big\} \tag{13}$$

$$\frac{\partial TC_1(T_1,T)}{\partial T_1} = 0 \quad and \quad \frac{\partial TC_1(T_1,T)}{\partial T} = 0$$

2.1.1. Solution procedure for optimum inventory level

$$\left[\frac{\partial^2 TC_1(T_1,T)}{\partial T_1^2}\right]_{at\ (T_1^*,T^*)} > 0\ ,\ \left[\frac{\partial^2 TC_1(T_1,T)}{\partial T^2}\right]_{at\ (T_1^*,T^*)} > 0\ ;\ \left[\left(\frac{\partial^2 TC_1(T_1,T)}{\partial T_1^2}\right)\left(\frac{\partial^2 TC_1(T_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_1(T_1,T)}{\partial T_1 \partial T}\right)^2\right] > 0$$

$\frac{\partial TC_1(T_1,T)}{\partial T_1} = 0 \quad and \quad \frac{\partial TC_1(T_1,T)}{\partial T} = 0$ Implies the optimal values of T^* and T_1^*

$$\frac{\partial TC_1(T_1,T)}{\partial T_1} = 0$$

$$\frac{1}{T}\Big\{(h+P\theta)\Big\{(\theta(P-a)+b)\left(\frac{-e^{\theta T_1}}{\theta^2}\right)+\frac{T_1}{\theta}\left[b\left(e^{\theta T_1}-\frac{1}{2}\right)\right]+\frac{1}{\theta^2}\left[\left(\theta(P-a)+b\left(e^{\theta T_1}-\frac{\theta T_1}{2}\right)\right)\right]\Big\}-\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}+PI_r\left\{-b\left(\frac{1-e^{\theta(T_1-M)}}{\theta^2}\right)-\frac{e^{\theta(T_1-M)}}{\theta^2}[\theta(P-a)+b(1-\theta T_1)]+\frac{\theta(P-a)+b}{\theta^2}-\frac{bT_1}{\theta}\right\}+c_d[a+bT_1]-\frac{pI_e T_1}{3}[3a+bT_1]-\frac{pI_e bT_1^2}{6}\Big\} = 0 \quad (14)$$

$$\frac{\partial TC_1(T_1,T)}{\partial T} = 0$$

$$\frac{1}{T}\left\{\frac{a(s+\delta\propto)(T-T_1)}{1+\delta(T-T_1)}\right\}-\frac{1}{T^2}\Big\{r+(h+P\theta)\left\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right]+\frac{T_1}{\theta^2}\left(\theta(P-a)+b\left(e^{\theta T_1}-\frac{\theta T_1}{2}\right)\right)\right\}+\frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1)-\log\left(1+\delta(T-T_1)\right)\right\}+PI_r\left\{\frac{1-e^{\theta(T_1-M)}}{\theta^3}[\theta(P-a)+b(1-\theta T_1)]+\frac{\theta(P-a)+b}{\theta^2}[T_1-M]-\frac{b}{2\theta}(T_1^2-M^2)\right\}+c_d\left[aT_1+\frac{bT_1^2}{2}\right]-\frac{pI_e T_1^2}{6}[3a+bT_1]\Big\} = 0 \quad (15)$$

Hence the optimum cycle length values are T_1^*, T^* and optimum average inventory cost is $TC_1(T_1,T)$

2.2. Scenario2: supplier's payment delay $\boldsymbol{T_1} < M$

Consignment inventory refers to a situation where the supplier stocks and maintains inventory at the buyer's premises, but the buyer does not pay for the inventory until it is consumed or sold. In this scenario, the buyer benefits from having access to the inventory without incurring any immediate costs. Consider a manufacturing company (buyer) that relies on a supplier for raw materials. The supplier agrees to provide consignment inventory to the buyer, ensuring a constant supply of raw materials. The agreement specifies that within the time interval [0, M), the buyer earns interest on the inventory held and does not pay any interest or carrying costs to the supplier during this period. During this time interval, the buyer can utilize the consignment inventory to fulfil production demands without paying for the materials upfront. This arrangement allows the buyer to effectively manage cash flow by deferring the payment until the inventory is consumed or transformed into finished goods. Meanwhile, the buyer can invest the funds that would have been allocated for purchasing inventory elsewhere, potentially earning interest on those funds during the consignment period. Overall, the consignment inventory arrangement exemplifies a real-life scenario where the buyer benefits from earning interest on inventory held within the specified time interval [0, M), without incurring any interest or carrying costs associated with the supplier's consigned inventory. Therefore we have,

$$IE_2 = pI_e\left\{\int_0^{T_1}(T_1-t)(a+bt)dt+(M-T_1)\int_0^{T_1}(a+bt)dt\right\}$$

$$= \frac{pI_e T_1^2}{6}[3a+bT_1]+\frac{pI_e T_1(M-T_1)}{2}[2a+bT_1] \quad (16)$$

The total average cost developed as

$$TC_2 = \frac{r + HC + DC + SC + OC + PC - IE_2}{T} \qquad (17)$$

$$= \frac{1}{T}\Big\{r + h\Big\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\Big(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\Big)\Big\} + P\theta\Big\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\Big(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\Big)\Big\} + s\left\{\frac{a}{\delta^2}\left[\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right]\right\} + \frac{a\alpha}{\delta}\left\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right\} + c_d\left[aT_1 + \frac{bT_1^2}{2}\right] - \frac{PI_e T_1^2}{6}\left[3a + bT_1\right] - \frac{pI_e T_1(M-T_1)}{2}\left[2a + bT_1\right]\Big\}$$

$$= \frac{1}{T}\Big\{r + (h+P\theta)\Big\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\Big(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\Big)\Big\} + \frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right\} + c_d\left[aT_1 + \frac{bT_1^2}{2}\right] - \frac{pI_e T_1^2}{6}\left[3a + bT_1\right] - \frac{pI_e T_1(M-T_1)}{2}\left[2a + bT_1\right]\Big\}$$

(18)

2.2.1. Solution procedure for optimum inventory level

To solve $\frac{\partial TC_2(T_1,T)}{\partial T_1} = 0$ *and* $\frac{\partial TC_2(T_1,T)}{\partial T} = 0$ and the sufficient conditions are

$$\left[\frac{\partial^2 TC_2(T_1,T)}{\partial T_1^2}\right]_{at\ (T_1^*,T^*)} > 0\ ,\ \left[\frac{\partial^2 TC_2(T_1,T)}{\partial T^2}\right]_{at\ (T_1^*,T^*)} > 0\ and\ \left[\left(\frac{\partial^2 TC_2(T_1,T)}{\partial T_1^2}\right)\left(\frac{\partial^2 TC_2(T_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC_2(T_1,T)}{\partial T_1 \partial T}\right)^2\right] > 0$$

$$\frac{1}{T}\Big\{(h+P\theta)\Big\{(\theta(P-a)+b)\left(\frac{-e^{\theta T_1}}{\theta^2}\right) + \frac{T_1}{\theta}\left[b\left(e^{\theta T_1} - \frac{1}{2}\right)\right] + \frac{1}{\theta^2}\left[\left(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\right)\right]\Big\} - \frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)} + c_d[a + bT_1] - pI_e\,(a+bT_1)(M-T_1)\Big\} = 0$$

(19)

$$\frac{\partial TC_2(T_1,T)}{\partial T} = 0$$

$$\frac{1}{T}\left\{\frac{a(s+\delta\alpha)(T-T_1)}{1+\delta(T-T_1)}\right\} - \frac{1}{T^2}\Big\{r + (h+P\theta)\Big\{\frac{\theta(P-a)+b}{\theta^3}\left[1-e^{\theta T_1}\right] + \frac{T_1}{\theta^2}\Big(\theta(P-a) + b\left(e^{\theta T_1} - \frac{\theta T_1}{2}\right)\Big)\Big\} + \frac{a(s+\delta\alpha)}{\delta^2}\left\{\delta(T-T_1) - \log\left(1+\delta(T-T_1)\right)\right\} + c_d\left[aT_1 + \frac{bT_1^2}{2}\right] - \frac{pI_e T_1^2}{6}\left[3a + bT_1\right] - \frac{pI_e T_1(M-T_1)}{2}\left[2a + bT_1\right]\Big\}$$

(20)

Hence the optimal values are $T_1{}^*$ and T^* and optimum average cost is $TC_2(T_1^*, T^*)$.

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3. NUMERICAL ANALYSIS

Table 3.1 Changes in different decision variables

		$\boldsymbol{\delta}$	
M	**2**	**3**	**4**
5 TC (T_1, T)	4.4963 X 10^{28}	9.2133 X 10^{26}	1.8418 X 10^{25}
$\boldsymbol{T_1^*}$	827.1952	778.6209	729.7397
$\boldsymbol{T^*}$	826.6952	778.2875	729.4897
10 TC (T_1, T)	3.0291 X 10^{21}	2.2013 X 10^{17}	1.3824 X 10^{13}
$\boldsymbol{T_1^*}$	620.8900	501.8787	381.1116
$\boldsymbol{T^*}$	620.3900	501.5454	380.8616
15 TC (T_1, T)	1.8893 X 10^{14}	7.8069 X 10^{7}	7.1454 X 10^{4}
$\boldsymbol{T_1^*}$	413.7335	230.4927	50.2504
$\boldsymbol{T^*}$	413.2335	230.1594	49.9543
20 TC (T_1, T)	1.0331 X 10^{7}	5.3775 X 10^{4}	7.5777 X 10^{3}
$\boldsymbol{T_1^*}$	205.0625	36.8832	168.5181
$\boldsymbol{T^*}$	204.5625	36.4306	168.2681
25 TC (T_1, T)	1.3602 X 10^{7}	7.2276 X 10^{6}	3.1132 X 10^{4}
$\boldsymbol{T_1^*}$	5.9916	200.5001	506.2087
$\boldsymbol{T^*}$	6.2550	200.1668	505.9587

Table 3.2 Changes in different decision variables

			$\boldsymbol{\delta}$		
M	**1**	**2**	**3**	**4**	**5**
5 TC (T_1,T)	4.1905 X 10^{43}	1.1096 X 10^{38}	6.2022 X 10^{33}	1.5567 X 10^{30}	9.3828 X 10^{26}
$\boldsymbol{T_1^*}$	1.4135 X 10^{3}	1.2301 X 10^{3}	1.0903 X 10^{3}	971.8674	866.0034
$\boldsymbol{T^*}$	1.4125 X 10^{3}	1.2296 X 10^{3}	1.0899 X 10^{3}	971.6174	865.8034
10 TC (T_1,T)	2.4093 X 10^{35}	3.7857 X 10^{27}	1.1996 X 10^{21}	1.5939 X 10^{15}	4.4645 X 10^{9}
	1.1425 X 10^{3}	885.9160	672.2785	479.2169	297.0809
$\boldsymbol{T_1^*}$	1.1415 X 10^{3}	885.4160	675.9451	478.9669	296.8809
$\boldsymbol{T^*}$					
30 TC (T_1,T)	5.1465 X 10^{35}	3.6045 X 10^{28}	7.5704 X 10^{21}	2.4025 X 10^{36}	2.4001 X 10^{49}
$\boldsymbol{T_1^*}$	30.4084	173.2545	698.5739	1.1754 X 10^{3}	1.6009 X 10^{3}
$\boldsymbol{T^*}$	29.3617	172.7545	698.2406	1.1751 X 10^{3}	1.6007 X 10^{3}

Table 3.3 Changes in different decision variables

			δ	
M		**1**	**2**	**3**
5	**TC (T₁, T)**	3.1050 X 10^{5}	9.2133 X 10^{26}	1.8418 X 10^{25}
	T_1^*	34.9379	778.6209	729.7397
	T^*	34.8524	778.2875	729.4897
10	**TC (T₁, T)**	2.4945 X 10^{5}	1.7809 X 10^{5}	2.6640 X 10^{4}
	T_1^*	27.8632	19.9712	13.8226
	T^*	26.7505	19.4182	13.4452
15	**TC (T₁, T)**	1.8503 X 10^{5}	9.6210 X 10^{4}	1.8597 X 10^{4}
	T_1^*	20.3688	10.3770	1.2340
	T^*	19.2046	9.7606	1.7273
25	**TC (T₁, T)**	2.2438 X 10^{4}	8.4983 X 10^{4}	1.7930 X 10^{5}
	T_1^*	4.1077	9.1159	20.3517
	T^*	0.0161	8.4776	19.9900
35	**TC (T₁, T)**	9.8379 X 10^{4}	2.4016 X 10^{5}	3.1132 X 10^{17}
	T_1^*	10.3926	27.3753	33.3314
	T^*	8.9951	26.8382	32.9819
45	**TC (T₁, T)**	2.3268 X 10^{5}	3.5302 X 10^{5}	3.2294 X 10^{5}
	T_1^*	26.1037	40.5510	37.2245
	T^*	24.9813	40.0281	36.8770
55	**TC (T₁, T)**	3.5347 X 10^{5}	4.2710 X 10^{5}	2.9820 X 10^{5}
	T_1^*	40.1939	49.1483	34.4051
	T^*	39.1227	48.6314	34.0561

Table 3.4 Changes in different decision variables

		M	
δ		**5**	**15**
1	**TC (T₁, T)**	4.0237 X 10^{30}	1.1256 X 10^{23}
	T_1^*	883.3441	666.0408
	T^*	882.3441	665.0408
2	**TC (T₁, T)**	2.7562 X 10^{28}	8.9539 X 10^{18}
	T_1^*	821.0799	548.1489
	T^*	820.5799	547.6489
3	**TC (T₁, T)**	6.5556 X 10^{26}	2.3585 X 10^{15}
	T_1^*	774.3687	445.2460
	T^*	774.0354	444.9126
4	**TC (T₁, T)**	3.0652 X 10^{25}	1.2527 X 10^{12}
	T_1^*	736.1033	351.1609
	T^*	735.8533	350.9109

5 TC (T_1, T)	2.1144 X 10^{24}	1.0747 X 10^{9}
T_1^*	702.6979	263.1678
T^*	702.4979	262.9678
10 TC (T_1, T)	2.3901 X 10^{19}	8.7991 X 10^{3}
T_1^*	560.4200	2.7822
T^*	560.3200	2.8177
20 TC (T_1, T)	3.6397 X 10^{9}	1.8342 X 10^{3}
T_1^*	278.3660	1.0174 X 10^{3}
T^*	278.3160	0.0174 X 10^{3}
25 TC (T_1, T) T_1^*	2.4325 X 10^{5}	1.0987 X 10^{3}
T^*	139.7781	2.1046 X 10^{3}
	139.7381	2.1046 X 10^{3}

4. CONCLUSION

The two scenarios presented in the paper shed light on different dynamics in the buyer-supplier relationship regarding permissible delay and interest rates. The findings highlight the importance of understanding and optimizing credit terms in inventory management strategies.

4.1 Results and Discussion

4.1.1. Scenario 1: Supplier's Permissible Delay M ≤ T1 In this scenario, where the supplier's permissible delay M is less than or equal to the total credit period T1, the analysis reveals a distinct pattern in interest earnings and payments for the buyer. During each credit cycle, the buyer earns interest (IE1) in the interval [0, T1) while paying interest (Ip) in the interval [M, T1). This model encourages the buyer to settle the payment within the permissible delay to avoid additional costs. The implications of this scenario suggest that the buyer can benefit from earning interest on the outstanding amount during the initial period of the credit term. However, if the buyer delays payment beyond the permissible delay, the buyer incurs interest charges, serving as a financial incentive for timely payment. This scenario emphasizes the importance of managing cash flow effectively to maximize interest earnings and minimize interest payments.

4.1.2. Scenario 2: Supplier's Permissible Delay T1 < M In this alternative scenario, the supplier's permissible delay T1 is less than the total credit period M. Notably, during the interval [0, M), denoted as IE2, the buyer earns interest at the Ie rate without paying any interest to the supplier. The discussion surrounding this scenario reveals a different dynamic in the buyer-supplier relationship. The buyer has the advantage of earning interest on the outstanding amount during the entire permissible delay period [0, M) without incurring any interest payments to the supplier. This situation presents an opportunity for the buyer to utilize the available cash resources strategically, potentially generating additional income through interest-earning investments. However, it is crucial for the buyer to manage the payment effectively and settle the outstanding amount within the permissible delay (M) to avoid additional interest charges. The findings suggest that the buyer's ability to optimize cash flow and leverage the interest-earning potential during the permissible delay can significantly impact the overall financial performance.

Overall, both scenarios emphasize the importance of effectively managing credit terms, permissible delays, and interest rates in inventory management. The findings provide valuable insights into the dynamics of the buyer-supplier relationship, enabling businesses to make informed decisions and optimize their inventory strategies based on the specific credit terms and financial incentives involved. Furthermore, the analysis of these two scenarios underscores the significance of aligning credit terms with the cash flow dynamics and financial goals of the buyer. The choice of permissible delay and the corresponding interest rates can have a substantial impact on the buyer's ability to earn interest, manage costs, and optimize working capital.

In Scenario 1, where the permissible delay is within or equal to the total credit period, the buyer faces the risk of incurring interest payments if the payment is delayed beyond the permissible limit. This setup encourages prompt payment and serves as a mechanism to incentivize the buyer to maintain a healthy cash flow and minimize financial costs. It highlights the importance of effective cash flow management and timely payment to leverage interest earnings while avoiding additional expenses. On the other hand, Scenario 2 presents a scenario where the permissible delay exceeds the total credit period. In this case, the buyer has the advantage of earning interest throughout the entire permissible delay without incurring any interest payments to the supplier. This scenario provides the buyer with an opportunity to strategically utilize cash resources and potentially earn additional income through interest-earning investments. However, it also emphasizes the importance of disciplined financial management to ensure timely payment within the permissible delay and avoid any negative consequences such as interest charges. Overall, the analysis of these scenarios emphasizes the need for businesses to carefully consider and negotiate credit terms with suppliers to align with their financial objectives. Optimizing credit terms can lead to improved cash flow, reduced financial costs, and enhanced profitability. By understanding the dynamics of permissible delay and interest rates, buyers can make informed decisions regarding inventory management, working capital allocation, and financial strategies. It is important to note that the specific implications and outcomes of these scenarios may vary depending on the industry, market conditions, and individual buyer-supplier relationships. Therefore, businesses should conduct a thorough analysis of their unique circumstances and consider the potential trade-offs and benefits associated with different credit terms and interest rate structures.

4.2 Research contribution

The objective of the developed analytical solution model is to determine the optimal production inventory cost by considering the cycle period and incorporating the interest earned during shortage periods. Through the model, the study aims to identify the most cost-effective production and inventory strategies. To illustrate the effectiveness of the model, a numerical example is provided to showcase the optimal values of the decision variables. This example demonstrates the practical application of the analytical solution in real-world scenarios and highlights the potential cost savings and efficiency improvements achievable through its implementation. Furthermore, the model has the potential for extension to accommodate additional factors such as price breaks and various holding costs. By incorporating these elements, the model can offer a more comprehensive and accurate representation of the inventory management problem, enabling businesses to make informed decisions and optimize their operations based on specific pricing and holding cost considerations.

Overall, the developed analytical solution model provides a valuable framework for evaluating and optimizing production inventory costs. Its flexibility to incorporate various factors and its potential for extension make it a versatile tool for businesses seeking to enhance their inventory management strategies and achieve cost savings.

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